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A CRITERION FOR GORENSTEIN ALGEBRAS TO BE REGULAR

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Abstract. In this paper we give a criterion for a left Gorenstein algebra to be AS-regular. Let A be a left Gorenstein algebra such that the trivial module ${}_Ak$ admits a finitely generated minimal free resolution. Then A is AS-regular if and only if its left Gorenstein index is equal to $-\inf\{i \mid \mathrm{Ext}_A^{\mathrm{depth}_A A}(k,k)_i \neq 0\}$. Furthermore, A is Koszul AS-regular if and only if its left Gorenstein index is $\mathrm{depth}_A A = -\inf\{i \mid \mathrm{Ext}_A^{\mathrm{depth}_A A}(k,k)_i \neq 0\}$.

As applications, we prove that the category of AS-regular algebras is a tensor category and that a left Noetherian p-Koszul, left Gorenstein algebra is AS-regular if and only if it is p-standard. This generalizes a result of Dong and the second author.

Introduction

A connected graded algebra is called left (resp. right) Gorenstein if the Ext-group from the trivial left (resp. right) module to itself is 1-dimensional. Gorenstein algebras have a multitude of connections to algebraic topology, representation theory and non-commutative algebraic geometry. In the last twenty years, Gorenstein algebras have been intensively studied in literature. AS-regular (resp. AS-Gorenstein) algebras, which were introduced by Artin and Schelter ([AS]), are Gorenstein algebras with finite global (resp. injective) dimension. AS-regular algebras are thought to be the coordinate rings of the corresponding non-commutative projective spaces in the non-commutative projective geometry. One of the central questions in non-commutative projective geometry is to classify non-commutative projective spaces, or equivalently, to classify the corresponding AS-regular algebras. In [DW], Dong and the second author proved that any Noetherian Koszul standard AS-Gorenstein algebra is AS-regular by using Catelnuovo-Mumford regularity. The motivation of this paper is to study when left Gorenstein algebras are AS-regular. The following result is Theorem 2.2 and Corollary 2.4.

Theorem A. *Let A be a left Gorenstein algebra such that the trivial module ${}_Ak$ admits a finitely generated minimal free resolution. Then A is AS-regular if and only if*

$$\mathrm{Gor}_A A = -\inf\{i \mid \mathrm{Ext}_A^{\mathrm{depth}_A A}(k,k)_i \neq 0\}.$$

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Furthermore, A is Koszul and AS-regular if and only if

$$\mathrm{Gor}_A A = \mathrm{depth}_A A = -\inf\{i \mid \mathrm{Ext}_A^{\mathrm{depth}_A A}(k,k)_i \neq 0\}.$$

For the definition of $\mathrm{Gor}_A A$, see Definition 2.1. Note that ${}_A k$ admits a finitely generated minimal free resolution of graded A-modules if A is a left Noetherian graded algebra. By using the above theorem, we prove that the tensor product of two AS-regular graded algebras is also AS-regular, and we prove the following proposition in Proposition 3.3.

Proposition B. *Let A be a left Noetherian p-Koszul left Gorenstein algebra ($p \geq 2$). Then A is AS-regular if and only if A is p-standard.*

This proposition is a generalization of [DW, Theorem 4.10]. The p-standard left Gorenstein algebra is defined in Definition 3.2.

1. Preliminaries

Throughout this paper, k is a fixed field. A k-algebra A is called $\mathbb{N}$-graded if A has a k-vector space decomposition $A = \bigoplus_{i\geq 0} A_i$ such that $A_i A_j \subseteq A_{i+j}$ for all $i, j \geq 0$. An $\mathbb{N}$-graded algebra is called connected graded if $A_0 = k$. In the following, A will always be a connected graded k-algebra if no special assumption is emphasized and $\mathfrak{m}$ will be its maximal graded ideal $\bigoplus_{i>0} A_i$.

A left A-module M, with a decomposition $M = \bigoplus_{i\in\mathbb{Z}} M_i$ of k-vector spaces, is called graded if $A_i M_j \subseteq M_{i+j}$ for all i, j. Graded right A-modules are defined similarly. Obviously the residue field k has a trivial graded left or right A-module structure via the canonical surjection $\varepsilon : A \to k$. The opposite algebra of A is denoted by A^{op}. Any graded right A-module can be identified with a graded left A^{op}-module.

A graded A-module M is called locally finite if each graded piece M_i is a finite-dimensional k-vector space; M is called bounded below if $M_i = 0$ for $i \ll 0$. Let M and N be two graded left A-modules. An A-module homomorphism $f : M \to N$ is said to be a graded homomorphism of degree l if $f(M_i) \subseteq N_{i+l}$ for all $i \in \mathbb{Z}$. Let $\mathrm{Gr}A$ be the category of graded left A-modules and graded homomorphisms of degree 0, and let $\mathrm{Hom}_{\mathrm{Gr}A}(-,-)$ be the hom-functor in the category $\mathrm{Gr}A$. If $M \in \mathrm{Gr}A$, $M(n)$ is the n-th shift of M, with $M(n)_i = M_{n+i}$. The graded vector space of all graded A-homomorphisms from M to N is denoted by

$$\underline{\mathrm{Hom}}_A(M,N) = \bigoplus_{n=-\infty}^{\infty} \mathrm{Hom}_{\mathrm{Gr}A}(M, N(n)).$$

For any complex X of graded A-modules and $d \in \mathbb{Z}$, we denote $X[d]$ as the d-th twisting of X such that $(X[d])^i = X^{d+i}$.

The derived category of $\mathrm{Gr}A$ is denoted by $\mathrm{D}(\mathrm{Gr}A)$. The full subcategories of $\mathrm{D}(\mathrm{Gr}A)$ consisting of objects which are cohomologically bounded below, bounded above and bounded respectively are denoted by $\mathrm{D}^+(\mathrm{Gr}A)$, $\mathrm{D}^-(\mathrm{Gr}A)$ and $\mathrm{D}^{\mathrm{b}}(\mathrm{Gr}A)$. The right derived functor of $\underline{\mathrm{Hom}}_A(-,-)$ is denoted by $R\mathrm{Hom}_A(-,-)$, and the left derived functor of $-\otimes_A -$ is denoted by $-{}^L\otimes_A -$. The Ext and Tor are defined as

$$\mathrm{Ext}_A^i(X,Y) = H^i(R\mathrm{Hom}_A(X,Y)) \quad \text{and} \quad \mathrm{Tor}_i^A(X,Y) = H^{-i}(X{}^L\otimes_A Y).$$

A bounded above complex L of graded free left A-modules is called minimal if $d_L^i(L^i) \subseteq \mathfrak{m}L^{i+1}$. In this case, the differentials in $\mathrm{Hom}_A(L,k)$ and $k \otimes_A L$ are zero.

A bounded below complex I of graded injective A-modules is called minimal if $\ker(d_I^i)$ is graded essential in I^i for each $i \in \mathbb{Z}$. In this case, the complex $\mathrm{Hom}_A(k, I)$ has zero differential.

For any graded A-module M, its depth is $\mathrm{depth}_A M = \inf\{i \mid \mathrm{Ext}_A^i(k, M) \neq 0\}$. The projective dimension and injective dimension of M are defined respectively as

$$\mathrm{pd}_A M = \sup\{i \mid \mathrm{Ext}_A^i(M, N) \neq 0 \text{ for some } N \in \mathrm{Gr}A\},$$
$$\mathrm{id}_A M = \sup\{i \mid \mathrm{Ext}_A^i(N, M) \neq 0 \text{ for some } N \in \mathrm{Gr}A\}.$$

If M is bounded below, then $\mathrm{pd}_A M = \sup\{i \mid \mathrm{Ext}_A^i(M, k) \neq 0\}$ and $\mathrm{pd}_A M$ coincides with the usual (ungraded) projective dimension of M by the existence of minimal free resolution. The graded global dimension of A, defined via the (graded) projective dimensions of graded modules, coincides with the usual global dimension of A, and it is well known that $\mathrm{gl.dim}(A) = \mathrm{pd}_A k = \mathrm{pd}_{A^{op}} k$.

2. Main theorem

Let A be a connected graded algebra. We say that A is left (resp. right) Gorenstein if $\dim_k \mathrm{Ext}_A^*(k, A) = 1$ (resp. $\dim_k \mathrm{Ext}_{A^{op}}^*(k, A) = 1$), where $\mathrm{Ext}_A^*(k, A) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Ext}_A^i(k, A)$. For a left Gorenstein graded algebra A, there is some integer l such that

$$\mathrm{Ext}_A^i(k, A) = \begin{cases} 0, & i \neq \mathrm{depth}_A A, \\ k(l), & i = \mathrm{depth}_A A. \end{cases} \tag{1}$$

A left (resp. right) Gorenstein graded algebra A is called left (resp. right) AS-Gorenstein (AS stands for Artin-Schelter) if $\mathrm{id}_A A < \infty$ (resp. $\mathrm{id}_{A^{op}} A < \infty$). If, further, $\mathrm{gl.dim} A < \infty$, then we say A is left (resp. right) AS-regular.

The usual definition of AS-regular algebras in the literature in the late 1980s and early 1990s requires the algebra to have finite Gelfand-Kirillov dimension. Note that our definition of AS-regular algebra differs from this. By [SZ, Theorem 2.4], any left (or right) Noetherian connected graded algebra with finite global dimension has finite Gelfand-Kirillov dimension. Hence there is no difference between our definition and the original definition of AS-regular algebra for left (or right) Noetherian connected graded algebras.

Definition 2.1. Let A be a left Gorenstein algebra. The number l as in (1) is called the left Gorenstein index of A, denoted by $\mathrm{Gor}_A A$.

For right Gorenstein graded algebras, we can define right Gorenstein index similarly. Note that the left (resp. right) Gorenstein index was called the Artin-Schelter index in [LPWZ]. For any left Gorenstein graded algebra A, $\mathrm{Gor}_A A$ is closely related to A's AS-regularity. The following theorem gives a criterion for a left Gorenstein graded algebra to be AS-regular.

Theorem 2.2. *Let A be a left Gorenstein algebra such that the trivial graded A-module ${}_A k$ admits a minimal free resolution consisting of finitely generated A-modules. Then the following are equivalent:*

1. *A is left AS-regular.*
2. *A is AS-regular.*
3. $\mathrm{Gor}_A A \geq -\inf\{i \mid \mathrm{Ext}_A^{\mathrm{depth}_A A}(k, k)_i \neq 0\}$.

(4) *The d-th part F_d of the minimal free resolution of ${}_Ak$ has a free basis concentrated in degrees $\leq \mathrm{Gor}_A A$, where $d = \mathrm{depth}_A A$.*
(5) *The d-th part F_d of the minimal free resolution of ${}_Ak$ is generated by one element in degree $\mathrm{Gor}_A A$, where $d = \mathrm{depth}_A A$.*

If one of the above holds, then $\mathrm{Gor}_A A = -\inf\{i \mid \mathrm{Ext}_A^{\mathrm{depth}_A A}(k,k)_i \neq 0\}$.

Proof. (1) ⇒ (5). Let A be a left AS-regular algebra with global dimension d. Then ${}_Ak$ admits a minimal free resolution

$$0 \to F_d \xrightarrow{\partial} F_{d-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} F_1 \xrightarrow{\partial} F_0 \xrightarrow{\varepsilon} {}_Ak \to 0,$$

where each F_i is finitely generated and $F_0 = A$. Since $\mathrm{pd}_A k = \mathrm{gl.dim}\, A = d$, it is easy to see that $\mathrm{depth}_A A = \mathrm{pd}_A k = d$. Let $\mathrm{Gor}_A A = l$. Then

$$0 \to (F_0)^\dagger \xrightarrow{(\partial)^\dagger} \cdots \xrightarrow{(\partial)^\dagger} (F_d)^\dagger \to k(l) \to 0$$

is a finitely generated minimal free resolution of $k_A(l)$, where $M^\dagger = \mathrm{Hom}_A(M, A)$ for any graded A-module M. Hence $(F_d)^\dagger \cong A(l)$ and $F_d \cong {}_AA(-l)$.

(5) ⇒ (4). Obvious.

(4) ⇔ (3). It follows from $\mathrm{Ext}_A^d(k,k) = \underline{\mathrm{Hom}}_A(F_d, k)$.

(3) ⇒ (2). Let A be a left Gorenstein algebra with $\mathrm{depth}_A A = d$ and $\mathrm{Gor}_A A = l \geq -\inf\{i \mid \mathrm{Ext}_A^{\mathrm{depth}_A A}(k,k)_i \neq 0\}$. By the assumption, ${}_Ak$ admits a finitely generated minimal free resolution

$$\cdots \xrightarrow{\partial} F_i \xrightarrow{\partial} \cdots \xrightarrow{\partial} F_1 \xrightarrow{\partial} A \xrightarrow{\varepsilon} {}_Ak \to 0, \tag{2}$$

where $F_i = A \otimes V_i$, each V_i is a finite-dimensional graded space and V_d is concentrated in degrees $\leq l$.

The short exact sequence

$$0 \to \mathfrak{m} \xrightarrow{\iota} A \xrightarrow{\varepsilon} k \to 0 \tag{3}$$

induces the following long exact sequence:

$$\cdots \to \mathrm{Ext}_A^d(k,\mathfrak{m})_{-l} \to \mathrm{Ext}_A^d(k,A)_{-l} \xrightarrow{\mathrm{Ext}_A^d(k,\varepsilon)_{-l}} \mathrm{Ext}_A^d(k,k)_{-l} \to \cdots.$$

As a sub-quotient of $\underline{\mathrm{Hom}}_A(F_d, \mathfrak{m})$, $\mathrm{Ext}_A^d(k,\mathfrak{m})$ is concentrated in degrees $> -l$. Hence $\mathrm{Ext}_A^d(k,\mathfrak{m})_{-l} = 0$. Since $\mathrm{Ext}_A^d(k,A)_{-l} \cong k_A \neq 0$, $\mathrm{Ext}_A^d(k,\varepsilon)_{-l} \neq 0$.

There exists $f \in \underline{\mathrm{Hom}}_A(F_d, A)$ such that $f\partial = 0$ and $\varepsilon f \neq 0$. As $\underline{\mathrm{Hom}}_A(F_d, k) \cong \underline{\mathrm{Hom}}_k(V_d, k)$, there exists $v \in V_d$ such that $\varepsilon f(v) = 1$. This implies that $f(v) = 1$ for some $v \in V_d$ and $f : F_d \to A$ is surjective. Therefore, f splits and $F_d = \ker f \oplus (A \otimes kv)$. Since $\ker f$ is graded free, $\ker f = A \otimes X_d$ for some graded space X_d. Then $F_d = A \otimes (kv \oplus X_d)$. Since $f\partial = 0$, we have $\partial(F_{d+1}) \subseteq A \otimes X_d$.

Let $Q^i = \underline{\mathrm{Hom}}_A(F_i, A)$ and $\delta = \underline{\mathrm{Hom}}_A(\partial, A)$. Since (2) is a finitely generated minimal free resolution, the complex

$$(Q^\bullet, \delta): \quad 0 \to Q^0 \xrightarrow{\delta} Q^1 \xrightarrow{\delta} \cdots \xrightarrow{\delta} Q^i \xrightarrow{\delta} \cdots$$

consisting of finitely generated graded free right A-modules is minimal. Obviously,

$$Q^d \cong (kf \otimes A) \oplus (\mathrm{Hom}_k(X_d, k) \otimes A) \quad \text{and} \quad \delta(f) = 0.$$

Since A is left Gorenstein, by (1), there is an integer d such that

$$H^i(Q^\bullet, \delta) = \mathrm{Ext}_A^i(k, A) = \begin{cases} 0, & i \neq d, \\ kf, & i = d. \end{cases}$$

In the following, we show by induction on $j\,(0 \leq j \leq d)$ that Q^{d-j} admits a decomposition

$$Q^{d-j} = (W^{d-j} \oplus R^{d-j}) \otimes A$$

with

$$\begin{cases} W^d = kf,\ R^d = \operatorname{Hom}_k(X_d, k), \\ \delta(W^{d-j}) \subseteq W^{d-j+1} \otimes A,\ \delta(R^{d-j}) \subseteq R^{d-j+1} \otimes A,\ j = 1, \cdots, d. \end{cases}$$

Suppose this is true for $j-1$. We choose a graded free right A-module decomposition $Q^{d-j} = (E \oplus F \oplus S) \otimes A$ of Q^{d-j} such that

$$\delta(E) \subseteq W^{d-j+1} \otimes A,\ \delta(F) \subseteq R^{d-j+1} \otimes A, \tag{4}$$

and $\dim_k(E \oplus F)$ is maximal among all such graded free right A-module decompositions of Q^{d-j} satisfying (4). Since Q^{d-j} is finitely generated, such a maximal decomposition exists. We claim that $S = 0$.

Suppose on the contrary that $S \neq 0$. Since S is finite-dimensional, there is a non-zero element $x \in S$ with the minimal degree, i.e., $|x| = \min\{|s| \mid s \in S\}$. Let

$$\delta(x) = \delta_1(x) + \delta_2(x), \text{ with } \delta_1(x) \in W^{d-j+1} \otimes A,\ \delta_2(x) \in R^{d-j+1} \otimes A.$$

Then $\delta(\delta_1(x)) = -\delta(\delta_2(x)) \in (W^{d-j+2} \otimes A) \cap (R^{d-j+2} \otimes A) = 0$. Hence both $\delta_1(x)$ and $\delta_2(x)$ are cocycles. When $j = 1$, $\delta_1(x) \in kf \otimes \mathfrak{m}$ and $\delta_2(x) \in \operatorname{Hom}_k(X_d, k) \otimes \mathfrak{m}$ by the minimality of $(Q^\bullet, \delta)$. Since $H^d(Q^\bullet, \delta) = kf$, both $\delta_1(x)$ and $\delta_2(x)$ are coboundaries. When $j \geq 2$, $\delta_1(x)$ and $\delta_2(x)$ are coboundaries since $H^{d-j+1}(Q^\bullet, \delta) = 0$. Hence there exist $\alpha_1, \alpha_2 \in Q^{d-j}$ such that $\delta(\alpha_1) = \delta_1(x)$ and $\delta(\alpha_2) = \delta_2(x)$.

For any $\alpha \in T \otimes A$, we denote $\alpha = \overline{\alpha} + \alpha'$ with $\overline{\alpha} \in T \otimes k$ and $\alpha' \in T \otimes \mathfrak{m}$. It follows from the minimality of $|x|$ that $\alpha_1' \in (E \oplus F) \otimes \mathfrak{m}$. Clearly, $\bar{\alpha}_1$ either belongs to $E \oplus F$ or it does not.

If $\bar{\alpha}_1 \in E \oplus F$, then $(k(x - \alpha_1) \otimes A) \cap ((E \oplus F \oplus S/kx) \otimes A) = 0$. Indeed, if for any $a \in A$, $(x - \alpha_1) \otimes a \in (E \oplus F \oplus S/kx) \otimes A$, then we have $x \otimes a = (x - \alpha_1) \otimes a + \bar{\alpha}_1 \otimes a + \alpha_1' \otimes a \in ((E \oplus F \oplus S/kx) \otimes A) \cap (kx \otimes A) = 0$. Hence $(k(x - \alpha_1) \oplus E \oplus F \oplus S/kx) \otimes A = (E \oplus F \oplus S) \otimes A$. But then $\delta(x - \alpha_1) = \delta_2(x) \in R^{d-j+1} \otimes A$ and $\dim_k(k(x - \alpha_1) \oplus E \oplus F) = \dim_k(E \oplus F) + 1$, which contradicts the maximality property of the chosen decomposition.

In the case of $\bar{\alpha}_1 \notin E \oplus F$, let $\bar{\alpha}_1 = e_1 + f_1 + s_1$, where $e_1 \in E, f_1 \in F$ and $0 \neq s_1 \in S$. Then $(k\alpha_1 \otimes A) \cap ((E \oplus F) \otimes A) = 0$. Indeed, if $\alpha_1 \otimes a \in (E \oplus F) \otimes A$, then $\bar{\alpha}_1 \otimes a = \alpha_1 \otimes a - \alpha_1' \otimes a \in (E \oplus F) \otimes A$, which implies that $s_1 \otimes a = (\bar{\alpha}_1 - e_1 - f_1) \otimes a \in ((E \oplus F) \otimes A) \cap (S \otimes A) = 0$. Hence $(k\alpha_1 \oplus E \oplus F) \otimes A$ is a graded free submodule of $(E \oplus F \oplus S) \otimes A$. Since $s_1 = \alpha_1 - \alpha_1' - e_1 - f_1 \in (k\alpha_1 \oplus E \oplus F) \otimes A$, there exists a homomorphism of graded right A-modules

$$g : (E \oplus F \oplus S) \otimes A \to (k\alpha_1 \oplus E \oplus F) \otimes A$$

$$\text{such that} \quad \begin{cases} g(e) = e, & \text{for any} \quad e \in E, \\ g(f) = f, & \text{for any} \quad f \in F, \\ g(s) = 0, & \text{for any} \quad s \in S/ks_1, \\ g(s_1) = s_1. \end{cases}$$

Then $g(\alpha_1) = \alpha_1$, and g is a surjective homomorphism of graded right A-modules which splits. It is easy to see that the inclusion map from $(k\alpha_1 \oplus E \oplus F) \otimes A$ to

$(E \oplus F \oplus S) \otimes A$ is a right inverse of g. Then

$$(E \oplus F \oplus S) \otimes A = (k\alpha_1 \oplus E \oplus F) \otimes A \oplus \ker(g).$$

Since $\ker(g)$ is bounded below, it is free. Now $\delta(\alpha_1) \in W^{d-j+1} \otimes A$ and

$$\dim_k(k\alpha_1 \oplus E \oplus F) = \dim_k(E \oplus F) + 1.$$

This also contradicts the maximality property of the chosen decomposition.

Therefore, $S = 0$.

Now let $W^{d-j} = E$ and $R^{d-j} = F$. Then Q^{d-j} admits a decomposition

$$Q^{d-j} = (W^{d-j} \oplus R^{d-j}) \otimes A,$$

with $\delta(W^{d-j}) \subseteq W^{d-j+1} \otimes A$ and $\delta(R^{d-j}) \subseteq R^{d-j+1} \otimes A$.

By induction, we have proved that for any $j = 0, \cdots, d$, Q^{d-j} admits a decomposition

$$Q^{d-j} = (W^{d-j} \oplus R^{d-j}) \otimes A,$$

with $\delta(W^{d-j}) \subseteq W^{d-j+1} \otimes A$ and $\delta(R^{d-j}) \subseteq R^{d-j+1} \otimes A$ for all $j \geq 1$.

For any $i = 0, \cdots, d$, let $P^i = W^i \otimes A$. Then the subcomplex

$$(P^\bullet, \delta):\ 0 \to P^0 \xrightarrow{\delta} P^1 \xrightarrow{\delta} \cdots \xrightarrow{\delta} P^{d-1} \xrightarrow{\delta} P^d \to 0$$

of $(Q^\bullet, \delta)$ satisfies that

$$H^i(P^\bullet, \delta) = \begin{cases} 0, & i \neq d, \\ kf, & i = d. \end{cases}$$

This shows that $(P^\bullet, \delta)$ is a minimal free resolution of k_A. Hence A is a left AS-regular graded algebra with gl.dim$A = d$. This implies that

$$0 \to F_d \xrightarrow{\partial} F_{d-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} F_1 \xrightarrow{\partial} A \xrightarrow{\varepsilon} {}_Ak \to 0$$

is a finitely generated minimal free resolution of ${}_Ak$. The left Gorensteinness of A implies that

$$0 \to (F_0)^\dagger \xrightarrow{(\partial)^\dagger} \cdots \xrightarrow{(\partial)^\dagger} (F_d)^\dagger \to k(l) \to 0$$

is a finitely generated minimal free resolution of $k(l)$ as a right A-module. Hence

$$\mathrm{Ext}^i_{A^{op}}(k(l), A) = \begin{cases} 0, & i \neq d, \\ {}_Ak, & i = d \end{cases} \quad \text{and} \quad \mathrm{Ext}^i_{A^{op}}(k, A) = \begin{cases} 0, & i \neq d, \\ {}_Ak(l), & i = d, \end{cases}$$

so A is AS-regular.

(2) $\Rightarrow$ (1). Obvious. □

The proof of (3) $\Rightarrow$ (2) in Theorem 2.2 is modified from the proof of [FM, Theorem 1]. For any left Noetherian connected graded algebra A, ${}_Ak$ admits a finitely generated minimal free resolution of graded A-modules. So we have the following corollary.

Corollary 2.3. *Let A be a left Noetherian, left Gorenstein algebra. Then A is AS-regular if and only if*

$$\mathrm{Gor}_A A = -\inf\{i \mid \mathrm{Ext}_A^{\mathrm{depth}_A A}(k, k)_i \neq 0\}.$$

A connected graded algebra A is called Koszul if for each $i \geq 0$, the i-th part F_i of the minimal free resolution of ${}_Ak$ is generated in degree i. For more details about Koszul algebras, please see [BGS] and [Sm].

Corollary 2.4. *Let A be a left Gorenstein algebra such that the trivial graded A-module ${}_Ak$ admits a finitely generated minimal free resolution. Then A is Koszul and AS-regular if and only if*

$$\mathrm{Gor}_A A = \mathrm{depth}_A A = -\inf\{i \mid \mathrm{Ext}_A^{\mathrm{depth}_A A}(k,k)_i \neq 0\}.$$

Proof. If A is Koszul and AS-regular, then by Theorem 2.2 and the Koszulity of A,

$$\mathrm{Gor}_A A = \mathrm{depth}_A A = -\inf\{i \mid \mathrm{Ext}_A^{\mathrm{depth}_A A}(k,k)_i \neq 0\}.$$

Conversely, if $\mathrm{Gor}_A A = \mathrm{depth}_A A = -\inf\{i \mid \mathrm{Ext}_A^{\mathrm{depth}_A A}(k,k)_i \neq 0\}$, then A is AS-regular by Theorem 2.2. We claim that A is Koszul. Let $d = \mathrm{gl.dim} A$. Then

$$\mathrm{Ext}_A^i(k,A) = \begin{cases} 0, & i \neq d, \\ k(d), & i = d. \end{cases}$$

By assumption, $d = -\inf\{i \mid \mathrm{Ext}_A^d(k,k)_i \neq 0\}$. Hence the trivial module ${}_Ak$ admits a finitely generated minimal free resolution

$$0 \to F_d \xrightarrow{\partial} F_{d-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} F_1 \xrightarrow{\partial} A \xrightarrow{\varepsilon} {}_Ak \to 0, \tag{5}$$

where F_d is generated in degrees $\leq d$. On the other hand, each F_i is generated in degrees $\geq i$ by the minimality of (5). Hence F_d is generated in degree d.

We show by induction on $n, n = 1, \cdots, d$, that F_{d-n} is generated in degree $d-n$. Suppose this is true for $n-1$. Then $\mathrm{Ext}_A^{d-n+1}(k,\mathfrak{m})$ is concentrated in degrees $> n-d-1$, since it is a sub-quotient of $\mathrm{Hom}_A(F_{d-n+1},\mathfrak{m})$. By the long exact sequence

$$\cdots \to \mathrm{Ext}_A^i(k,\mathfrak{m})_j \to \mathrm{Ext}_A^i(k,A)_j \to \mathrm{Ext}_A^i(k,k)_j \to \mathrm{Ext}_A^{i+1}(k,\mathfrak{m})_j \to \cdots$$

induced from the short exact sequence (3), we conclude that $\mathrm{Ext}_A^{d-n}(k,k)$ is concentrated in degrees $> n-d-1$. So F_{d-n} is generated in degrees $\leq d-n$. On the other hand, F_{d-n} is generated in degrees $\geq d-n$ by the minimality of (5). Hence F_{d-n} is generated in degree $d-n$. By induction, every F_i is generated in degree i. Therefore A is Koszul. □

3. Applications

In [DW, Theorem 4.10], Dong and Wu proved that any Noetherian Koszul standard AS-Gorenstein algebra is AS-regular by using Catelnuovo-Mumford regularity. Now we generalize it to the higher Koszul case. First, we recall the definition of p-Koszul algebras ([Be], [HL], [YZ]).

Let $p > 1$ be an integer. Denote $\alpha_p : \mathbb{N} \to \mathbb{N}$ as the map

$$\alpha_p(n) = \begin{cases} pq, & n = 2q, \\ pq+1, & n = 2q+1. \end{cases}$$

Definition 3.1. A connected graded algebra is called p-Koszul if for each $i \geq 0$, the graded vector space $\mathrm{Ext}_A^i(k,k)$ is concentrated in degree $-\alpha_p(i)$.

A connected graded algebra A is p-Koszul if and only if the i-th part F_i of the minimal free resolution of ${}_Ak$ is generated in degree $\alpha_p(i)$ for each $i \geq 0$. If $p = 2$, then a p-Koszul algebra is just a Koszul algebra.

Definition 3.2. A left Gorenstein algebra A is *p-standard if* $\mathrm{Gor}_A A = \alpha_p(\mathrm{depth}_A A)$.

Obviously, 2-standard left Gorenstein algebra is just the standard left Gorenstein algebra as defined in [DW].

Proposition 3.3. *Let A be a left Noetherian, p-Koszul algebra. Then A is p-standard left Gorenstein if and only if A is AS-regular.*

Proof. If A is p-standard left Gorenstein, then $\mathrm{Gor}_A A = \alpha_p(\mathrm{depth}_A A)$. Since A is p-Koszul, the graded vector space $\mathrm{Ext}_A^{\mathrm{depth}_A A}(k,k)$ is concentrated in degree $-\alpha_p(\mathrm{depth}_A A)$. Hence $\mathrm{Gor}_A A = -\inf\{i \mid \mathrm{Ext}_A^{\mathrm{depth}_A A}(k,k)_i \neq 0\}$. By Corollary 2.3, A is AS-regular.

Conversely, if A is AS-regular, then $\mathrm{Gor}_A A = -\inf\{i \mid \mathrm{Ext}_A^{\mathrm{depth}_A A}(k,k)_i \neq 0\}$ by Corollary 2.3. Since A is p-Koszul, the graded vector space $\mathrm{Ext}_A^{\mathrm{depth}_A A}(k,k)$ is concentrated in degree $-\alpha_p(\mathrm{depth}_A A)$. Hence, $\mathrm{Gor}_A A = -\alpha_p(\mathrm{depth}_A A)$. This implies that A is p-standard left Gorenstein. $\square$

It is easy to see that Proposition 3.3 is a generalization of [DW, Theorem 4.10]. In the remainder of this section, we prove that the tensor product of two AS-regular algebras is AS-regular. First, we have the following proposition on the left Gorensteinness of algebras under tensor product.

Proposition 3.4. *Let A and B be two left Gorenstein algebras. If either A is finite-dimensional or ${}_Bk$ has a finitely generated minimal free resolution of finite length, then $A \otimes B$ is left Gorenstein with*

$$\mathrm{depth}_{A\otimes B} A \otimes B = \mathrm{depth}_A A + \mathrm{depth}_B B$$

and

$$\mathrm{Gor}_{A\otimes B} A \otimes B = \mathrm{Gor}_A A + \mathrm{Gor}_B B.$$

Proof. Let $(F^\bullet, \delta_A)$ and $(G^\bullet, \delta_B)$ be minimal free resolutions of ${}_Ak$ and ${}_Bk$ respectively. Then the tensor product $(P^\bullet, \delta)$ of $(F^\bullet, \delta_A)$ and $(G^\bullet, \delta_B)$ is a minimal complex of graded $A \otimes B$-modules, and it is a minimal free resolution of k as a graded $A \otimes B$-module. By assumption, we have

$$\dim_k \mathrm{H}(\mathrm{Hom}_A(F^\bullet, A)) = \dim_k \mathrm{H}(\mathrm{Hom}_B(G^\bullet, B)) = 1,$$

$$\mathrm{Hom}_A(F^\bullet, A) \simeq k_A[-\mathrm{depth}_A A](\mathrm{Gor}_A A)$$

and

$$\mathrm{Hom}_B(G^\bullet, B) \simeq k_B[-\mathrm{depth}_B B](\mathrm{Gor}_B B).$$

Here we use the twisting of complexes and the shift of graded modules. The readers can see the explanations of this two notations in Section 1. If either $\dim_k A < \infty$ or $(G^\bullet, \delta_B)$ is a bounded complex of finitely generated B-modules, then

$$\mathrm{Hom}_B(G^\bullet, A \otimes B) \cong A \otimes \mathrm{Hom}_B(G^\bullet, B).$$

Hence

$$\begin{aligned}
\mathrm{H}(R\mathrm{Hom}_{A\otimes B}(k, A \otimes B)) &= \mathrm{H}(\mathrm{Hom}_{A\otimes B}(F^\bullet \otimes G^\bullet, A \otimes B)) \\
&\cong \mathrm{H}(\mathrm{Hom}_A(F^\bullet, \mathrm{Hom}_B(G^\bullet, A \otimes B))) \\
&\cong \mathrm{H}(\mathrm{Hom}_A(F^\bullet, A \otimes \mathrm{Hom}_B(G^\bullet, B))) \\
&\cong \mathrm{H}(\mathrm{Hom}_A(F^\bullet, A[-\mathrm{depth}_B B](\mathrm{Gor}_B B))) \\
&\cong k[-\mathrm{depth}_A A - \mathrm{depth}_B B](\mathrm{Gor}_A A + \mathrm{Gor}_B B).
\end{aligned}$$

Hence $A \otimes B$ is left Gorenstein with

$$\mathrm{depth}_{A\otimes B} A \otimes B = \mathrm{depth}_A A + \mathrm{depth}_B B$$

and

$$\mathrm{Gor}_{A\otimes B} A \otimes B = \mathrm{Gor}_A A + \mathrm{Gor}_B B.$$

□

For any connected graded algebra A, $\dim_k \mathrm{Ext}_A^*(k,k) < \infty$ if and only if A has finite global dimension and ${}_Ak$ has a finitely generated minimal free resolution. If A is left Gorenstein with finite global dimension, i.e., A is left AS-regular, then ${}_Ak$ has a finitely generated minimal free resolution by [SZ, Proposition 3.1], and so $\mathrm{Ext}_A^*(k,k)$ is finite dimensional. Using Proposition 3.4, we can prove the following proposition.

Proposition 3.5. *If A and B are two AS-regular algebras, then $A\otimes B$ is AS-regular with*

$$\mathrm{gl.dim}(A \otimes B) = \mathrm{gl.dim}A + \mathrm{gl.dim}B.$$

Proof. Since A and B are AS-regular, both ${}_Ak$ and ${}_Bk$ admit bounded finitely generated minimal free resolutions $(F^\bullet, \delta_A)$ and $(G^\bullet, \delta_B)$ respectively. Therefore $\dim_k \mathrm{Ext}_A^*(k,k)$ and $\dim_k \mathrm{Ext}_B^*(k,k)$ are finite, and $(P^\bullet, \delta) = F^\bullet \otimes G^\bullet$ is a minimal free resolution of k as a graded $A \otimes B$-module. Let $\mathrm{gl.dim}\, A = d_A$ and $\mathrm{gl.dim}\, B = d_B$. Then

$$d_A + d_B = -\inf\{i \mid P^i \neq 0\} = \mathrm{gl.dim}(A \otimes B).$$

Since A and B are AS-regular, by Theorem 2.2,

$$d_A = \mathrm{depth}_A A, \quad \mathrm{Gor}_A A = -\inf\{i \mid \mathrm{Ext}_A^{d_A}(k,k)_i \neq 0\}$$

and

$$d_B = \mathrm{depth}_B B, \quad \mathrm{Gor}_B B = -\inf\{i \mid \mathrm{Ext}_B^{d_B}(k,k)_i \neq 0\}.$$

By Proposition 3.4, $A \otimes B$ is left Gorenstein with

$$\mathrm{depth}_{A\otimes B} A \otimes B = \mathrm{depth}_A A + \mathrm{depth}_B B = d_A + d_B$$

and

$$\begin{aligned}\mathrm{Gor}_{A\otimes B} A \otimes B &= \mathrm{Gor}_A A + \mathrm{Gor}_B B \\ &= -\inf\{i \mid \mathrm{Ext}_A^{d_A}(k,k)_i \neq 0\} - \inf\{i \mid \mathrm{Ext}_B^{d_B}(k,k)_i \neq 0\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}-\inf\{i \mid \mathrm{Ext}_{A\otimes B}^{d_A+d_B}(k,k)_i \neq 0\} &= -\inf\{i \mid \mathrm{Hom}_{A\otimes B}(F^{-d_A} \otimes G^{-d_B}, k)_i \neq 0\} \\ &= -\inf\{i \mid \mathrm{Ext}_A^{d_A}(k,k)_i \neq 0\} - \inf\{i \mid \mathrm{Ext}_B^{d_B}(k,k)_i \neq 0\}.\end{aligned}$$

Hence $\mathrm{Gor}_{A\otimes B} A \otimes B = -\inf\{i \mid \mathrm{Ext}_{A\otimes B}^{d_A+d_B}(k,k)_i \neq 0\}$. □

Proposition 3.5 indicates that the category of AS-regular algebras is a tensor category.

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