PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 139, Number 5, May 2011, Pages 1841–1850
S 0002-9939(2010)10597-X
Article electronically published on October 8, 2010

BERNSTEIN-TYPE THEOREMS IN SEMI-RIEMANNIAN WARPED PRODUCTS

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(Communicated by Richard A. Wentworth)

ABSTRACT. This paper deals with complete hypersurfaces immersed in the $(n+1)$-dimensional hyperbolic and steady state spaces. By applying a technique of S. T. Yau and imposing suitable conditions on both the r-th mean curvatures and on the norm of the gradient of the height function, we obtain Bernstein-type results in each of these ambient spaces.

1. INTRODUCTION

In this paper we are concerned with complete, connected Riemannian hypersurfaces of bounded mean curvature in a class of $(n+1)$-dimensional semi-Riemannian warped product spaces which includes the hyperbolic space $\mathbb{H}^{n+1}$ and the steady state space $\mathcal{H}^{n+1}$ (cf. Remark 3.2).

Related to our work, L. J. Alías and M. Dajczer [3] studied properly immersed complete surfaces of the 3-dimensional hyperbolic space contained between two horospheres, obtaining a Bernstein-type result for the case of constant mean curvature between -1 and 1.

L. J. Alías, M. Dajczer and J. Ripoll [4] extended the classical theorem of Bernstein for minimal graphs (that is, with zero mean curvature) in $\mathbb{R}^3$ to complete minimal surfaces in Riemannian ambient spaces of non-negative Ricci curvature and endowed with a Killing field. This was done under the assumption that the sign of the angle function between a global Gauss map and the Killing field remains unchanged along the surface.

More recently, A. L. Albujer and L. J. Alías [1] have proved that if a complete spacelike hypersurface with constant mean curvature is bounded away from the infinity of the steady state space $\mathcal{H}^{n+1}$, then its mean curvature must be identically 1. As a consequence of this result, they concluded that the only complete spacelike surfaces with constant mean curvature in $\mathcal{H}^3$ which are bounded away from the infinity are the totally umbilical flat surfaces.

Received by the editors November 6, 2009 and, in revised form, March 29, 2010 and May 18, 2010 .

2010 *Mathematics Subject Classification.* Primary 53C42; Secondary 53B30, 53C50, 53Z05, 83C99.

Key words and phrases. Semi-Riemannian manifolds, Lorentz geometry, hyperbolic space, steady state space, spacelike hypersurfaces, mean curvature, Bernstein-type theorems.

The second author is partially supported by CNPq.

The third author is partially supported by PPP/FAPESQ/CNPq.

In [8], the second and third authors have studied complete vertical graphs of constant mean curvature in the hyperbolic and steady state spaces. They first derived suitable formulas for the Laplacians of the height function h and of a support-like function naturally attached to the graph. Then, under appropriate restrictions on the values of the mean curvature and the growth of the height function, they obtained necessary conditions for the existence of such a graph. Further, in the 3-dimensional case, they proved Bernstein-type results in each of these ambient spaces.

Motivated by the works described above and using some of the analytical framework of [19], we obtain in Section 3 Bernstein-type results for complete, connected Riemannian hypersurfaces of the semi-Riemannian warped product space $\epsilon I \times_{e^t} M^n$ (cf. Section 2), under suitable conditions on both the mean curvature and the norm of the gradient of the height function of the hypersurface.

As we said in the beginning, the spaces $\epsilon I \times_{e^t} M^n$ include the $(n+1)$-dimensional steady state space $\mathcal{H}^{n+1}$ as well as the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$, and we quote below the corresponding particular cases of the general results of Theorems 3.1 and 3.4. In what follows, h stands for the height function of the hypersurfaces in question.

Theorem 1.1. *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete, connected spacelike hypersurface bounded away from the infinity on $\mathcal{H}^{n+1}$ with* (*not necessarily constant*) *mean curvature $H \geq 1$. If ∇h has integrable norm on Σ^n, then Σ^n is a hyperplane of $\mathcal{H}^{n+1}$.*

Theorem 1.2. *Let $\psi : \Sigma^n \to \mathbb{H}^{n+1}$ be a complete, connected hypersurface of $\mathbb{H}^{n+1}$, bounded away from the infinity and with* (*not necessarily constant*) *mean curvature $0 < H \leq 1$. If ∇h has integrable norm on Σ^n, then Σ^n is a horosphere of $\mathbb{H}^{n+1}$.*

Finally, we also extend the previous theorems to the context of the r-th mean curvatures under an additional assumption on the norm of the second fundamental form (cf. Theorem 3.6 and Theorem 3.7).

2. Semi-Riemannian warped products

Let $\overline{M}^{n+1}$ be a connected semi-Riemannian manifold with metric $\overline{g} = \langle\ ,\ \rangle$ of index $\nu \leq 1$ and semi-Riemannian connection $\overline{\nabla}$. For a vector field $X \in \mathfrak{X}(\overline{M})$, let $\epsilon_X = \langle X, X\rangle$; X is a *unit* vector field if $\epsilon_X = \pm 1$ and is *timelike* if $\epsilon_X = -1$.

In all that follows, we consider *Riemannian immersions* $\psi : \Sigma^n \to \overline{M}^{n+1}$, namely, immersions from a connected, n-dimensional orientable differentiable manifold Σ^n into $\overline{M}$ such that the induced metric $g = \psi^*(\overline{g})$ turns Σ into a Riemannian manifold (in the Lorentz case $\nu = 1$, we refer to (Σ^n, g) as a *spacelike* hypersurface of $\overline{M}$), with Levi-Civita connection ∇. We orient Σ^n by the choice of a unit normal vector field N on it.

In this setting, if we let A denote the corresponding shape operator, then, at each $p \in \Sigma^n$, A restricts to a self-adjoint linear map $A_p : T_p\Sigma \to T_p\Sigma$.

For $0 \leq r \leq n$, let $S_r(p)$ denote the r-th elementary symmetric function on the eigenvalues of A_p. This way one gets n smooth functions $S_r : \Sigma^n \to \mathbb{R}$ such that

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by definition. If $p \in \Sigma^n$ and $\{e_k\}$ is a basis of $T_p\Sigma$ formed by eigenvectors of A_p, with corresponding eigenvalues $\{\lambda_k\}$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \ldots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[X_1, \ldots, X_n]$ is the r-th elementary symmetric polynomial on the indeterminates $X_1, \ldots, X_n$.

Also, we define the r-th mean curvature H_r of ψ, $0 \leq r \leq n$, by

$$\binom{n}{r} H_r = \epsilon_N^r S_r = \sigma_r(\epsilon_N \lambda_1, \ldots, \epsilon_N \lambda_n).$$

We observe that $H_0 = 1$ and H_1 is the usual mean curvature H of Σ^n.

For $0 \leq r \leq n$, one defines the r-th Newton transformation P_r on Σ^n by setting $P_0 = I$ (the identity operator) and, for $1 \leq r \leq n$, via the recurrence relation

$$P_r = \epsilon_N^r S_r I - \epsilon_N A P_{r-1}. \tag{2.1}$$

A trivial induction shows that

$$P_r = \epsilon_N^r (S_r I - S_{r-1} A + S_{r-2} A^2 - \cdots + (-1)^r A^r),$$

so that the Cayley-Hamilton theorem gives $P_n = 0$. Moreover, since P_r is a polynomial in A for every r, it is also self-adjoint and commutes with A. Therefore, all bases of $T_p\Sigma$ diagonalizing A at $p \in \Sigma^n$ also diagonalize all of the P_r at p. Let $\{e_k\}$ be such a basis. Denoting by A_i the restriction of A to $\langle e_i \rangle^\perp \subset T_p\Sigma$, it is easy to see that

$$\det(tI - A_i) = \sum_{k=0}^{n-1} (-1)^k S_k(A_i) t^{n-1-k},$$

where

$$S_k(A_i) = \sum_{\substack{1 \leq j_1 < \ldots < j_k \leq n \\ j_1, \ldots, j_k \neq i}} \lambda_{j_1} \cdots \lambda_{j_k}.$$

It is also immediate to check that $P_r e_i = \epsilon_N^r S_r(A_i) e_i$, so that an easy computation (cf. Lemma 2.1 of [6]) gives the following:

Lemma 2.1. *With the above notation, the following formulas hold:*

(a) $S_r(A_i) = S_r - \lambda_i S_{r-1}(A_i)$.
(b) $\mathrm{tr}(P_r) = \epsilon_N^r \sum_{i=1}^n S_r(A_i) = \epsilon_N^r (n-r) S_r = b_r H_r$.
(c) $\mathrm{tr}(AP_r) = \epsilon_N^r \sum_{i=1}^n \lambda_i S_r(A_i) = \epsilon_N^r (r+1) S_{r+1} = \epsilon_N b_r H_{r+1}$.
(d) $\mathrm{tr}(A^2 P_r) = \epsilon_N^r \sum_{i=1}^n \lambda_i^2 S_r(A_i) = \epsilon_N^r (S_1 S_{r+1} - (r+2) S_{r+2})$,

where $b_r = (n-r)\binom{n}{r}$.

Associated to each Newton transformation P_r one has the second order linear differential operator $L_r : \mathcal{D}(\Sigma) \to \mathcal{D}(\Sigma)$, given by

$$L_r(f) = \mathrm{tr}(P_r \, \mathrm{Hess}\, f).$$

In particular, $L_0 = \Delta$ and, if $\overline{M}$ has constant sectional curvature, H. Rosenberg proved in [17] that $L_r f = \mathrm{div}(P_r \nabla f)$, where div stands for the divergence on Σ.

For a smooth $\varphi : \mathbb{R} \to \mathbb{R}$ and $h \in \mathcal{D}(\Sigma)$, it follows from the properties of the Hessian of functions that

$$L_r(\varphi \circ h) = \varphi'(h) L_r(h) + \varphi''(h) \langle P_r \nabla h, \nabla h \rangle. \tag{2.2}$$

In order to study semi-Riemannian warped products, we define conformal fields vectors. A vector field V on $\overline{M}^{n+1}$ is said to be *conformal* if

$$\mathcal{L}_V\langle\ ,\ \rangle = 2\phi\langle\ ,\ \rangle \tag{2.3}$$

for some function $\phi \in C^\infty(\overline{M})$, where $\mathcal{L}$ stands for the Lie derivative of the metric of $\overline{M}$. The function ϕ is called the *conformal factor* of V.

Since $\mathcal{L}_V(X) = [V,X]$ for all $X \in \mathfrak{X}(\overline{M})$, it follows from the tensorial character of $\mathcal{L}_V$ that $V \in \mathfrak{X}(\overline{M})$ is conformal if and only if

$$\langle\overline{\nabla}_X V, Y\rangle + \langle X, \overline{\nabla}_Y V\rangle = 2\phi\langle X, Y\rangle, \tag{2.4}$$

for all $X, Y \in \mathfrak{X}(\overline{M})$. In particular, V is a Killing vector field relative to $\overline{g}$ if $\phi \equiv 0$.

Let M^n be a connected, n-dimensional oriented Riemannian manifold, $I \subseteq \mathbb{R}$ an interval and $f : I \to \mathbb{R}$ a positive smooth function. In the product differentiable manifold $\overline{M}^{n+1} = I \times M^n$, let π_I and π_M denote the projections onto the I and M factors, respectively. A particular class of semi-Riemannian manifolds having conformal fields is the one obtained by furnishing $\overline{M}$ with the metric

$$\langle v, w\rangle_p = \epsilon\langle(\pi_I)_* v, (\pi_I)_* w\rangle + f(p)^2\langle(\pi_M)_* v, (\pi_M)_* w\rangle,$$

for all $p \in \overline{M}$ and all $v, w \in T_p\overline{M}$, where $\epsilon = \epsilon_{\partial_t}$ and ∂_t is the standard unit vector field tangent to I. Moreover (cf. [13] and [14]), the vector field

$$V = (f \circ \pi_I)\partial_t$$

is conformal and closed (in the sense that its dual 1-form is closed), with conformal factor $\phi = f' \circ \pi_I$, where the prime denotes differentiation with respect to $t \in I$. Such a space is a particular instance of a semi-Riemannian *warped product*, and, from now on, we shall denote it by $\overline{M}^{n+1} = \epsilon I \times_f M^n$.

If $\psi : \Sigma^n \to \epsilon I \times_f M^n$ is a Riemannian immersion, with Σ oriented by the unit vector field N, one obviously has $\epsilon = \epsilon_{\partial_t} = \epsilon_N$. We let h denote the (vertical) height function naturally attached to Σ^n, namely, $h = (\pi_I)|_\Sigma$.

Let $\overline{\nabla}$ and ∇ denote gradients with respect to the metrics of $\epsilon I \times_f M^n$ and Σ^n, respectively. A simple computation shows that the gradient of π_I on $\epsilon I \times_f M^n$ is given by

$$\overline{\nabla}\pi_I = \epsilon\langle\overline{\nabla}\pi_I, \partial_t\rangle = \epsilon\partial_t, \tag{2.5}$$

so that the gradient of h on Σ^n is

$$\nabla h = (\overline{\nabla}\pi_I)^\top = \epsilon\partial_t^\top = \epsilon\partial_t - \langle N, \partial_t\rangle N. \tag{2.6}$$

In particular, we get

$$|\nabla h|^2 = \epsilon\left(1 - \langle N, \partial_t\rangle^2\right), \tag{2.7}$$

where $|\ \ |$ denotes the norm of a vector field on Σ^n.

In the Lorentz setting, the following result is a particular case of one obtained by L. J. Alías and A. G. Colares (cf. [2], Lemma 4.1). For sake of completeness, we present an alternative proof in a semi-Riemannian version.

Lemma 2.2. *Let $\psi : \Sigma^n \to \epsilon I \times_f M^n$ be a Riemannian immersion. If $h = (\pi_I)|_\Sigma : \Sigma^n \to I$ is the height function of Σ^n, then*

$$L_r(h) = (\log f)'(\epsilon\mathrm{tr}P_r - \langle P_r\nabla h, \nabla h\rangle) + \langle N, \partial_t\rangle\mathrm{tr}(AP_r). \tag{2.8}$$

Proof. Fix $p \in M$, $v \in T_pM$ and write $v = w + \epsilon\langle v, \partial_t\rangle\partial_t$ so that $w \in T_p\overline{M}$ is tangent to the fiber of $\overline{M}$ passing through p. Therefore, by repeated use of the formulas of item (2) of Proposition 7.35 of [16], we get

$$\begin{aligned}\overline{\nabla}_v\partial_t &= \overline{\nabla}_w\partial_t + \epsilon\langle v, \partial_t\rangle\overline{\nabla}_{\partial_t}\partial_t = \overline{\nabla}_w\partial_t \\ &= (\log f)'w = (\log f)'(v - \epsilon\langle v, \partial_t\rangle\partial_t).\end{aligned}$$

Thus, from (2.6), we obtain that

$$\begin{aligned}\nabla_v\nabla h &= \overline{\nabla}_v\nabla h - \epsilon\langle Av, \nabla h\rangle N \\ &= \overline{\nabla}_v(\epsilon\,\partial_t - \langle N, \partial_t\rangle N) - \epsilon\langle Av, \nabla h\rangle N \\ &= \epsilon(\log f)'w - v(\langle N, \partial_t\rangle)N + \langle N, \partial_t\rangle Av - \epsilon\langle Av, \nabla h\rangle N \\ &= \epsilon(\log f)'w + (\langle Av, \partial_t\rangle - \langle N, \overline{\nabla}_v\partial_t\rangle)N + \langle N, \partial_t\rangle Av - \epsilon\langle Av, \nabla h\rangle N \\ &= \epsilon(\log f)'w + (\langle Av, \partial_t^\top\rangle - \langle N, (\log f)'w\rangle)N + \langle N, \partial_t\rangle Av - \epsilon\langle Av, \nabla h\rangle N \\ &= \epsilon(\log f)'w + \epsilon(\log f)'\langle v, \partial_t\rangle\langle N, \partial_t\rangle N + \langle N, \partial_t\rangle Av \\ &= \epsilon(\log f)'\{v - \langle v, \partial_t\rangle(\epsilon\,\partial_t - \langle N, \partial_t\rangle N)\} + \langle N, \partial_t\rangle Av \\ &= (\log f)'(\epsilon\, v - \epsilon\langle v, \partial_t^\top\rangle\nabla h) + \langle N, \partial_t\rangle Av \\ &= (\log f)'(\epsilon\, v - \langle v, \nabla h\rangle\nabla h) + \langle N, \partial_t\rangle Av.\end{aligned}$$

Now, by fixing $p \in \Sigma$ and an orthonormal frame $\{e_i\}$ at $T_p\Sigma$, one gets

$$\begin{aligned}L_r(h) &= \mathrm{tr}(P_r\mathrm{Hess}(h)) = \sum_{i=1}^n\langle\nabla_{e_i}\nabla h, P_re_i\rangle \\ &= \sum_{i=1}^n\langle(\log f)'(\epsilon\, e_i - \langle e_i, \nabla h\rangle\nabla h) + \langle N, \partial_t\rangle Ae_i, P_re_i\rangle \\ &= (\log f)'(\epsilon\mathrm{tr}P_r - \langle P_r\nabla h, \nabla h\rangle) + \langle N, \partial_t\rangle\mathrm{tr}(AP_r).\end{aligned}$$

□

For $t_0 \in \mathbb{R}$, we orient the *slice* $\Sigma^n_{t_0} = \{t_0\} \times M^n$ by using the unit normal vector field ∂_t. According to [5], Σ_{t_0} has constant r-th mean curvature $H_r = -\epsilon\left(\dfrac{f'(t_0)}{f(t_0)}\right)^r$ with respect to ∂_t. Since our applications in the next sections all deal with semi-Riemannian warped products with warping function $f(t) = e^t$, all slices will have r-th mean curvature $H_r = -\epsilon$ with respect to ∂_t.

In order to prove our Bernstein-type results, we shall use the following result of S. T. Yau (cf. Corollary on page 660 of [19]).

Lemma 2.3. *Let Σ^n be an n-dimensional complete Riemannian manifold. If $g : \Sigma^n \to \mathbb{R}$ is a smooth subharmonic or superharmonic function whose gradient norm is integrable on Σ^n, then g must be actually harmonic.*

In [9], the first and the second authors jointly with P. Sousa proved an extension of above lemma to the context of the L_r operators.

Lemma 2.4. *Let $\overline{M}^{n+1}$ have constant sectional curvature, and let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be an n-dimensional complete Riemannian immersion with bounded second fundamental form. Also let $g : \Sigma^n \to \mathbb{R}$ be a smooth function whose gradient norm is integrable on Σ^n. If L_rg does not change sign on Σ^n, then $L_rg = 0$ on Σ^n.*

3. Applications

In this section we assemble all of the above to obtain Bernstein-type theorems in semi-Riemannian warped products $\epsilon I \times_{e^t} M^n$, where M^n is a complete Riemannian manifold.

First of all, we consider (according to [1]; see also Remark 3.2) *steady state-type spacetimes*, i.e., Lorentzian warped products

$$-\mathbb{R} \times_{e^t} M^n, \tag{3.1}$$

where M^n is an n-dimensional complete, connected Riemannian manifold (see Remark 3.8).

As we pointed out by the end of Section 2, each slice $\Sigma_{t_0} = \{t_0\} \times M^n$ is a complete, connected spacelike hypersurface with r-th mean curvature equal to 1 if we take the orientation given by the unit normal vector field $N = \partial_t$.

Following [1], we say that a spacelike hypersurface $\psi : \Sigma^n \to -\mathbb{R} \times_{e^t} M^n$ is *bounded away from the future infinity* of $-\mathbb{R} \times_{e^t} M^n$ if there exists $\overline{t} > 0$ such that

$$\psi(\Sigma) \subset \{(t, x) \in -\mathbb{R} \times_{e^t} M^n; t \leq \overline{t}\}.$$

Analogously, we say that Σ^n is *bounded away from the past infinity* of $-\mathbb{R} \times_{e^t} M^n$ if there exists $\underline{t} > 0$ such that

$$\psi(\Sigma) \subset \{(t, x) \in -\mathbb{R} \times_{e^t} M^n; t \geq \underline{t}\}.$$

Finally, Σ^n is said to be *bounded away from the infinity* of $-\mathbb{R} \times_{e^t} M^n$ if it is both bounded away from the past and future infinity of $-\mathbb{R} \times_{e^t} M^n$. In other words, Σ^n is bounded away from the infinity if there exist $0 < \underline{t} < \overline{t}$ such that $\psi(\Sigma)$ is contained in the slab bounded by the slices $\{\underline{t}\} \times M^n$ and $\{\overline{t}\} \times M^n$.

Now, we present our Bernstein-type theorem in the steady state-type space. As before, h is the height function of Σ.

Theorem 3.1. *Let $\psi : \Sigma^n \to -\mathbb{R} \times_{e^t} M^n$ be a complete, connected spacelike hypersurface bounded away from the infinity on $-\mathbb{R} \times_{e^t} M^n$ with (not necessarily constant) mean curvature $H \geq 1$. If ∇h has integrable norm on Σ^n, then Σ^n is a slice of $-\mathbb{R} \times_{e^t} M^n$.*

Proof. If N is the Gauss map such that $\langle N, \partial_t \rangle < 0$, then, by applying the reverse Cauchy inequality, we have

$$\langle N, \partial_t \rangle \leq -1. \tag{3.2}$$

On the other hand, standard computations give $\Delta e^h = e^h(|\nabla h|^2 + \Delta h)$. Thus, with the aid of Lemmas 2.1, 2.2 and inequality (3.2), we get

$$\Delta e^h = -ne^h(1 + H\langle N, \partial_t \rangle) \geq 0, \tag{3.3}$$

for $H \geq 1$ on Σ. Now, since $|\nabla h|$ is integrable and h is bounded on Σ (this last assertion is due to the fact that Σ^n is bounded away from the infinity of $-\mathbb{R} \times_{e^t} M^n$), we get $|\nabla e^h| = e^h |\nabla h|$ also integrable on Σ.

Consequently, e^h is a subharmonic function on Σ whose gradient has integrable norm. Since Σ is complete, it follows from Lemma 2.3 that e^h is actually harmonic. Back to formula (3.3) we get

$$-1 = H\langle N, \partial_t \rangle \leq -H \leq -1,$$

so that $\langle N, \partial_t \rangle = -1$ and $H \equiv 1$.

Finally, it follows from (2.7) and the connectedness of Σ^n that the hypersurface is a slice of $-\mathbb{R} \times_{e^t} M^n$. $\square$

Remark 3.2. An interesting special case is that of the $(n+1)$-dimensional *steady state space*, i.e., the warped product

$$\mathcal{H}^{n+1} = -\mathbb{R} \times_{e^t} \mathbb{R}^n,$$

which is isometric to an open subset of the de Sitter space $\mathbb{S}_1^{n+1}$. In this case, the slice Σ_{t_0} is isometric to $\mathbb{R}^n$ and is called a *hyperplane* of $\mathcal{H}^{n+1}$.

The importance of considering $\mathcal{H}^{n+1}$ comes from the fact that, in cosmology, $\mathcal{H}^4$ is the steady state model of the universe proposed by H. Bondi and T. Gold [7], and F. Hoyle [11], when looking for a model of the universe which looks the same, not only at all points and in all directions (that is, spatially isotropic and homogeneous), but also at all times (cf. [18], Section 14.8, and [10], Section 5.2).

Among other interesting results related to ours, S. Montiel (cf. [15]) proved that, under an appropriate restriction on their hyperbolic Gauss maps (i.e., on N), complete spacelike hypersurfaces immersed in the de Sitter space and with constant mean curvature greater than or equal to 1 must actually have mean curvature 1.

Remark 3.3. Let M^n be of nonnegative sectional curvature. As a consequence of the classical Bonnet-Myers theorem, if a complete spacelike hypersurface $\psi : \Sigma^n \to -\mathbb{R} \times_{e^t} M^n$ has (not necessarily constant) mean curvature H satisfying

$$|H| \le c < \frac{2\sqrt{n-1}}{n}$$

(c a positive real constant), then Σ^n has to be compact. In fact, if we let Ric_Σ stand for the Ricci tensor of Σ^n, then inequality (16) of [1], together with the nonnegativity of the sectional curvature of M and the above bound on H, gives

$$\mathrm{Ric}_\Sigma \ge (n-1) - \frac{n^2H^2}{4} > 0.$$

We observe that $\frac{2\sqrt{n-1}}{n} \le 1$ for $n \ge 2$.

However, in the case $M^n = \mathbb{R}^n$ (so that $-\mathbb{R} \times_{e^t} M^n = \mathcal{H}^{n+1}$), if Σ^n is bounded away from the future infinity, then Lemma 1 of [1] assures that Σ^n is diffeomorphic to $\mathbb{R}^n$; in particular, $\mathcal{H}^{n+1}$ does not possess any compact (without boundary) spacelike hypersurface.

On the other hand, it follows from the classification of totally umbilical spacelike hypersurfaces of the de Sitter space (cf. [12], Example 1) that there exists no totally umbilical complete immersed spacelike hypersurfaces with mean curvature $0 \le H < 1$ in the steady state space.

It follows from all of the above that, in a certain sense, it is natural to restrict our attention to mean curvature $H \ge 1$.

In analogy to the Lorentz case, we now turn our attention to *hyperbolic-type spaces*, i.e., warped products

$$\mathbb{R} \times_{e^t} M^n,$$

where M^n is a complete, connected Riemannian manifold (see Remarks 3.5 and 3.8). According to the material of Section 2, these hypersurfaces have constant mean curvature 1 if we take the orientation given by the unit normal vector field $N = -\partial_t$.

Similarly to the Lorentz case, we say that a complete hypersurface $\psi : \Sigma^n \to \mathbb{R} \times_{e^t} M^n$ is *bounded away from the infinity* of $\mathbb{R} \times_{e^t} M^n$ if there exist $\underline{t} < \overline{t}$ such that $\psi(\Sigma)$ is contained in the slab bounded by the slices $\Sigma_{\underline{t}}$ and $\Sigma_{\overline{t}}$.

We can finally state and prove, in the Riemannian setting, the analogue of Theorem 3.1.

Theorem 3.4. *Let $\psi : \Sigma^n \to \mathbb{R} \times_{e^t} M^n$ be a complete, connected hypersurface of $\mathbb{H}^{n+1}$, bounded away from the infinity and with (not necessarily constant) mean curvature $0 < H \leq 1$. If ∇h has integrable norm on Σ^n, then Σ^n is a slice of $\mathbb{R} \times_{e^t} M^n$.*

Sketch of the proof. If we choose the Gauss map N of Σ in such a way that $\langle N, \partial_t \rangle < 0$, then Cauchy-Schwarz inequality gives

$$\langle N, \partial_t \rangle \geq -1. \tag{3.4}$$

Since the mean curvature H of Σ satisfies $0 < H \leq 1$, inequality (3.4) gives $1 + H\langle N, \partial_t \rangle \geq 0$. From here, the same arguments that led to (3.3) give

$$\Delta e^h = ne^h(1 + H\langle N, \partial_t \rangle) \geq 0.$$

Finally, if we follow essentially the same arguments employed in the last part of the proof of Theorem 3.1, we get that Σ^n is a slice of $\mathbb{R} \times_{e^t} M^n$. $\square$

Remark 3.5. A motivation for considering the spaces $\mathbb{R} \times_{e^t} M^n$ comes from the fact that the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$ is isometric to $\mathbb{R} \times_{e^t} \mathbb{R}^n$, with an explicit isometry between the half-space model and this warped product model as found in [3]. It can easily be seen from such isometry that the slices $\Sigma_{t_0} = \{t_0\} \times \mathbb{R}^n$ of the warped product model of the hyperbolic space are precisely the horospheres.

At this point, a natural idea would be to extend the above results to the r-th mean curvatures of the hypersurface, and we do this below. However, since the analytical tool at our disposal (i.e., Lemma 2.4) asks $\epsilon\mathbb{R} \times_{e^t} M^n$ to have constant sectional curvature, it follows from Proposition 7.42 of [16] that the sectional curvatures of M^n must vanish identically. Moreover, since our hypersurfaces are to be complete, Remark 3.8 shows that M^n must be also complete; i.e., M^n must be a space form of zero sectional curvature. This being said, we have the following results.

Theorem 3.6. *Let M^n be a Riemmanian space form of zero sectional curvature and $\psi : \Sigma^n \to -\mathbb{R} \times_{e^t} M^n$ be a complete, connected spacelike hypersurface with bounded second fundamental form and bounded away from the infinity of $-\mathbb{R} \times_{e^t} M^n$. If $|\nabla h|$ in integrable and $0 < H_r \leq H_{r+1}$ on Σ^n, then Σ^n is a slice of $-\mathbb{R} \times_{e^t} M^n$.*

Sketch of the proof. Again, we assume Σ^n is oriented by normal unit vector field N such that $\langle N, \partial_t \rangle < 0$. The assumptions on H_r and H_{r+1}, together with the reverse Cauchy-Schwarz inequality and formulas (2.2) and (2.8), give

$$L_r(e^h) = -b_r e^h(H_r + \langle N, \partial_t \rangle H_{r+1}) \geq 0.$$

Since h and the second fundamental form are bounded on Σ^n, it follows from Lemma 2.4 that $L_r(e^h) = 0$ on Σ^n. Therefore,

$$H_r = -\langle N, \partial_t \rangle H_{r+1} \geq -\langle N, \partial_t \rangle H_r \geq H_r,$$

so that $\langle N, \partial_t \rangle = -1$ and Σ^n must be a slice. $\square$

We omit even a sketch of the proof of our last result, due to the fact that it closely parallels the proof of the previous result.

Theorem 3.7. *Let M^n be a Riemannian space form of zero sectional curvature and let $\psi : \Sigma^n \to \mathbb{R} \times_{e^t} M^n$ be a complete, connected hypersurface with bounded second fundamental form and bounded away from the infinity of $\mathbb{R} \times_{e^t} M^n$. If $|\nabla h|$ is integrable and $H_r \geq H_{r+1} > 0$ on Σ^n, then Σ^n is a slice of $\mathbb{R} \times_{e^t} M^n$.*

Remark 3.8. As the referee pointed out to us, according to Lemma 7 of [1], if $\epsilon\mathbb{R} \times_{e^t} M^n$ is to admit a complete hypersurface bounded away from the infinity, then M^n must necessarily be complete.

Acknowledgements

This work was finished while the first author was visiting the mathematics department of the Universidade Federal do Ceará, Fortaleza, Brazil. She would like to thank that institution for its hospitality.

The authors would like to thank the referee for having made several comments that improved the final version of this paper.

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