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ON TRUNCATED WIENER-HOPF OPERATORS AND $BMO(\mathbb{Z})$

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Abstract. We give a tractable estimate for the norm of a truncated Wiener-Hopf operator in terms of the discrete BMO-space. We also improve earlier norm estimates as well as obtain new, more tractable, criteria for compactness.

1. Introduction

The Wiener-Hopf operators are defined by the expression

$$L^2(\mathbb{R}^+) \ni F \mapsto \int_0^\infty \Phi(y)F(x+y)dy \in L^2(\mathbb{R}^+),$$

where $\Phi \in L^1(\mathbb{R})$ or, more generally, certain distributions. Given an interval $I \subset \mathbb{R}$, let $C_I^\infty(\mathbb{R}) \subset C^\infty(\mathbb{R})$ denote the set of C^∞-functions with support within I, and let $\mathcal{D}'(I)$ denote the set of distributions on I. The truncated Wiener-Hopf operators, $W_{\Phi,a} : L^2([0,a]) \to L^2([0,a])$, are defined for any $a \in \mathbb{R}^+$ and any distribution $\Phi \in \mathcal{D}'((-a,a))$ by the expression

$$C^\infty_{(0,a)}(\mathbb{R}) \ni F \mapsto \Phi(F(\cdot + x)), \quad 0 < x < a, \tag{1.1}$$

whenever Φ is such that this extends to a bounded operator on $L^2([0,a])$. Whenever a is of no importance we will omit it from the notation. We abbreviate by saying that W_Φ is a TWH-operator. These operators (or rather, unitarily equivalent ones) also go under the name finite interval convolution operators, truncated Hankel operators or Toeplitz operators on the Paley-Wiener space. See e.g. [1, 2, 8, 9] and [10]. We will in this paper see that the properties of TWH-operators are more similar to those of Hankel operators rather than Toeplitz operators. Let $\mathcal{F}$ denote the Fourier transform on $L^2(\mathbb{R})$, defined as follows:

$$\mathcal{F}(f)(x) = \int_{-\infty}^\infty f(y)\mathrm{e}^{-\mathrm{i}xy}dy. \tag{1.2}$$

We will also use the notation $\hat{f} = \mathcal{F}(f)$ and $\check{f} = \mathcal{F}^{-1}(f)$. We let $L^2_{[0,a]}(\mathbb{R})$ denote the subspace of $L^2(\mathbb{R})$ of functions with (essential) support in $[0,a]$, and we let $P_{[0,a]} : L^2(\mathbb{R}) \to L^2_{[0,a]}(\mathbb{R})$ be the orthogonal projection onto $L^2_{[0,a]}(\mathbb{R})$. Finally, let

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$\breve{L^\infty} \subset \mathcal{D}'(\mathbb{R})$ be the image of $L^\infty(\mathbb{R})$ under $\mathcal{F}^{-1}$, where the transform is interpreted in the distributional sense. Given $\Phi \in \breve{L^\infty}$ it follows by standard results that

$$W_\Phi F = P_{[0,a]}\mathcal{F}(\hat{\Phi}\check{F}). \tag{1.3}$$

(In the above formula and the rest of the paper, we will without comment identify $L^2([0,a])$ and $L^2_{[0,a]}(\mathbb{R})$.) Let $\Phi|_{(-a,a)}$ denote the restriction of Φ to $C^\infty_{(-a,a)}(\mathbb{R})$. In particular, setting

$$C_\Phi = \inf\left\{\|\hat{\Psi}\|_{L^\infty} : \Psi \in \breve{L^\infty} \text{ and } \Psi|_{(-a,a)} = \Phi|_{(-a,a)}\right\} \tag{1.4}$$

we immediately obtain

$$\|W_\Phi\| \leqslant C_\Phi.$$

It was shown by R. Rochberg that the two quantities above are comparable. This fact has also recently appeared in [3], where compressions of Toeplitz operators are studied in a more general setting. Based on results by Farforovskaya and Nikolskaya we will improve the constant as follows.

Theorem 1.1. *Given any $\Phi \in \mathcal{D}'((-a,a))$, the operator W_Φ is bounded if and only if $\Phi = \Psi|_{(-a,a)}$ for some $\Psi \in \breve{L^\infty}$. In this case we have that*

$$W_\Phi F = P_{[0,a]}\mathcal{F}(\hat{\Psi}\check{F}),$$

the infimum in (1.4) *is attained and*

$$\frac{C_\Phi}{3} \leqslant \|W_\Phi\| \leqslant C_\Phi.$$

However, C_Φ is not easy to compute. Another norm estimate is given in [11], which involves splitting Φ into 3 parts: left, center and right. Loosely speaking, the result says that $\|W_A\|$ is comparable with the BMO-norm of the Fourier transform of certain translations of the left and right part, plus the L^∞-norm of the Fourier transform of the center part. In [11] there is also given a norm estimate involving breaking up W_Φ into two pieces and the discrete $BMO(\mathbb{Z})$-space, defined below. The issue of finding a more tractable norm estimate was raised in [11], and this is our next objective. Define $BMO(\mathbb{Z})$ as the space of all sequences σ such that the following semi-norm is finite:

$$\|\sigma\|_{BMO} = \sup_I \left\{ |I|^{-1} \sum_{k \in I} |\sigma(k) - \sigma_I| \right\}.$$

Here $I \subset \mathbb{Z}$ ranges over all sets of the form $\{K_1 < k \leqslant K_2\}$ $(K_1, K_2 \in \mathbb{Z})$, $|I| = K_2 - K_1$ and $\sigma_I = |I|^{-1}\sum_{k\in I}\sigma(k)$.

Theorem 1.2. *Given any $\Phi \in L^1((-a,a))$ set $\Phi(x) = \sum_{k=-\infty}^{\infty} \phi_k e^{i\pi kx/a}$. There exist constants $C_1, C_2 > 0$ such that*

$$C_1 a\|((-1)^k\phi_k)_k\|_{BMO(\mathbb{Z})} \leqslant \|W_\Phi\| \leqslant C_2 a\|((-1)^k\phi_k)_k\|_{BMO(\mathbb{Z})}.$$

Remark 1. The restriction to L^1 is for the introduction only, as there are some complications involved in defining the Fourier series for general elements of $L^\infty \cap \mathcal{D}'(-a,a)$, which is treated in Section 3.

Remark 2. The weight $(-1)^k$ is an effect of the definition of W_Φ such that the midpoint of Φ is at 0 and can easily be removed by a translation of Φ to $(0, 2a)$.

As an example, consider $\Phi = \delta_x$, where δ_x denotes the Dirac distribution at x. With $x = 0$ we obtain $W_\Phi(F) = F$, so $\|W_\Phi\| = 1$. With a suitable interpretation of its Fourier series we get $\phi_k = \frac{1}{2a}$ and $a\|((-1)^k\phi_k)_k\|_{BMO(\mathbb{Z})} = 1/2$. On the other hand, setting $x = a$ we obtain $W_\Phi = 0$ as well as $\phi_k = \frac{(-1)^k}{2a}$, so $\|((-1)^k\phi_k)_k\|_{BMO(\mathbb{Z})} = 0$.

We also show that W_Φ is compact if and only if $((-1)^k\phi_k)_k$ is in $CMO(\mathbb{Z})$, the closure in $BMO(\mathbb{Z})$ of the sequences with finite support. The proof of this result and Theorem 1.2 goes via a new class of "discrete" Hankel operators, which we now introduce. Let $\mathbb{T}$ denote the unit circle and set $\mathbb{T}^+ = \{\zeta \in \mathbb{T} : \mathsf{Im}\ \zeta > 0\}$, $\mathbb{T}^- = \{\zeta \in \mathbb{T} : \mathsf{Im}\ \zeta < 0\}$. Let $\mathcal{F}^{-1}$ denote the inverse Fourier transform $\mathcal{F}^{-1} : L^2(\mathbb{T}) \to l^2(\mathbb{Z})$ defined via $\mathcal{F}^{-1}(f)(k) = \int_0^{2\pi} f(e^{it})e^{ikt}dt/(2\pi)$. (Note that we use the same symbol as in (1.2); the type of f will determine which one is intended.) The common denominator of the various Hardy spaces is that the Fourier transform of the elements is in some sense one-sided. It therefore makes some sense to define the discrete Hardy space $H^2(\mathbb{Z})$ as $\mathcal{F}^{-1}(L^2_{\mathbb{T}^+}(\mathbb{T}))$ and similarly $H^2_-(\mathbb{Z}) = \mathcal{F}^{-1}(L^2_{\mathbb{T}^-}(\mathbb{T}))$ (where $L^2(\mathbb{T})$ is defined with normalized arc-length measure and $L^2_{\mathbb{T}^+}(\mathbb{T})$ denotes the subspace of functions with support in $\mathbb{T}^+$). In analogy with the classical definition, given $\sigma \in l^\infty(\mathbb{N})$ we define the Hankel operator $H_\sigma : H^2(\mathbb{Z}) \to H^2_-(\mathbb{Z})$ via

$$H_\sigma(f) = P_{H^2_-(\mathbb{Z})}(\sigma \cdot f), \tag{1.5}$$

where $(\sigma \cdot f)(k) = \sigma(k)f(k)$, $\forall k \in \mathbb{Z}$. (A more general definition includes certain unbounded symbols, but we omit this in the introduction.) Using the notation of Theorem 1.2, we show that $\frac{1}{2a}W_\Phi$ is equivalent with $H_{((-1)^k\phi_k)}$ under unitary transformations. Moreover, we show

Theorem 1.3. *H_σ is bounded if and only if $\sigma \in BMO(\mathbb{Z})$ and the norms are comparable.*

The proof relies on a characterization in [11] of the $BMO(\mathbb{Z})$-norm of a given σ in terms of the operator-norm of an "infinite matrix" R_σ whose "(i,j)" 'th entry is given by $\frac{\sigma(i)-\sigma(j)}{i-j}$.

2. A Nehari-type theorem for truncated Wiener-Hopf operators

Given $N \in \mathbb{N}$ and $\phi \in \mathbb{C}^{\{-N,\dots,N\}}$, we define the Toeplitz matrix by

$$T_\phi = \begin{pmatrix} \phi_0 & \phi_1 & \phi_2 & \cdots & \phi_N \\ \phi_{-1} & \phi_0 & \phi_1 & \cdots & \phi_{N-1} \\ \phi_{-2} & \phi_{-1} & \phi_0 & \cdots & \phi_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{-N} & \phi_{-N+1} & \phi_{-N+2} & \cdots & \phi_0 \end{pmatrix}.$$

We introduce yet a third meaning of $\mathcal{F}$; when acting on $\phi \in L^2(\mathbb{T})$ we set $\mathcal{F}(\phi)(k) = \int_{\mathbb{T}} \phi(z)z^{-k}dm(z)$, where m denotes the normalized arc-length measure on the unit circle $\mathbb{T}$. We will without comment let the type of a function/sequence/distribution s determine the meaning of $\mathcal{F}(s) = \hat{s}$ and $\check{s} = \mathcal{F}^{-1}(s)$.

For convenience, we provide a table of the various Fourier transforms used in this paper:

$$\begin{cases} \mathcal{F}: L^2(\mathbb{R}) \to L^2(\mathbb{R}), \mathcal{F}(F)(x) = \int_{\mathbb{R}} F(y)e^{-ixy}dy; \mathcal{F}^{-1}(F)(x) = \int_{\mathbb{R}} F(y)e^{ixy}\frac{dy}{2\pi} \\ \mathcal{F}: L^2(\mathbb{T}) \to l^2(\mathbb{Z}), \mathcal{F}(F)(k) = \int_{\mathbb{T}} F(z)z^{-k}dm(z); \mathcal{F}^{-1}(\sigma)(z) = \sum_k \sigma(k)z^k \\ \mathcal{F}: l^2(\mathbb{Z}) \to L^2(\mathbb{T}), \mathcal{F}(\sigma)(z) = \sum_k \sigma(k)z^{-k}; \mathcal{F}^{-1}(F)(k) = \int_{\mathbb{T}} F(z)z^k dm(z) \end{cases}$$

For any ϕ as above we set

$$C_\phi = \inf\{\|g\|_{L^\infty(\mathbb{T})} : g \in L^\infty(\mathbb{T}) \text{ and } \hat{g}(k) = \phi_k\}. \tag{2.1}$$

We recall the following theorem by Yu. B. Farforovskaya and L. N. Nikolskaya [7].

Theorem 2.1.

$$\frac{1}{3}C_\phi \leqslant \|T_\phi\| \leqslant C_\phi.$$

This theorem was first obtained, albeit with a different constant, in [11]. Note that the right inequality is immediate because T_ϕ is the compression of the Toeplitz operator with symbol g, for any g that appears in (2.1). It is an open problem whether 1/3 is the best possible constant, but it is known that it is not 1 (see [7]). We will now prove Theorem 1.1. It is easy to see that it suffices to show Theorem 1.1 for a fixed value of a, so we set $a = 1$. For $p \geqslant 1$ and any $N \in \mathbb{N}$ let the sampling operator $\mathcal{S}_N : C([-1,1]) \to \mathbb{C}^{\{-N+1,\dots,N-1\}}$ be defined by

$$\mathcal{S}_N F = \left(\frac{1}{N}F\left(\frac{k}{N}\right)\right)_{k=-N+1}^{N-1}. \tag{2.2}$$

Let $\chi(S,\cdot)$ denote the characteristic function of a set S. For each $N \in \mathbb{N}$, set

$$b_k^N(x) = \sqrt{N}\chi\left(\left[\frac{k}{N}, \frac{k+1}{N}\right), x\right)$$

and let $P_N : L^2([0,1]) \to L^2([0,1])$ be the orthogonal projection on the subspace spanned by $\{b_k^N\}_{k=0}^{N-1}$. Note that $\{b_k^N\}_{k=0}^{N-1}$ is an orthonormal set in $L^2([0,1])$. Define $\mathcal{I}_N : \mathbb{C}^{\{0,\dots,N-1\}} \to L^2([0,1])$ by

$$\mathcal{I}_N(\phi) = \sum \phi_k b_k^N. \tag{2.3}$$

Note that $\mathcal{I}_N\mathcal{I}_N^* P_N = P_N$ and that, given $\Phi \in C([-1,1])$, the compression of the operator $\mathcal{I}_N T_{\mathcal{S}_N\Phi}\mathcal{I}_N^*$ to the subspace $\mathsf{Ran}\ P_N$ is represented by the matrix $T_{\mathcal{S}_N\Phi}$ in the basis $\{b_k^N\}_{k=0}^{N-1}$. We shall show that for $\Phi \in C^1([-1,1])$, the operators $\mathcal{I}_N T_{\mathcal{S}_N\Phi}\mathcal{I}_N^*$ converge to W_Φ as $N \to \infty$. In order to simplify the notation we set

$$W_\Phi^N = \mathcal{I}_N T_{\mathcal{S}_N\Phi}\mathcal{I}_N^*.$$

For any $\epsilon > 0$ let $\rho_\epsilon : C^1([-1,1]) \to C([-1,1])$ be defined by

$$\rho_\epsilon(\Phi)(x) = \sup\{|\Phi'(y)| : |x-y| \leqslant \epsilon\}.$$

Proposition 2.2. *Let $\Phi \in C^1([-1,1])$ be given. Then*

$$\|W_\Phi - W_\Phi^N\| \leqslant \left(1 + \sqrt{\frac{1}{3}}\right)\frac{2}{N}\sqrt{\int_{-1}^{1}\left(\rho_{2/N}(\Phi)(x)\right)^2 dx}.$$

Proof. Assume that Φ is real-valued. We shall first give an estimate of $W_\Phi - W_\Phi^N$ restricted to $\mathsf{Ran}\ P_N$. For each fixed $0 \leqslant x \leqslant 1$ we have

$$W_\Phi b_k^N(x) = \sqrt{N}\int_{\frac{k}{N}-x}^{\frac{k+1}{N}-x} \Phi(y)dy = N^{-1/2}\Phi(y_x - x), \tag{2.4}$$

for some $\frac{k}{N} \leqslant y_x \leqslant \frac{k+1}{N}$ by the mean value theorem. On the other hand, note that $\mathcal{I}_N^* b_k^N = e_k$ (where $(e_k)_{k=0}^{N-1}$ denotes the standard basis in $\mathbb{C}^{\{0,\ldots,N-1\}}$), so

$$W_\Phi^N b_k^N(x) = (\mathcal{I}_N T_{\mathcal{S}_N\Phi} e_k)(x) = \sum_{l=0}^{N} N^{-1}\Phi\left(\frac{k-l}{N}\right) b_l^N(x) = N^{-1/2}\Phi\left(\frac{k-l_x}{N}\right),$$

where $l_x \in \mathbb{N}$ is such that $l_x/N \leqslant x < (l_x+1)/N$. Since $|y_x - x - \frac{k-l_x}{N}| \leqslant 1/N$ we have

$$\left|W_\Phi b_k^N(x) - W_\Phi^N b_k^N(x)\right| = \frac{1}{\sqrt{N}}\left|\Phi(y_x - x) - \Phi\left(\frac{k-l_x}{N}\right)\right| \leqslant \frac{1}{N^{3/2}}\rho_{1/N}(\Phi)\left(\frac{k}{N} - x\right). \tag{2.5}$$

Now, let $a \in \mathbb{C}^{\{0,\ldots,N-1\}}$ be arbitrary but satisfy $\|a\| = 1$. Then

$$\begin{aligned}
\left|\left(\left(W_\Phi - W_\Phi^N\right)\left(\sum a_k b_k^N\right)\right)(x)\right| &= \left|\sum a_k\left(\left(W_\Phi - W_\Phi^N\right) b_k^N\right)(x)\right| \\
\leqslant N^{-3/2}\left|\sum a_k \rho_{1/N}(\Phi)(\frac{k}{N} - x)\right| &= \frac{1}{N}\sqrt{\frac{1}{N}\sum\left(\rho_{1/N}(\Phi)(\frac{k}{N} - x)\right)^2} \\
\leqslant \frac{1}{N}\sqrt{\sum\int_{\frac{k}{N}}^{\frac{k+1}{N}}\left(\rho_{2/N}(\Phi)(y-x)\right)^2 dy} &= \frac{1}{N}\sqrt{\int_0^1\left(\rho_{2/N}(\Phi)(y-x)\right)^2 dy}.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
\|(W_\Phi - W_\Phi^N)\left(\sum a_k b_k^N\right)\|_{L^2} &\leqslant \frac{1}{N}\sqrt{\int_0^1\int_0^1\left(\rho_{2/N}(\Phi)(y-x)\right)^2 dydx} \\
= \frac{1}{N}\sqrt{\int_{\substack{0<u+v<2\\0<u-v<2}}\left(\rho_{2/N}(\Phi)(v)\right)^2\frac{1}{2}dudv} &\leqslant \frac{1}{N}\sqrt{\int_{-1}^1\left(\rho_{2/N}(\Phi)(v)\right)^2 dv},
\end{aligned}$$

which, upon noting that $\|\sum a_k b_k\|_{L_2} = 1$, yields

$$\|(W_\Phi - W_\Phi^N)P_N\| \leqslant \frac{1}{N}\sqrt{\int_{-1}^1\left(\rho_{2/N}(\Phi)(x)\right)^2 dx}. \tag{2.6}$$

We turn to the estimate for $(W_\Phi - W_\Phi^N)(I - P_N) = W_\Phi(I - P_N)$. Define subsets $S_{k,i,j}^N \subset [\frac{k}{N}, \frac{k+1}{N}]$ for $0 \leqslant k < N$, $j \geqslant 1$ and $0 \leqslant i \leqslant 2^j - 1$ by

$$S_{k,i,j}^N = \left[\frac{k}{N} + \frac{i}{2^j N}, \frac{k}{N} + \frac{i+1}{2^j N}\right],$$

and let $d_{k,i,j}^N$ be functions defined by

$$d_{k,i,j}^N(x) = \sqrt{2^{j-1}N}\left(\chi(S_{k,2i,j}^N, x) - \chi(S_{k,2i+1,j}^N, x)\right)$$

for $0 \leqslant k < N$, $j \geqslant 1$ and $0 \leqslant i \leqslant 2^{j-1} - 1$. It is easy to see that (for N fixed),

$$\mathsf{Span}\ (\{b_k^N\} \cup \{d_{k,i,j}^N\}) = \mathsf{Span}\ (\{\chi_{S_{k,i,j}^N}\}),$$

and it follows from basic integration theory that the right hand side is dense in $L^2([0,1])$. Moreover, $\{b_k^N\} \cup \{d_{k,i,j}^N\}$ is clearly an orthonormal set, and hence it is a basis for $L^2([0,1])$. Thus $\mathsf{Ran}\ (I-P_N) = \mathsf{Span}\ \{d_{k,i,j}^N\}$. In a similar fashion as in (2.4) and (2.5) we obtain that

$$|W_\Phi d_{k,i,j}^N(x)| = \frac{\sqrt{2^{j-1}N}}{2^jN}\left|\Phi\left(\frac{k}{N}+\frac{2i}{2^jN}+\delta_1-x\right)-\Phi\left(\frac{k}{N}+\frac{2i+1}{2^jN}+\delta_2-x\right)\right|$$
$$\leqslant \frac{1}{\sqrt{2^{j+1}N}}\left|\frac{1}{2^{j-1}N}\rho_{1/N}(\Phi)\left(\frac{k}{N}-x\right)\right| = \frac{\sqrt{2}}{(2^jN)^{3/2}}\left|\rho_{1/N}(\Phi)\left(\frac{k}{N}-x\right)\right|,$$

where $0 \leqslant \delta_1, \delta_2 \leqslant (2^jN)^{-1}$. Now let $a_{k,i,j} \in \mathbb{C}$ be any numbers (indexed by the same index set as $\{d_{k,i,j}^N\}$) such that $\sum |a_{k,i,j}|^2 = 1$. By repeated use of the Cauchy-Schwarz and Minkowski inequalities we get

$$\left\|W_\Phi\left(\sum a_{k,i,j}d_{k,i,j}^N\right)\right\| \leqslant \sum_{k=0}^{N-1}\sum_{j=1}^{\infty}\sum_{i=0}^{2^{j-1}-1}|a_{k,i,j}|\frac{\sqrt{2}}{(2^jN)^{3/2}}\left\|\rho_{1/N}(\Phi)\left(\frac{k}{N}-\cdot\right)\right\|$$
$$\leqslant \sum_{k=0}^{N-1}\sum_{j=1}^{\infty}\sqrt{\sum_i |a_{k,i,j}|^2}\,2^{(j-1)/2}\frac{\sqrt{2}}{(2^jN)^{3/2}}\left\|\rho_{1/N}(\Phi)\left(\frac{k}{N}-\cdot\right)\right\|$$
$$\leqslant \sum_{k=0}^{N-1}\sqrt{\sum_{j,i}|a_{k,i,j}|^2}\sqrt{\sum_{j\geqslant 1}2^{-2j}}\frac{1}{N^{3/2}}\left\|\rho_{1/N}(\Phi)\left(\frac{k}{N}-\cdot\right)\right\|$$
$$\leqslant \sqrt{\sum_{k,j,i}|a_{k,i,j}|^2}\sqrt{\frac{1}{1-1/4}-1}\,\frac{1}{N^{3/2}}\sqrt{\sum_k\int_0^1\left(\rho_{1/N}(\Phi)\left(\frac{k}{N}-x\right)\right)^2dx}$$
$$\leqslant \frac{1}{\sqrt{3}N}\sqrt{\int_0^1\int_0^1\left(\rho_{2/N}(\Phi)(y-x)\right)^2dydx} \leqslant \frac{1}{\sqrt{3}N}\sqrt{\int_{-1}^1(\rho_{2/N}(\Phi)(x))^2dx}.$$

It follows that

$$\|(W_\Phi - W_\Phi^N)(I-P_N)\| \leqslant \frac{1}{\sqrt{3}N}\sqrt{\int_{-1}^1\left(\rho_{2/N}(\Phi)(x)\right)^2dx}, \tag{2.7}$$

which combined with (2.6) yields the first part of the proposition in the case when Φ is real-valued, but with constant $(1+1/\sqrt{3})$. For the general case, write $\Phi = \Phi_1 + i\Phi_2$ where Φ_1 and Φ_2 are real-valued. Then

$$\|W_\Phi - W_\Phi^N\| \leqslant \|W_{\Phi_1} - W_{\Phi_1}^N\| + \|W_{\Phi_2} - W_{\Phi_2}^N\|$$
$$\leqslant \left(1+\sqrt{\frac{1}{3}}\right)\frac{1}{N}\left(\sqrt{\int_{-1}^1\left(\rho_{2/N}(\Phi_1)(x)\right)^2dx}+\sqrt{\int_{-1}^1\left(\rho_{2/N}(\Phi_2)(x)\right)^2dx}\right)$$
$$\leqslant \left(1+\sqrt{\frac{1}{3}}\right)\frac{2}{N}\sqrt{\int_{-1}^1\left(\rho_{2/N}(\Phi)(x)\right)^2dx},$$

as desired. □

Corollary 2.3. *For every $\Phi \in C^1([-1,1])$ there exists a $C > 0$ such that*

$$\|T_{\mathcal{S}_N\Phi}\| \leqslant \|W_\Phi\| + \frac{C}{N}.$$

We are now in a position to prove the lower estimate in Theorem 1.1 for $\Phi \in C^1([-1,1])$. For standard results and definitions concerning distributions, we refer to [9]. We omit the proofs of simple results such as that $L^\infty(\mathbb{R})$ can be considered as temperate distributions, so that $\widecheck{L^\infty}$ is well defined.

Proposition 2.4. *Given $a > 0$ and $\Phi \in C^1([-a,a])$ there exists a $\Psi \in \widecheck{L^\infty}$ with $\Psi|_{(-a,a)} = \Phi$ and*

$$\frac{1}{3}\|\hat{\Psi}\| \leqslant \|W_{\Phi,a}\|.$$

Proof. As noted earlier, it suffices to consider $a = 1$. By Corollary 2.3 we have $\|T_{\mathcal{S}_N\Phi}\| \leqslant \|W_\Phi\| + \frac{C}{N}$ for some C and every $N \in \mathbb{N}$, and by Theorem 2.1 we get that there exists $\psi_N \in L^\infty(\mathbb{T})$ such that

$$\widehat{\psi_N}(k) = \frac{1}{N}\Phi(\frac{k}{N}), \quad -N+1 \leqslant k \leqslant N-1 \tag{2.8}$$

and

$$\|\psi_N\|_{L^\infty} \leqslant 3\|T_{\mathcal{S}_N\Phi}\| \leqslant 3\|W_\Phi\| + \frac{3C}{N}.$$

Let $\Psi_N \in \mathcal{D}'(\mathbb{R})$ be defined by $\Psi_N = \sum_{k=-\infty}^{\infty} \widehat{\psi_N}(k)\delta_{k/N}$, where δ_x is the Dirac distribution at x, and note that

$$\widehat{\Psi_N}(t) = \sum_{k=-\infty}^{\infty} \widehat{\psi_N}(k)e^{-itk/N} = \psi_N(e^{-it/N}),$$

so in fact $\Psi_N \in \widecheck{L^\infty(\mathbb{R})}$ and $\|\widehat{\Psi_N}\|_{L^\infty} \leqslant 3\|W_\Phi\| + \frac{3C}{N}$. By standard theorems of functional analysis there exists a subsequence of $(\widehat{\Psi_{N_j}})_{j=1}^\infty$ convergent in the weak-star topology to some $\widehat{\Psi} \in L^\infty(\mathbb{R})$ with $\|\widehat{\Psi}\|_{L^\infty} \leqslant 3\|W_\Phi\|$. Given $F \in C^\infty_{(-1,1)}(\mathbb{R})$ we have, using (2.8),

$$\int \Phi F = \lim_{j\to\infty} \int \Psi_{N_j} F = \lim_{j\to\infty} \int \widehat{\Psi_{N_j}} \check{F} = \int \hat{\Psi}\check{F} = \int \Psi F, \tag{2.9}$$

which shows that $\Psi|_{(-1,1)} = \Phi$, and the proof is complete. Note that we use the notation $\int \Psi_N F$ even if Ψ_N is not a function, as opposed to the formally correct $\Psi_N(F)$ or $\langle \Psi_N, F\rangle$. We will in the remainder do this without comment. $\square$

Theorem 2.5. *Given any $a > 0$ and $\Phi \in \mathcal{D}'((-a,a))$, the operator $W_{\Phi,a}$ is bounded if and only if $\Phi = \Psi|_{(-a,a)}$ for some $\Psi \in \widecheck{L^\infty}$. In this case,*

$$W_\Phi F = P_{[0,a]}\mathcal{F}(\hat{\Psi}\check{F}),$$

the infimum in (1.4) *is attained and*

$$\frac{C_\Phi}{3} \leqslant \|W_\Phi\| \leqslant C_\Phi.$$

Proof. All claims in the statement follow by standard arguments once we show that given a $\Phi \in \mathcal{D}'((-1,1))$ such that W_Φ is bounded, there exists a $\Psi \in \widecheck{L^\infty}$ such that $\|\widehat{\Psi}\|_{L^\infty} \leqslant 3\|W_\Phi\|$ and $\Psi|_{(-1,1)} = \Phi$. Take a positive function $\eta \in C^\infty_{(-1,1)}(\mathbb{R})$ that is symmetric around 0 and satisfies $\|\eta\|_{L^1} = 1$, take a sequence of positive functions $\gamma_k \in C^\infty_{(-1,1)}(\mathbb{R})$ such that $\gamma_k(x) = 1$ for $-1 + 4^{-k} < x < 1 - 4^{-k}$, set $\eta_k(x) = 4^k\eta(4^k x)$ and define $\Phi_k \in C^\infty(\mathbb{R})$ via

$$\Phi_k(x) = (\gamma_k\Phi) * \eta_k,$$

which is well-defined as $\gamma_k\Phi$ has a natural extension to $\mathbb{R}$ that is "zero on $\mathbb{R}\backslash(-1,1)$" (see [9], Sec 2.3 for details). We have, for any $F \in L^2_{[2^{-k},1-2^{-k}]}(\mathbb{R})$, that $\eta_k * F \in C^\infty_{(4^{-k},1-4^{-k})}(\mathbb{R})$ and $\hat{\eta_k}\check{F} = \widecheck{\eta_k * F}$, which follows by the symmetry of η. Moreover, by standard properties of distributions with compact support (see [9], Chs. 2 and 7) we get $\widehat{\Phi_k} = \widehat{(\gamma_k\Phi)}\widehat{\eta_k}$, where $\widehat{(\gamma_k\Phi)}$ is a function that grows polynomially since distributions with compact support have finite order. Thus, for $F \in C^\infty_{(-1,1)}(\mathbb{R})$ we get

$$W_{\Phi_k}(F) = P_{[0,1]}\mathcal{F}\big(\widehat{(\gamma_k\Phi)}\hat{\eta_k}\check{F}\big) = P_{[0,1]}\mathcal{F}\big(\widehat{(\gamma_k\Phi)}(\widecheck{\eta_k * F})\big)$$
$$= W_{\gamma_k\Phi}(\eta_k * F) = W_\Phi(\eta_k * F).$$

Now let $\tilde{\Phi}_k(t) = \Phi_k(2^{-k} + t)$ for $|t| < 1 - 2^{1-k}$. The above identity yields, for any $F \in L^2_{[0,1-2^{1-k}]}(\mathbb{R})$, the following estimate:

$$\|W_{\tilde{\Phi}_k,(1-2^{1-k})}(F)\| \leqslant \|W_{\Phi_k,1}(F(\cdot - 2^{-k}))\| = \|W_{\Phi,1}(\eta_k * F(\cdot - 2^{-k}))\|$$
$$\leqslant \|W_{\Phi,1}\|\|\eta_k\|_{L^1}\|F\|_{L^2}.$$

In particular, $\|W_{\tilde{\Phi}_k,(1-2^{1-k})}\| \leqslant \|W_{\Phi,1}\|$. By Proposition 2.4 we get the existence of $\Psi_k \in \widecheck{L^\infty}$ with $\Psi_k(t) = \Phi_k(2^{-k} + t)$ for $|t| \leqslant 1 - 2^{1-k}$ and $\|\widehat{\Psi}_k\|_{L^\infty} \leqslant 3\|W_{\Phi,1}\|$. For any $F \in C^\infty_{(-1,1)}(\mathbb{R})$ and k large enough that $\mathsf{supp}\ F \subset [-1 + 2^{1-k}, 1 - 2^{1-k}]$ we get

$$\int \Psi_k F = \int \Phi_k F(\cdot - 2^{-k}) = \Phi\Big(\gamma_k\big(\eta_k * (F(\cdot - 2^{-k}))\big)\Big) = \Phi\big(\eta_k * (F(\cdot - 2^{-k}))\big).$$

Let $\mathcal{D}(-1,1)$ denote the set of test functions on $(-1,1)$ with the usual topology. It is a standard matter to check that $\eta_k * (F(\cdot - 2^{-k}))$ goes to F in $\mathcal{D}((-1,1))$ as $k \to \infty$, and hence the Ψ_k converge to Φ in $\mathcal{D}'((-1,1))$. The proof is now easily completed with a similar calculation as in (2.9); we omit the details. $\square$

3. Technicalities

We will in this section collect a number of definitions and results, mainly from distribution theory, that would otherwise disrupt the flow of the text. The first 7 chapters of [9] contain all the necessary background material.

Lemma 3.1. *Given any $\Phi \in \mathcal{D}'(-a,a)$ such that W_Φ is bounded, there exists a sequence $\Psi_1, \Psi_2, \ldots$ such that $\Psi_k \in C^\infty(\mathbb{R})$, $\|\widehat{\Psi_k}\|_{L^\infty} \leqslant 3\|W_\Phi\|$ and $\lim_{k\to\infty}\Psi_k = \Phi$ in $\mathcal{D}'(-a,a)$.*

Proof. The proof of Theorem 2.5 almost provides such a sequence; the only issue is that we do not know that the Ψ_k's produced there are functions near the edges. However, it is easily seen that this issue can be resolved by considering the sequence $(\Psi_k * \eta_k)_{k=1}^\infty$ instead. $\square$

Given an open interval I, let $\overline{I}$ denote its closure and let $\mathcal{H}^1(\overline{I})$ denote the usual Sobolev space, i.e. the space of functions in $L^2(\overline{I})$ whose first (weak) derivative is in $L^2(\overline{I})$ (for basic references see e.g. [6] or [9]). We use $\mathcal{H}^1$ as opposed to H^1 to avoid confusion with the Hardy spaces. Note that each equivalence class in $\mathcal{H}^1(\overline{I})$ has a continuous representative on $\overline{I}$, so we will treat the elements of $\mathcal{H}^1(\overline{I})$ as continuous functions on $\overline{I}$. Let $\mathcal{H}^1_{\overline{I}}(\mathbb{R})$ denote the set of functions with support within $\overline{I}$ such that $f|_{\overline{I}} \in \mathcal{H}^1(\overline{I})$, and let $\mathcal{H}^1_{\overline{I},0}(\mathbb{R}) \subset \mathcal{H}^1_{\overline{I}}(\mathbb{R})$ be those functions that are continuous at the endpoints of I. In particular, $\mathcal{H}^1_{\overline{I},0}(\mathbb{R}) \subset \mathcal{H}^1(\mathbb{R})$, and we have that $C_I^\infty(\mathbb{R})$ is dense in $\mathcal{H}^1_{\overline{I},0}(\mathbb{R})$. We omit the proof of the following basic lemma.

Lemma 3.2. *Given any interval (a,b), $G \in L^2([a,b])$ is the weak derivative of $F \in \mathcal{H}^1([a,b])$ if and only if*

$$F(x) = F(a) + \int_a^x G(y)dy, \quad x \in (a,b).$$

Let $|I|$ denote the length of the interval I.

Lemma 3.3. *Given any interval I and $F \in \mathcal{H}^1_{\overline{I},0}(\mathbb{R})$ we have $\hat{F} \in L^1(\mathbb{R})$ and*

$$\|\hat{F}\|_{L^1} \leqslant (2 + |I|^{3/2})\|F'\|_{L^2}.$$

Proof. For $F \in \mathcal{H}^1_{\overline{I},0}(\mathbb{R})$ we have $F(x) = \int_{-\infty}^x F'(y)dy$, so $\|F\|_{L^\infty} \leqslant \|F'\|_{L^2}\sqrt{|I|}$ by Hölder's inequality and therefore $\|\hat{F}\|_{L^\infty} \leqslant |I|^{3/2}\|F'\|_{L^2}$. Moreover $\widehat{F'}(\xi) = i\xi\hat{F}(\xi)$, so by Parseval's formula and the Cauchy-Schwarz inequality we get

$$\int_{|\xi|>1/2} |\hat{F}(\xi)|d\xi = \int_{|\xi|>1/2} |\widehat{F'}(\xi)/\xi|d\xi \leqslant \sqrt{2\int_{1/2}^\infty \xi^{-2}d\xi}\|\widehat{F'}\|_{L^2} \leqslant 2\|F'\|_{L^2}.$$

These combine easily to give the desired inequality. □

Lemma 3.4. *Let I be an open interval and let $\Phi \in \mathcal{D}'(I)$ be such that $\Phi = \Psi|_I$ for some $\Psi \in \breve{L^\infty}$. Then Ψ determines a continuous extension of Φ to $\mathcal{H}^1_{\overline{I},0}(\mathbb{R})$ via*

$$\Phi(F) = \int_{\mathbb{R}} \hat{\Psi}\check{F}.$$

This extension is independent of Ψ.

Proof. By Lemma 3.3 we clearly have that $\mathcal{H}^1_{\overline{I},0} \ni F \mapsto \int \hat{\Psi}\check{F}$ defines a continuous linear functional on the space $\mathcal{H}^1_{\overline{I},0}$. Since $\{F'|_{\overline{I}} : F \in \mathcal{H}^1_{\overline{I},0}(\mathbb{R})\}$ has codimension 1 in $L^2(\overline{I})$ and $\|F\|_{\mathcal{H}^1_{\overline{I},0}(\mathbb{R})}$ is comparable with $\|F'\|_{L^2(\overline{I})}$, it follows that there exists a function $\nu \in L^2(\overline{I})$ such that

$$\int_{\mathbb{R}} \hat{\Psi}\check{F} = \int_{\overline{I}} F'\nu \tag{3.1}$$

for all $F \in \mathcal{H}^1_{\overline{I},0}$. Moreover, ν is clearly uniquely defined by Ψ modulo the constant functions on $\overline{I}$. Since we clearly have $\Phi(F) = \int F'\nu$ for all $F \in C_I^\infty(\mathbb{R})$ and $C_I^\infty(\mathbb{R})$ is dense in $\mathcal{H}^1_{\overline{I},0}$, we see that ν is uniquely defined by Φ (modulo constants). □

Given $\Phi \in \mathcal{D}'(I)$ such that $\Phi = \Psi|_I$ for some $\Psi \in \breve{L^\infty}$, due to the above lemma we will in the future identify Φ with its extension to $\mathcal{H}^1_{\overline{I},0}(\mathbb{R})$.

Lemma 3.5. *Let $\Psi \in \breve{L^\infty}$ be given and set $\Phi = \Psi|_{(-a,a)}$. For any $F, G \in \mathcal{H}^1_{[0,a]}(\mathbb{R})$ we have*

$$\int F(\cdot + y)G(y)dy \in \mathcal{H}^1_{[-a,a],0}(\mathbb{R})$$

and

$$\langle W_\Phi F, \bar{G}\rangle_{L^2([0,a])} = \Psi\left(\int F(\cdot + y)G(y)dy\right).$$

Proof. Let us write $F \circ G$ for $\int F(\cdot + y)G(y)dy$. By the location of supports of F, G, it is easy to see that $F \circ G$ is continuous on $\mathbb{R}$ with support in $[-a, a]$. Let $H \in L^2_{[0,1]}(\mathbb{R})$ be such that $H|_{[0,1]}$ is the weak derivative of $G|_{[0,1]}$. By Lemma 3.2, $F \circ G \in \mathcal{H}^1_{[-a,a],0}(\mathbb{R})$ follows once we establish that

$$F \circ G(x) = \int_{-\infty}^{x} (-F \circ H(y) + F(a+y)G(a) - F(y)G(0))\, dy, \tag{3.2}$$

which follows by careful use of Fubini's theorem. For the second part, we first note that when $F \in C^\infty_{(0,a)}(\mathbb{R})$, then $\frac{d}{dx} F \circ G = F' \circ G$; so using Lemma 3.4, we can show the formula by approximating the integrals with Riemann sums. We omit the details and assume that this has been established. To get the general statement, let $(F_k)_{k=1}^\infty$ be a sequence in $C^\infty_{(0,a)}(\mathbb{R})$ converging to F. By (3.2) it is easily seen that $F_k \circ G \longrightarrow F \circ G$ in $\mathcal{H}^1_{[-a,a],0}(\mathbb{R})$, so by the continuity of Ψ (Lemma 3.4) we get

$$\langle W_\Phi F, \bar{G} \rangle = \lim_{k\to\infty} \langle W_\Phi F_k, \bar{G} \rangle = \lim_{k\to\infty} \Psi(F_k \circ G) = \Psi(F \circ G),$$

as desired. $\square$

It will be convenient to move the discussion to the circle, so we will introduce a new class of operators, unitarily equivalent with the TWH-operators, which resemble Hankel operators. Recall that m denotes the normalized arc-length measure on the unit circle $\mathbb{T}$. As before we will let the meaning of expressions such as $\mathcal{F}^{-1}(f)$ be determined by the type of f; if $f \in l^2(\mathbb{Z})$, then $\mathcal{F}^{-1}(f)(z) = \sum f(k) z^k$, $z \in \mathbb{T}$, and if $f \in L^2(\mathbb{T})$, then $\mathcal{F}^{-1}(f)(k) = \int_{\mathbb{T}} f z^k dm$, $k \in \mathbb{Z}$. Given $\Theta \in \mathcal{D}'(\mathbb{T}\backslash\{1\})$ we define the operator $\Gamma_\Theta : L^2_{\mathbb{T}^+}(\mathbb{T}) \to L^2_{\mathbb{T}^-}(\mathbb{T})$ by

$$F \ni C^\infty_{\mathbb{T}^+}(\mathbb{T}) \to \begin{cases} \int \Theta(\zeta) F(z\zeta) dm(\zeta), & z \in \mathbb{T}^-, \\ 0, & z \in \mathbb{T}^+ \end{cases} \tag{3.3}$$

whenever this extends to a bounded operator on $L^2(\mathbb{T}^+)$. Formally, we should write $\Theta(F(z\ \cdot))$ instead of $\int \Theta(\zeta)F(z\zeta)dm(\zeta)$, but we believe that the latter notation is more readable and therefore we will continue to abuse notation in the above way. The reader should keep in mind that Θ is not necessarily a function.

We now show that this new class is equivalent under unitary transformations with the set of TWH-operators (for any fixed a). It will be convenient to set the value of a to $1/2$. Thus let $\Phi \in \mathcal{D}'(-1/2, 1/2)$ be given such that $W_\Phi : L^2_{[0,1/2]}(\mathbb{R}) \to L^2_{[0,1/2]}(\mathbb{R})$ is bounded. Set

$$\Theta(e^{2\pi i y}) = \Phi(1/2 - y), \quad 0 < y < 1. \tag{3.4}$$

Given any $F \in L^2_{[0,1/2]}(\mathbb{R})$ we also define $\tilde{F} \in L^2_{\mathbb{T}^+}(\mathbb{T})$ via $\tilde{F}(e^{2\pi i t}) = F(1/2 - t)$ $(0 \leqslant t < 1)$ and note that this transformation is unitary. Moreover, for any $0 < x < 1/2$ and $F \in C^\infty_{(0,1/2)}(\mathbb{R})$ we have

(3.5)

$$(W_\Phi F)(x) = \int_0^{1/2} \Phi(y - x) F(y) dy = \int_0^{1/2} \Phi(1/2 - y - x) F(1/2 - y) dy$$

$$= \int_0^{1/2} \Theta(e^{2\pi i x} e^{2\pi i y}) \tilde{F}(e^{2\pi i y}) dy = \int_{\mathbb{T}^+} \Theta(\bar{z}\zeta) \tilde{F}(\zeta) dm(\zeta) = (\Gamma_\Theta, \tilde{F})(z),\ z = e^{-2\pi i x}.$$

Thus Γ_Θ and W_Φ are equivalent under simple unitary transformations. Endow $C^1(\mathbb{T})$ with the norm

$$\|F\|_{C^1(\mathbb{T})} = \|F\|_{L^\infty} + \|F'\|_{L^\infty}.$$

Here, and in the remainder of the paper, $F'(z)$ denotes $\lim_{t\to 0}(F(ze^{it}) - F(z)/t)$ whenever F is a function on $\mathbb{T}$. By Theorem 2.5, and the proof of Lemma 3.4 we conclude that it is no restriction to assume that Θ is of the form

$$\Theta(F) = \int F'\mu \; dm, \quad \mu \in L^2(\mathbb{T}), \tag{3.6}$$

in the definition of Γ_Θ, (3.3). The expression (3.6) clearly defines an element of $(C^1(\mathbb{T}))^*$, which we will call the canonical extension of Θ, and we shall use the same symbol for it. Whenever Φ and Θ are related as above, we shall write $\mathcal{C}(\Phi)$ for the canonical extension of Θ. The following lemma is immediate from the above construction.

Lemma 3.6. *Let $\Psi \in \breve{L^\infty}$ be given and set $\Theta = \mathcal{C}(\Psi|_{(-1/2,1/2)})$. Given $\{F \in C^1(\mathbb{T}) : F(1) = 0\}$ we define $\tilde{F}$ via $\tilde{F}(1/2 - t) = F(e^{2\pi i t})$. Then*

$$\Psi(\tilde{F}) = \Theta(F),$$

where $\Psi(\tilde{F})$ is defined as in Lemma 3.4.

Let $z : \mathbb{T} \to \mathbb{T}$ denote the identity function $z(\zeta) = \zeta$. Given a distribution $\Theta \in \mathcal{D}'(\mathbb{T})$ we define the Fourier transform as the sequence $\hat{\Theta} \in \mathbb{C}^{\mathbb{Z}}$ given by

$$\hat{\Theta}(j) = \Theta(z^{-j}),$$

and similarly $\check{\Theta}(j) = \Theta(z^j)$. For $\Theta \in \mathcal{D}'(\mathbb{T}\backslash\{1\})$ such that Γ_Θ is bounded, we define $\hat{\Theta}$ to be the Fourier transform of the canonical extension. Note that in the case when Θ is a function in $L^1(\mathbb{T})$, this definition can disagree with the traditional definition by a constant sequence. Similarly, given $\Phi(x) = \sum_{k=-\infty}^{\infty} \phi_k e^{2\pi i k x}$ and $\Theta = \mathcal{C}(\phi|_{(-1/2,1/2)})$, a short calculation shows that

$$(-1)^k \phi_{-k} = \hat{\Theta}(k) \mod \mathbf{1}, \tag{3.7}$$

where $\mathbf{1}$ is the constant sequence. For distributions Φ such that W_Φ is bounded, one can use (3.7) to define its Fourier series. Again, for functions Φ this definition can be off by a constant sequence with respect to the usual definition. Note that this formula combined with the material in the next section proves Theorem 1.2 for $a = 1/2$ and that the corresponding statement for any a easily follows from this one. Let $\mathcal{D}'^k$ denote the set of distributions of order k. We omit the proof of the next result.

Lemma 3.7. *If $\Theta \in \mathcal{D}'^k(\mathbb{T})$, then there exists $C > 0$ such that $|\hat{\Theta}(j)| < C|j|^k$. Conversely, if $\sigma = (\sigma_j)_{j=-\infty}^{\infty} \in \mathbb{C}^{\mathbb{Z}}$ satisfies $|\sigma_j| < C|j|^k$ for some $C > 0$ and $k \in \mathbb{N}$, then there is a unique $\Theta \in \mathcal{D}'^{k+2}(\mathbb{T})$ such that $\sigma = \hat{\Theta}$.*

Sequences σ as in the above lemma will be called polynomially bounded.

4. Discrete Hardy spaces, Hankel operators and $BMO(\mathbb{Z})$

Recall the definition of $H^2(\mathbb{Z})$ and the discrete Hankel operators defined in (1.5). We extend this definition to polynomially bounded sequences σ by setting

$$H_\sigma(f) = P_{H^2_-(\mathbb{Z})}(\sigma \cdot f), \quad f \in \mathcal{F}^{-1}\left(C^\infty_{\mathbb{T}+}(\mathbb{T})\right), \tag{4.1}$$

whenever this extends to a bounded operator from $H^2(\mathbb{Z})$ to $H^2_-(\mathbb{Z})$. Given $\Theta \in \mathcal{D}'(\mathbb{T})$ and $F \in C^\infty_{\mathbb{T}+}(\mathbb{T})$, we will in the next calculation use the notation $\Theta(F(z\,\cdot)) = \Theta(F(z\zeta))$; i.e., we think of Θ as acting on functions in the variable ζ. We have

$$\begin{aligned}
&\mathcal{F}^{-1}\big(\Gamma_\Theta(F)\big) = \mathcal{F}^{-1}P_{L^2_{\mathbb{T}-}(\mathbb{T})}\big(\Theta(F(z\zeta))\big) = P_{H^2_-(\mathbb{Z})}\mathcal{F}^{-1}\big(\Theta(F(z\zeta))\big)\\
&= P_{H^2_-(\mathbb{Z})}\left(\left(\int \Theta\Big(F(z\zeta)\Big)z^k\,dm(z)\right)_{k=-\infty}^{\infty}\right)\\
&= P_{H^2_-(\mathbb{Z})}\left(\left(\Theta\Big(\int F(z\zeta)z^k\,dm(z)\Big)\right)_{k=-\infty}^{\infty}\right)\\
&= P_{H^2_-(\mathbb{Z})}\left(\big(\Theta\big(\zeta^{-k}\check{F}(k)\big)\big)_{k=-\infty}^{\infty}\right) = P_{H^2_-(\mathbb{Z})}\left(\hat{\Theta}\cdot\check{F}\right),
\end{aligned}\tag{4.2}$$

which gives

$$\mathcal{F}^{-1}\Gamma_\Theta\mathcal{F} = H_{\hat{\Theta}} \tag{4.3}$$

in analogy with the classical theory. Conversely, given H_σ for some polynomially bounded sequence σ, we can define a distribution $\Theta \in \mathcal{D}'(\mathbb{T})$ via Lemma 3.7 and by (4.2) we get $\mathcal{F}^{-1}\Gamma_\Theta\mathcal{F} = H_\sigma$. We summarize a number of elementary observations in the following three propositions.

Proposition 4.1. *Let $\Phi \in \mathcal{D}'((-1/2,1/2))$ be such that W_Φ is bounded, set $\Theta = \mathcal{C}(\Phi)$ and set $\sigma = \hat{\Theta}$. Then W_Φ, Γ_Θ and H_σ are equivalent via (3.5) and (4.3). Moreover $\Phi \in \mathcal{D}'^1((-1/2,1/2))$. If Γ_Θ is bounded for some $\Theta \in \mathsf{Ran}\,\mathcal{C}$ or if $\sigma \in \mathbb{C}^{\mathbb{Z}}$ satisfies $\lim_{k\to\pm\infty}\sigma(k)/k = 0$, $\sigma(0) = 0$ and H_σ is bounded, then there exists a unique $\Phi \in \mathcal{D}'((-1/2,1/2))$ such that the first statement is true.*

Proof. The only part of the proposition that is not immediate from the previous developments is that the first statement is true if $\lim_{k\to\pm\infty}\sigma(k)/k = 0$, $\sigma(0) = 0$ and H_σ is bounded. By Lemma 3.7 and (4.3) there exists a $\Theta \in \mathcal{D}'(\mathbb{T})$ such that Γ_Θ and H_σ are unitarily equivalent, with $\sigma = \hat{\Theta}$. Moreover, by (3.5) and Theorem 2.5 there is a $\Psi \in \widecheck{L^\infty}$, $\Phi = \Psi|_{(-1/2,1/2)}$ such that W_Φ is equivalent with Γ_Θ via (3.5). But setting $\tau = \widehat{\mathcal{C}(\Phi)}$, W_Φ is also equivalent with $\Gamma_{\mathcal{C}(\Phi)}$ and H_τ. Let μ be the function that gives $\mathcal{C}(\Phi)$ via (3.6). By the Riemann-Lebesgue lemma it follows that $\lim_{k\to\pm\infty}\hat{\mu}(k) = 0$. Thus

$$\lim_{k\to\pm\infty}\frac{\tau(k)}{k} = \lim_{k\to\pm\infty}\frac{\widehat{\mathcal{C}(\Phi)}(k)}{k} = \lim_{k\to\pm\infty} -i\hat{\mu}(k) = 0.$$

We are done if we show that $\sigma = \tau$. Since $\Gamma_\Theta = \Gamma_{\mathcal{C}(\Phi)}$, it follows by Proposition 4.2 below that $\sigma = \tau + c_0\mathbf{1}$ for some $c_0 \in \mathbb{C}$, (where $\mathbf{1} = (\ldots,1,1,1,1,\ldots)$), and since $\sigma(0) = \tau(0) = 0$, we get $c_0 = 0$. $\square$

Proposition 4.2. *Let $\Theta \in \mathcal{D}'(\mathbb{T})$ be such that Γ_Θ is bounded. Then there exists a unique $\tilde{\Theta} \in \mathsf{Ran}\,\mathcal{C}$ such that $\Gamma_\Theta = \Gamma_{\tilde{\Theta}}$. Moreover, there is an $N \in \mathbb{N}$ and $c_0,\ldots,c_N$ such that*

$$\Theta - \tilde{\Theta} = c_0\delta_1 + c_1\delta_1' + \cdots + c_N\delta_1^{(N)}.$$

Proof. Any distribution on a compact set is a distribution of finite order (Theorem 2.3.1 in [9]). If $\Gamma_\Theta = \Gamma_{\tilde\Theta}$, then clearly $(\Theta - \tilde\Theta)|_{\mathbb{T}\setminus\{1\}} = 0$, so $\Theta - \tilde\Theta$ is a finite order distribution with support in $\{1\}$. By Theorem 2.3.4 in [9], it is necessarily of the form $c_0\delta_1 + c_1\delta_1' + \cdots + c_N\delta_1^{(N)}$. $\square$

The next proposition follows immediately from the previous two.

Proposition 4.3. *Let $\sigma \in \mathbb{C}^{\mathbb{Z}}$ be polynomially bounded such that H_σ is bounded. Then there exists a unique $\tilde\sigma$ such that $\lim_{k\to\pm\infty}\tilde\sigma(k)/k = 0$, $\tilde\sigma(0) = 0$ and $H_{\tilde\sigma} = H_\sigma$. Moreover, there is an $N \in \mathbb{N}$ and $c_0, \ldots, c_N$ such that*

$$\sigma - \tilde\sigma = c_0(1)_{k=-\infty}^{\infty} + c_1(k)_{k=-\infty}^{\infty} + \ldots + c_N\left(\delta_1^{(N)}(z^{-k})\right)_{k=-\infty}^{\infty}.$$

We will now begin the proof that $\|\Gamma_\Theta\|$ and $\|\hat\Theta\|_{BMO}$ are comparable for $\Theta \in \mathsf{Ran}\ \mathcal{C}$. Define $R_\Theta : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ via

$$\langle R_\Theta a, b\rangle = \sum \frac{\hat\Theta(i) - \hat\Theta(j)}{i-j} a(i)\overline{b(j)}.$$

In the above formulas we interpret $0/0$ as 0. Let $\mathbb{Z}_e$ and $\mathbb{Z}_o$ be the even/odd integers respectively, and let $\{e_k\}_{k\in\mathbb{Z}}$ denote the standard basis for $l^2(\mathbb{Z})$, (i.e. $e_k(j) = \delta(k-j)$, where δ is the Kronecker symbol). Define $P_e : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ via $P_e(\sum_{k\in\mathbb{Z}} a(k)e_k) = \sum_{k\in\mathbb{Z}_e} a(k)e_k$ and set $P_o = I - P_e$. Define $f_l^+ \in L^2_{\mathbb{T}^+}(\mathbb{T})$ via $f_l^+(z) = \sqrt{2}z^{-l}\chi(\mathbb{T}^+, z)$ for all $l \in \mathbb{Z}$ and define $f_l^- \in L^2_{\mathbb{T}^-}(\mathbb{T})$ analogously. Note that each of the sets $\{f_k^+\}_{k\in\mathbb{Z}_e}$, $\{f_k^+\}_{k\in\mathbb{Z}_o}$ forms an orthonormal basis for $L^2_{\mathbb{T}^+}(\mathbb{T})$ and the same is true if $+$ is exchanged with $-$ everywhere.

Lemma 4.4. *Given $\Theta \in \mathsf{Ran}\ \mathcal{C}$, $a \in \mathsf{Ran}\ P_o$ and $b \in \mathsf{Ran}\ P_e$ we have*

$$\left\langle \Gamma_\Theta \sum a_l f_l^+, \sum b_l f_l^- \right\rangle = \frac{i}{\pi}\langle R_\Theta a, b\rangle.$$

The corresponding formula with o and e switched also holds. In particular,

$$\|P_o R_\Theta P_e\| = \|P_e R_\Theta P_o\| = \pi\|\Gamma_\Theta\|.$$

Proof. Assume first that $\Theta \in L^1(\mathbb{T})$. These formulas can of course be obtained by evaluating some multiple integrals, but this provides little intuition for what is going on. We therefore prefer the following argument. By (4.3) and some simple calculations we have

$$\left\langle \Gamma_\Theta f_k^+, f_l^- \right\rangle = \left\langle \hat\Theta \cdot \breve{f_k^+}, \breve{f_l^-} \right\rangle = \sum_m \left(\hat\Theta \cdot \breve{f_k^+} \cdot \overline{\breve{f_l^-}}\right)(m) = \sum_m \left(\hat\Theta \cdot \breve{f_k^+} \cdot \breve{f_l^+}\right)(m). \tag{4.4}$$

The assumption $\Theta \in L^1(\mathbb{T})$ ensures that the sum is absolutely convergent. We have $\breve{f_0^+} = \frac{1}{\sqrt2}e_0 + \sum_{m\in\mathbb{Z}_o}\frac{\sqrt2 i}{\pi m}e_m$ or, written out as a sequence,

$$\breve{f_0^+} = \sqrt2\left(\ldots, \frac{-i}{5\pi}, 0, \frac{-i}{3\pi}, 0, \frac{-i}{\pi}, \frac12, \frac{i}{\pi}, 0, \frac{i}{3\pi}, 0, \frac{i}{5\pi}, \ldots\right).$$

By the formula $\breve{f_k^+}(m) = \breve{z^{-k}f_0^+}(m) = \breve{f_0^+}(m-k)$ for all $m, k \in \mathbb{Z}$ we see that any $\breve{f_k^+}$ is obtained by a translation of $\breve{f_0^+}$. We thus get

$$\breve{f_k^+} \cdot \breve{f_l^+} = \frac{i}{\pi}\frac{e_k - e_l}{k-l} \tag{4.5}$$

whenever $k-l$ is an odd number, which combined with (4.4) yields the desired formula.

If Θ is not in L^1, let $\Phi \in \mathcal{D}'((-1/2,1/2))$ be such that $\Theta = \mathcal{C}(\Phi)$ and W_Φ is bounded. Lemma 3.1 shows that there exists a sequence $\Psi_1, \Psi_2, \ldots \in C^\infty(\mathbb{R})$ such that $\|\widehat{\Psi_m}\|_{L^\infty} \leqslant 3\|W_\Phi\|$, and $\Phi(F) = \lim_{m\to\infty}\int \Psi_m F$ for all $F \in C^\infty_{(-1/2,1/2)}(\mathbb{R})$. By standard functional analysis, we can choose a subsequence $(\Psi_{m_j})_{j=1}^\infty$ such that $(\widehat{\Psi_{m_j}})_{j=1}^\infty$ is convergent in the weak*-topology of L^∞. Denote the limit by $\hat{\Psi}$ and note that $\Phi = \Psi|_{(-1/2,1/2)}$. Put $\Theta_m = \mathcal{C}\big(\Psi_m|_{(-1/2,1/2)}\big)$ and note that $\Theta_m \in L^1$. Moreover, for any $l \in \mathbb{Z}$, we have by Lemmas 3.3 and 3.6 that

$$
\begin{aligned}
&\widehat{\Theta_{m_j}}(l) - \widehat{\Theta_{m_j}}(0) = \Theta_{m_j}(z^{-l}-1) = \int_{-1/2}^{1/2} \Psi_{m_j}(x)\big(e^{-2\pi i l(1/2-x)} - 1\big) \\
&= \int \hat{\Psi}_{m_j}\mathcal{F}^{-1}\big((e^{-2\pi i l(1/2-x)} - 1)\chi_{(-1/2,1/2)}(x)\big) \\
&\qquad \to \int \hat{\Psi}\mathcal{F}^{-1}\big((e^{-2\pi i l(1/2-x)} - 1)\chi_{(-1/2,1/2)}(x)\big) \\
&= \Psi\big((e^{-2\pi i l(1/2-x)} - 1)\chi_{(-1/2,1/2)}(x)\big), \qquad \text{as } j \to \infty.
\end{aligned}
$$

But $\Theta = \mathcal{C}(\Phi)$, so the right hand side equals $\hat{\Theta}(l) - \hat{\Theta}(0)$, again by Lemma 3.6. Let $k \in \mathbb{Z}_o$ and $l \in \mathbb{Z}_e$ be fixed. By a calculation similar to (3.5), it is easy to see that there exist functions $A, B \in \mathcal{H}^1_{[0,1/2]}$ such that $\langle \Gamma_\Theta f_k^+, f_l^- \rangle = \langle W_\Phi A, B\rangle$. By Lemmas 3.3, 3.5, and the first part of the proof we thus get

$$
\begin{aligned}
&\langle \Gamma_\Theta f_k^+, f_l^- \rangle = \langle W_\Phi A, B\rangle = \Psi\left(\int A(\cdot+y)\overline{B(y)}dy\right) = \int \hat{\Psi}\mathcal{F}^{-1}\left(\int A(\cdot+y)\overline{B(y)}dy\right) \\
&= \lim_{j\to\infty}\int \widehat{\Psi_{m_j}}\mathcal{F}^{-1}\left(\int A(\cdot+y)\overline{B(y)}dy\right) = \lim_{j\to\infty}\langle \Gamma_{\Theta_{m_j}} f_k^+, f_l^-\rangle \\
&= \lim_{j\to\infty}\frac{i}{\pi}\frac{\widehat{\Theta_{m_j}}(k) - \widehat{\Theta_{m_j}}(l)}{k-l} \\
&= \lim_{j\to\infty}\frac{i}{\pi}\frac{\widehat{\Theta_{m_j}}(k) - \widehat{\Theta_{m_j}}(0) - \widehat{\Theta_{m_j}}(l) + \widehat{\Theta_{m_j}}(0)}{k-l} = \frac{i}{\pi}\frac{\hat{\Theta}(k) - \hat{\Theta}(l)}{k-l}. \qquad \square
\end{aligned}
$$

The calculation leading to (4.5) also shows why the "*oo*" and "*ee*" cases are not part of Lemma 4.4; infinitely many terms would appear on the right hand side of (4.5). Nevertheless we have

Lemma 4.5. *Given $\Theta \in \mathrm{Ran}\,\mathcal{C}$ we have*

$$\|P_e R_\Theta P_e\| \leqslant \frac{3\pi+\pi^2}{2}\|\Gamma_\Theta\|.$$

Upon replacing e with o, the formula is still true.

Proof. We only do the ee case; the other is identical. As earlier, we interpret $x/0$ as 0, regardless of x. Let $a, b \in \mathsf{Ran}\ P_e$ be arbitrary. Then

$$
\begin{aligned}
|\langle R_\Theta P_e a, P_e b\rangle| &= \left|\sum_{i,j\in\mathbb{Z}_e} \frac{\hat{\Theta}(i)-\hat{\Theta}(j)}{i-j} a(i)\overline{b(j)}\right| \\
&= \left|\sum_{i,j\in\mathbb{Z}_e} \frac{i-1-j}{i-j}\frac{\hat{\Theta}(i-1)-\hat{\Theta}(j)}{i-1-j} a(i)\overline{b(j)} + \sum_{i,j\in\mathbb{Z}_e} \frac{\hat{\Theta}(i)-\hat{\Theta}(i-1)}{i-j} a(i)\overline{b(j)}\right| \qquad (4.6)\\
&\leqslant \sup_{i,j\in\mathbb{Z}_e}\left\{\frac{i-1-j}{i-j}\right\}\|P_e R_\Theta P_o\|\|a\|\|b\| + \frac{1}{2}\left|\sum_{l,k\in\mathbb{Z}}\frac{1}{l-k}\tilde{a}(l)\overline{\tilde{b}(k)}\right|,
\end{aligned}
$$

where $\tilde{a}(l) = a(2l)(\hat{\Theta}(2l) - \hat{\Theta}(2l-1))$ and $\tilde{b}(k) = b(2k)$. Clearly $\|\tilde{b}\| = \|b\|$ and, moreover, by Lemma 4.4 we have $\sup_{l\in\mathbb{Z}}\{|\hat{\Theta}(2l)-\hat{\Theta}(2l-1)|\} \leqslant \pi\|\Gamma_\Theta\|$, so

$$\|\tilde{a}\| \leqslant \pi\|\Gamma_\Theta\|\|a\|.$$

Let $\mathsf{Im}\ \log$ denote the imaginary part of the logarithm defined in the right half plane and note that $\sum_k -z^k/k = 2i\mathsf{Im}\ \log(1-z)$ for $z \in \mathbb{T}$. By these calculations and Lemma 4.4 we may continue the above calculation as follows:

$$
\begin{aligned}
&|\langle R_\Theta P_e a, P_e b\rangle| \leqslant \frac{3}{2}\pi\|\Gamma_\Theta\|\|a\|\|b\| + \left|\left\langle i\mathsf{Im}\ \log(1-z)\check{\tilde{a}}(z), \check{\tilde{b}}(z)\right\rangle_{L^2(\mathbb{T})}\right| \\
&\leqslant \frac{3\pi}{2}\|\Gamma_\Theta\|\|a\|\|b\| + \|\mathsf{Im}\ \log(1-z)\|_{L^\infty(\mathbb{T})}\|\check{\tilde{a}}\|_{L^2(\mathbb{T})}\|\check{\tilde{b}}\|_{L^2(\mathbb{T})} \leqslant \frac{3\pi+\pi^2}{2}\|\Gamma_\Theta\|\|a\|\|b\|.
\end{aligned}
$$

□

Theorem 4.6. *There exist $C_1, C_2 > 0$ such that $C_1\|\Gamma_\Theta\| \leqslant \|\hat{\Theta}\|_{BMO} \leqslant C_2\|\Gamma_\Theta\|$ for all $\Theta \in \mathsf{Ran}\ \mathcal{C}$.*

Proof. By Lemma 4.4 we have $\pi\|\Gamma_\Theta\| \leqslant \|R_\Theta\|$. Conversely,

$$R_\Theta = P_e R_\Theta P_o + P_o R_\Theta P_e + P_e R_\Theta P_e + P_o R_\Theta P_o, \qquad (4.7)$$

so

$$\|R_\Theta\| \leqslant \sum_{x\in\{o,e\};y\in\{o,e\}} \|P_y R_\Theta P_x\| \leqslant (5\pi+\pi^2)\|\Gamma_\Theta\|$$

by Lemmas 4.4 and 4.5. The theorem now follows from Theorem 6.2 in [11], which states that $\|R_\Theta\|$ and $\|\hat{\Theta}\|_{BMO(\mathbb{Z})}$ are bounded by each other. □

5. Compactness

We first recall some standard results on Hadamard-Schur multipliers. Let $\mathcal{L}(l^2(\mathbb{Z}))$ denote all bounded operators on $l^2(\mathbb{Z})$, which we identify with matrices via the canonical basis $\{e_k\}_{k\in\mathbb{Z}}$. Given $A = (a_{ij}) \in \mathcal{L}(l^2(\mathbb{Z}))$ and $B = (b_{ij}) \in \mathcal{L}(l^2(\mathbb{Z}))$ we define the Hadamard-Schur product via

$$A \diamond B = (a_{ij}b_{ij}).$$

If $A \diamond B \in \mathcal{L}(l^2(\mathbb{Z}))$ for all $B \in \mathcal{L}(l^2(\mathbb{Z}))$, then A is called a Hadamard-Schur multiplier. In this case we write

$$\|A\|_{HS} = \sup_{B\in\mathcal{L}(l^2(\mathbb{Z}))} \frac{\|A\diamond B\|_{\mathcal{L}(l^2(\mathbb{Z}))}}{\|B\|_{\mathcal{L}(l^2(\mathbb{Z}))}}.$$

Given $\omega \in L^1(\mathbb{T})$ we define $A_\omega = (a_{ij}) \in \mathcal{L}(l^2(\mathbb{Z}))$ via $a_{ij} = \int_{\mathbb{T}} \omega(z) z^{j-i} dm(z)$.

Lemma 5.1. *Let $\omega \in L^1(\mathbb{T})$ be given. Then*

$$\|A_\omega\|_{HS} \leqslant \|\omega\|_{L^1(\mathbb{T})},$$

and $A_\omega \diamond B$ is compact whenever B is.

Proof. Let $D_z \in \mathcal{L}(l^2(\mathbb{Z}))$ be given by $D_z(a) = (a_j z^j)_{j=-\infty}^{\infty}$. It is not hard to see that

$$A_\omega \diamond B = \int_{\mathbb{T}} \omega(z) D_{\bar{z}} B D_z dm(z), \tag{5.1}$$

where the integral is interpreted in the WOT-sense (see [10]). Thus

$$\|A_\omega \diamond B\| \leqslant \int_{\mathbb{T}} |\omega(z)| \|D_{\bar{z}} B D_z\| dm(z) = \|\omega\|_{L^1(\mathbb{T})} \|B\|.$$

Moreover, a short argument shows that when B is compact, (5.1) holds as a Bochner integral. The compactness of $A_\omega \diamond B$ thus follows as the set of compact operators is closed in $\mathcal{L}(l^2(\mathbb{Z}))$. □

Given an interval $I \subset \mathbb{Z}$ and $\sigma \in \mathbb{C}^{\mathbb{Z}}$, let $c(I)$ denote its midpoint and set $Osc(\sigma, I) = |I|^{-1} \sum_{k \in I} |\sigma(k) - \sigma_I|$, where σ_I is the average of σ over I. Following [11] we define $CMO(\mathbb{Z}) \subset BMO(\mathbb{Z})$ by requiring that

$$\lim_{|I| \text{fixed},\, c(I) \to \pm\infty} Osc(\sigma, I) = 0, \quad \lim_{|I| \to \infty} \sup_{|I| \text{fixed},} Osc(\sigma, I) = 0.$$

Note that by [11] we know that $CMO(\mathbb{Z})$ is the closure in $BMO(\mathbb{Z})$ of the sequences with finite support.

Theorem 5.2. *Let $\Theta \in \mathsf{Ran}\ \mathcal{C}$ be given. Then Γ_Θ is compact if and only if $\hat{\Theta} \in CMO(\mathbb{Z})$.*

Proof. We first note that the above theorem is true if Γ_Θ is replaced with R_Θ, by Theorem 6.2 in [11]. By Lemma 4.4, Γ_Θ is unitarily equivalent with both $P_e R_\Theta P_o$ and $P_o R_\Theta P_e$. The "if"-part of the theorem thus follows easily. Conversely, by (4.7) one sees that the "only if" part follows once we establish the following claim: If Γ_Θ is compact, then the same is true for $P_e R_\Theta P_e$ and $P_o R_\Theta P_o$. To this end, we define $\iota_e, \iota_o : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ via $\iota_e(a) = \sum_{k=-\infty}^{\infty} a(k) e_{2k}$ and $\iota_o(a) = \sum_{k=-\infty}^{\infty} a(k) e_{2k-1}$. Given a sequence $\sigma \in \mathbb{C}^{\mathbb{Z}}$ we define the "diagonal operator" D_σ via $D_\sigma(a) = (\sigma(k) a(k))_{k=-\infty}^{\infty}$. Let $a, b \in l^2(\mathbb{Z})$ be arbitrary. Returning to the second line of (4.6), a careful calculation show that it can be rewritten as follows:

$$\begin{aligned} \langle R_\Theta \iota_e a, \iota_e b \rangle &= \sum_{i,j \in \mathbb{Z}} \left(1 - \frac{1}{2(i-j)}\right) \frac{\hat{\Theta}(2i-1) - \hat{\Theta}(2j)}{2i-1-2j} a(i) \overline{b(j)} \\ &\quad + \sum_{i,j \in \mathbb{Z}} \frac{\hat{\Theta}(2i) - \hat{\Theta}(2i-1)}{2(i-j)} a(i) \overline{b(j)} \\ &= \left\langle \left(\iota_e^* R_\Theta \iota_o - \frac{1}{2} A_{2i \mathsf{Im}\ \log(1-z)} \diamond (\iota_e^* R_\Theta \iota_o) \right) a, b \right\rangle \\ &\quad + \left\langle \frac{1}{2} A_{2i \mathsf{Im}\ \log(1-z)} D_{(\hat{\Theta}(2i) - \hat{\Theta}(2i-1))_{i=-\infty}^{\infty}} a, b \right\rangle. \end{aligned}$$

Now, $\iota_e^* R_\Theta \iota_o$ is essentially the same object as $P_e R_\Theta P_o$ (unitarily equivalent), so by Lemma 5.1 it follows that the operator in the first bracket is compact. By Lemma 4.4 we have

$$|\hat{\Theta}(2i) - \hat{\Theta}(2i-1)| = \pi \left|\langle \Gamma_\Theta f_{2i}^+, f_{2i-1}^- \rangle\right| \leqslant \pi \|\Gamma_\Theta f_{2i}^+\|,$$

which by standard facts about compact operators show that $\lim_{i\to\pm\infty} |\hat{\Theta}(2i) - \hat{\Theta}(2i-1)| = 0$, and hence the operator in the second bracket is compact as well. $\square$

A few simple estimates show that $\sigma \in CMO(\mathbb{Z})$ whenever σ is a sequence such that $\lim_{k\to\pm\infty} |\sigma(k)| = 0$. Thus, by the Riemann-Lebesgue lemma, we conclude in particular that Γ_Θ is a compact operator when Θ coincides with an absolutely continuous measure on $\mathbb{T}$. On the other hand, by the examples in the beginning we have that Γ_Θ can be a unitary operator when Θ is a singular measure. For further information on TWH-operators, we refer to Section 3 of [1], where questions concerning theorems of Adamyan, Arov, Krein-type for TWH-operators are numerically investigated.

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