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ON THE TOPOLOGICAL KOLMOGOROV PROPERTY OF THE CHACON AND PETERSEN SUBSHIFTS

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(Communicated by Bryna Kra)

Abstract. Basic properties of a K-relation, the topological analogue of the classical Kolmogorov definition, are investigated. It is shown that the Petersen subshift is a topological K-system and that the Chacon subshift is not.

Introduction

Kolmogorov dynamical systems (K-systems) play an important role in the theory of chaos of dynamical systems. The classical results of Rokhlin and Sinai say that the K-property is equivalent to the absence of zero entropy factors and is also equivalent to the K-mixing property. The former property implies a very strong chaotic behaviour in a dynamical system.

If the phase space of a dynamical system is equipped only with a topology there are some different concepts of the topological K-property. Two such types of properties have been defined and investigated by F. Blanchard ([1]). These are systems with uniform positive entropy (u.p.e.) and systems with completely positive entropy (c.p.e.).

Systems with u.p.e. have some properties which are similar to the measure-theoretical K-systems. Among other things they have positive topological entropy and are disjoint with all distal systems. However, in contrast to the measure-theoretic case, they are only weakly mixing and need not be strongly mixing (in the topological sense).

Systems with c.p.e. also have positive entropy and have an invariant measure with a full support, but they need not even be transitive.

Another concept of a K-property, defined as an analogue of a classical Kolmogorov definition (by the use of invariant measurable partitions) in ergodic theory, has been defined by Kamiński, Siemaszko and Szymański in [3]. This concept is defined by the use of invariant equivalence relations. It appears that if these K-systems are minimal they are weakly mixing and the set of asymptotic pairs is dense. It is known that any topological dynamical system which is a K-system with respect to an invariant measure with full support (in particular any u.p.e. system)

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is a topological K-system in the sense of [3]. However, there exist K-systems in this sense which have zero entropy.

The aim of this paper is to start investigating this phenomenon by considering two important classes of topological systems, the Chacon and Petersen subshifts (cf. [7], [5]). It is well known that they are strictly ergodic and have zero entropy. The first is weakly mixing and the second is strongly mixing.

We show that the Petersen subshift is a K-system and the Chacon subshift is not. It would be interesting to characterize all minimal K-systems with zero entropy.

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1. The asymptotic relation and the K-relation

Let (X, d) be a compact metric space and let T be a homeomorphism of X onto itself. We shall call the pair (X, T) *a topological dynamical system.* For a given $x \in X$ we denote by $O_T(x), O_T^+(x)$ *the orbit* and *the positive semiorbit of* x *w.r.t.* T, respectively, i.e.

$$O_T(x) = \{T^n x \ : \ n \in \mathbb{Z}\}, \ \ O_T^+(x) = \{T^n x \ : \ n \in \mathbb{N}_0\}, \ \text{ where } \ \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

A pair $(x, x') \in X \times X$ is said to be *asymptotic* if

$$\lim_{n \to +\infty} d(T^n x, T^n x') = 0.$$

We denote by $\mathcal{AS}$ *the asymptotic relation* (the set of asymptotic pairs) and by $CER(X)$ the family of all relations $R \subset X \times X$ which are equivalence relations and are closed subsets of $X \times X$. Obviously, the asymptotic relation need not belong to $CER(X)$. The diagonal relation is denoted by Δ.

A relation $R \in CER(X)$ is said to be *invariant* if $(T \times T)(R) \subset R$ and *totally invariant* if $(T \times T)(R) = R$.

R is called *generating* if

$$\bigcap_{n \in \mathbb{Z}} (T \times T)^n (R) = \Delta. \tag{1.1}$$

If

$$\overline{\bigcup_{n \in \mathbb{Z}} (T \times T)^{-n} (R)} = X \times X, \tag{1.2}$$

then we say that R is *orbitally dense.*

Any relation from $CER(X)$ which is invariant, generating and orbitally dense is said to be a K*-relation.* Due to the invariance of R in the definition of a K-relation we can replace $\mathbb{Z}$ with $\mathbb{N}_0$ in (1.1) and (1.2).

Let $\xi = \xi_R$ denote the partition whose elements are classes of R. It is easy to see that the invariance of R and its generating property imply that for any Borel probability measure μ of X invariant w.r.t. T the partition ξ is measurable and

$$T^{-1}\xi \le \xi, \quad \bigvee_{n \in \mathbb{Z}} T^n \xi = \bigvee_{n \in \mathbb{N}_0} T^n \xi = \varepsilon,$$

where ε is the partition into points, i.e. ξ is an exhaustive partition. However the orbital density of R does not imply the triviality of the tail partition generated by ξ, i.e.

$$\bigwedge_{n\in\mathbb{Z}} T^n\xi = \bigwedge_{n\in\mathbb{N}_0} T^{-n}\xi \neq \nu,$$

where ν denotes the trivial partition.

In other words, the partition ξ_R induced by R need not be a K-partition.

A topological dynamical system admitting a K-relation is called *a topological Kolmogorov system (K-system).*

It has been shown in [4] that for any topological dynamical system (X,T) and an invariant probability measure μ there exists an invariant and generating relation $R \in CER(X)$ with

$$E_\mu(X,T) \cup S(\mu) \subset \overline{\bigcup_{n\in\mathbb{Z}} (T\times T)^n(R)} \subset \Pi_\mu(T),$$

where $E_\mu(X,T)$ is the set of entropy pairs, $S(\mu) = \{(x,x) : x \in \mathrm{Supp}(\mu)\}$ and $\Pi_\mu(T)$ is the Pinsker relation of T w.r.t. μ. In particular, if (X,T) is uniquely ergodic the above inclusions imply that the smallest relation (belonging to $CER(X)$) containing all relations $(T\times T)^n(R)$, $n \in \mathbb{Z}$, is equal to $\Pi(T)$, the Pinsker relation of T. For the definition of a Pinsker relation see [2]. In the sequel we shall need four simple observations.

Remark 1.1. Every closed, invariant and generating relation R is contained in $\mathcal{AS}$.

Proof. Suppose $(x,y) \in R$ is not asymptotic. Hence there exist $\varepsilon > 0$ and an increasing sequence of natural numbers $\{n_k\}_{k\in\mathbb{N}}$ such that

$$d(T^{n_k}x, T^{n_k}y) > \varepsilon,\ k \in \mathbb{N}.$$

Taking if necessary a subsequence of $\{n_k\}_{k\in\mathbb{N}}$, we may assume that

$$\lim_{k\to+\infty} T^{n_k}x = x_0\,, \quad \lim_{k\to+\infty} T^{n_k}y = y_0.$$

Obviously, $d(x_0,y_0) \geq \varepsilon$; hence $(x_0,y_0) \notin \Delta$. Because R is invariant,

$$(T^{n_l}x, T^{n_l}y) \in (T\times T)^{n_k}R \ \text{ for } \ l \geq k.$$

As $(T\times T)^{n_k}R$ is closed, $(x_0,y_0) \in (T\times T)^{n_k}R$. Since the last statement is true for every $k \in \mathbb{N}$, $\{n_k\}_{k\in\mathbb{N}}$ grows to $+\infty$ and R is invariant, we obtain that $(x_0,y_0) \in \bigcap_{n\in\mathbb{Z}}(T\times T)^n(R)$. Because R is generating, this intersection has to be equal to Δ, which is a contradiction. Therefore, $R \subset \mathcal{AS}$. ☐

Now let $\mathcal{A} = \{0,1\}$ and $S = \mathcal{A}^{\mathbb{Z}}$, and suppose that S is equipped with the natural product topology. *The shift transformation* $\sigma : S \to S$ is defined, as usual, by

$$(\sigma x)_n = x_{n+1},\ n \in \mathbb{Z}.$$

The pair (S,σ) is a topological dynamical system, as is a pair (Λ,σ) for an arbitrary closed, σ-invariant set $\Lambda \subset S$. Systems of this form will be called *subshifts.*

Suppose we are given a subshift (Λ,σ). Let us define the relation

$$\mathcal{AS}_0(\Lambda) = \{(x,y) \in \Lambda\times\Lambda :\ x_i = y_i \ \text{ for } \ i \geq 0\}.$$

When it is clear which subshift is taken under consideration, we shall write $\mathcal{AS}_0$ instead of $\mathcal{AS}_0(\Lambda)$. It is easy to see the following:

Remark 1.2. For an arbitrary subshift Λ, $\mathcal{AS}_0$ is a closed, invariant, generating equivalence relation and

$$\mathcal{AS} = \bigcup_{n\in\mathbb{Z}} (\sigma\times\sigma)^n \mathcal{AS}_0.$$

Hence we have

Remark 1.3. A subshift (Λ,σ) is a K-system iff $\mathcal{AS}_0(\Lambda)$ is a K-relation.

Proof. If $\mathcal{AS}_0$ is a K-relation, then (Λ,σ) is a K-system by definition. Suppose there exists a K-relation $R\subset\Lambda\times\Lambda$. By Remarks 1.1 and 1.2 and the obvious fact that $\mathcal{AS}$ is totally invariant, we have

$$\bigcup_{n\in\mathbb{Z}}(\sigma\times\sigma)^n R\subset\bigcup_{n\in\mathbb{Z}}(\sigma\times\sigma)^n\mathcal{AS}=\mathcal{AS}=\bigcup_{n\in\mathbb{Z}}(\sigma\times\sigma)^n\mathcal{AS}_0.$$

Therefore, $\mathcal{AS}_0$ is orbitally dense. If $(x,y)\in(\sigma\times\sigma)^n\mathcal{AS}_0$, then $x_i=y_i$ for $i\geq -n$. If this statement holds for every $n\in\mathbb{N}$, then $x=y$. Therefore

$$\bigcap_{n\in\mathbb{Z}}(\sigma\times\sigma)^n\mathcal{AS}_0\subset\Delta.$$

The obvious opposite inclusion and Remark 2 give us the conclusion. □

Summarizing we obtain

Remark 1.4. A subshift (Λ,σ) is a K-system iff $\mathcal{AS}_0(\Lambda)$ is orbitally dense.

2. The Chacon and Petersen subshifts

Now we briefly recall the definitions of the Chacon and Petersen subshifts. These two systems (especially the Chacon subshift) were examined by many authors. The definition of the Petersen subshift which we present here is the original definition given by Petersen in [5]. In this paper, Lemma 2.3 is also formulated and proved (Lemma 3.1, p. 606). The definition of the Chacon subshift presented below is almost the same as in [6] (see p. 217). The reader can also find in this book the arguments at the beginning of the proof of Theorem 2.1 (concerning the block structure of elements of the Chacon subshift) as Lemma 5.5 (see p. 217). We include these results here for completeness and convenience.

Let $\mathcal{A}^k$ be the kth Cartesian power of $\mathcal{A}$ and let $\mathcal{A}^\infty=\bigcup_{k\in\mathbb{N}}\mathcal{A}^k$. Any element $B=b_1b_2...b_k\in\mathcal{A}^k$ is said to be *a block* and k is called its *length*. For $B=b_1b_2...b_k$ and $1\leq i\leq k$ we put

$$B[i]=b_ib_{i+1}...b_kb_1...b_{i-1}.$$

For $B=b_1b_2...b_k,\ C=c_1c_2...c_l\in\mathcal{A}^\infty$ we denote by BC the block that is the *concatenation* of B and C, i.e. the block

$$BC=b_1...b_kc_1...c_l.$$

We say that B *appears at the nth place in* C if $l\geq k,\ n\leq l-k$ and $c_n=b_1,\ c_{n+1}=b_2,\ ...,\ c_{n+k-1}=b_k$.

We say that $B=b_1b_2...b_k\in\mathcal{A}^\infty$ *appears in a sequence* $x\in S$ *at the* n*th place* if

$$x_n=b_1,\ x_{n+1}=b_2,\ ...,\ x_{n+k-1}=b_k.$$

B is said to be *the central block in* x if k is odd, $k=2m+1$ and $n=-m$.

The subshifts of Chacon and Petersen will be defined as the closures of orbits of special sequences called the Chacon and Petersen sequences, respectively.

In order to recall the definition of the Chacon sequence w^C, we consider the following sequence (B_k) of blocks:

$$B_0 = 0,\ B_1 = 0010,\ B_{k+1} = B_k B_k 1 B_k,\ k \geq 1. \tag{2.1}$$

Now let $v^k \in S,\ k \geq 1$ be the sequence such that $v_0^k = 1,\ v_{-2L_k}^k ... v_{-1}^k = B_k B_k,\ v_1^k ... v_{L_k}^k = B_k,\ L_k = |B_k|,\ k \geq 1$, and the other entries are equal to 0. The sequence (v^k) is convergent, and the Chacon sequence w^C and Chacon subshift X^C are defined as

$$w^C = \lim_{k \to \infty} v^k \text{ and } X^C = \overline{O_\sigma(w^C)}.$$

Observe that if we take $\overline{v}^k \in S,\ k \geq 1$, such that $\overline{v}_{-L_k+1}^k ... \overline{v}_{L_k}^k = B_k B_k$ and the other entries are equal to 0, then again the sequence $(\overline{v}^k)$ is convergent and its limit, denoted by v^C, is an element of X^C. In fact we could use v^C to define X^C.

In order to construct the Petersen sequence w^P, we shall build two sequences of blocks (A_k) and (B_k) with lengths $L_k = |A_k| = |B_k|$ growing to infinity. The first sequence will consist of central blocks of w^P and the second sequence will help us to build the first one. We shall also need a sequence $\mathcal{U}(k)$ of subsets of $\mathbb{N}$ defined in the following way:

$$\mathcal{U}(k) = \{j\ :\ 1 \leq j \leq L_k \text{ and } b_j = 1\},$$

where $B_k = b_1 b_2 ... b_{L_k}$.

Now, we put $A_0 = 101$ and $B_0 = 111$; hence $\mathcal{U}(0) = \{1, 2, 3\}$.

Suppose $A_k,\ B_k$ have been defined and $\mathcal{U}(k) = \{i_1, i_2, ..., i_{m_k}\}$ with $1 = i_1 < i_2 < ... < i_{m_k}$. Then we define

$$A_{k+1} = A_k A_k[i_{m_k}] A_k ... A_k A_k[i_2] A_k A_k[i_2] A_k ... A_k A_k[i_{m_k}] A_k$$

and

$$B_{k+1} = E_k B_k[i_{m_k}] E_k ... E_k B_k[i_2] E_k B_k[i_2] E_k ... E_k B_k[i_{m_k}] E_k,$$

where E_k is the L_k-block all of whose entries are 0 except for the first one, which is 1.

It is obvious that the numbers L_k are odd, so we define w^k as the element of S whose central block is A_k and all of whose other entries are 0. Finally, we put

$$w^P = \lim_{k \to \infty} w^k \text{ and } X^P = \overline{O_\sigma(w^P)}.$$

Theorem 2.1. *The Chacon subshift is not a K-system.*

Proof. We claim that

$$\mathcal{AS}_0(X^C) = \bigcup_{n \in \mathbb{N}} (\sigma \times \sigma)^n \{(w^C, v^C),\ (v^C, w^C)\} \cup \Delta^C, \tag{2.2}$$

where $\Delta^C = \{(x, x)\ :\ x \in X^C\}$. Suppose that $(z', z'') \in \mathcal{AS}_0(X^C) \setminus \Delta^C$. Then by the definition of $\mathcal{AS}_0$ there exist $n \in \mathbb{N}$ and $(z^1, z^2) \in \mathcal{AS}_0$ with $z_k^1 = z_k^2$ for $k \in \mathbb{N}$ and $z_0^1 \neq z_0^2$ such that $(z', z'') = (\sigma \times \sigma)^n (z^1, z^2)$. Suppose

$$z^1 = y^1 1 x\,,\ \text{ and }\ z^2 = y^2 0 x,$$

where y^1, y^2 are left infinite 0-1 sequences, x is a right infinite 0-1 sequence, and the shown single symbols 1 in z^1 and 0 in z^2 appear in those sequences at the position 0. Because of (2.1) it is easy to see that for every given $k \in \mathbb{N}$, every sequence $z \in X^C$ is built out of B_k's separated possibly at some places with single 1's. In other words, every sequence z has a k-structure, i.e. a representation as an infinite concatenation of B_k's and single 1's. This k-structure is unique, and, even more, the places in the k-structure of z where the B_k's appear are the only places where B_k may appear in z. To see this, suppose first that $B_1 = 0010$ appears in $B_1 B_1$. Because of the possible locations of the 1's, they may appear only at two obvious positions. We obtain the same conclusion if we suppose that B_1 appears in $B_1 1 B_1$. Proceeding by induction and analyzing the possible locations of an additional 1 (spacer) in the same way as for $k = 1$, we see that the only places where B_k appears in z are those given by the k-structure of z. Let us now come back to z^1, z^2. Because of the above considerations, their k-structures coincide on x. If for some $k \in \mathbb{N}$, x had not begun with B_k, then the 0th positions of z^1 and z^2 would fall at the same place of a certain B_k. As $z_0^1 \neq z_0^2$, x has to begin with B_k, for every $k \in \mathbb{N}$. This fact, the obvious observation that every B_k starts and ends with 0, and the structure of z^1 and z^2 together imply that $z^1 = w^C$ and $z^2 = v^C$. This proves (2.2).

Now, since w^C and v^C have equal positive coordinates and $w_n^C = v_{n+1}^C$ for negative n, then the closure of the orbit of $\mathcal{AS}_0(X^C)$ is equal to

$$O_{\sigma\times\sigma}(w^C, v^C) \cup O_{\sigma\times\sigma}(v^C, w^C) \cup \Delta^C \cup (\sigma \times Id)(\Delta^C) \cup \ (Id \times \sigma)(\Delta^C).$$

As a union of a countable set and three graphs, it is of course different from $X^C \times X^C$. Hence the Chacon subshift is not a topological K-system. □

Theorem 2.2. *The Petersen subshift is a K-system.*

To prove this theorem we shall need one more definition and the following lemma. Let

$$\mathcal{B}(k) = \{1 \le j \le L_k \ : \ \text{there is } i \in \mathcal{U}(k) \setminus \{1\} \text{ such that 1 appears at the } j\text{th place in } B_k[i]\}.$$

We have

Lemma 2.3. *For each $k = 0, 1, 2, ...,$ $\mathcal{B}(k) = \{1, 2, ..., L_k\}$.*

Proof. For $k = 0$ the above equality is obvious. Suppose now that for a certain $k \ge 0$,

$$\mathcal{B}(k) = \{1, 2, ..., L_k\}. \tag{2.3}$$

Let $n \in \mathbb{N}$ with $1 \le n \le L_{k+1}$ be given and set $n = n' + pL_k$, where $1 \le n' \le L_k$. By (2.3), $n' \in \mathcal{B}(k)$. Hence 1 appears at the n'th place in some $B[i]$, $i \in \mathcal{U}(k) \setminus \{1\}$. This $B[i]$ appears at some place, say r, in B_{k+1}. Now, let us move back in B_{k+1} by pL_k positions (possibly in the cyclic way); i.e., let us consider the position $q = r - pL_k$ when $r - pL_k \ge 1$, or $q = r - pL_k + L_{k+1}$ otherwise. As a consequence of the definition, in B_{k+1} at the position q there appears either some $B[j], j \in \mathcal{U}(k)$ or E_k. In any case, we see that 1 appears in B_{k+1} at the place q. Moreover, by the construction of q and the definition of $B_k[i]$, we see that 1 appears at the place n in $B_{k+1}[q]$. Since by the definition of B_{k+1} there are two such r's, at least one of the corresponding q's is different from 1; hence $n \in \mathcal{B}(k+1)$. Induction gives us the lemma. □

Proof of Theorem 2.2. Because of Remark 1.4 it is enough to show that $\mathcal{AS}_0(X^P)$ is orbitally dense. Let C, D be arbitrary blocks that appear in w^P and let $k \in \mathbb{N}$ be such that both C and D appear in A_k at the places r_C and r_D, respectively. Then for arbitrary $m > k$ we have

$$1 < L_m - r_C \leq L_m;$$

hence $L_m - r_C \in \mathcal{B}(m)$. This means that we can build $A_{m+1}[j_1]$ from A_{m+1} with j_1 of the form $pL_m + (L_m - r_C)$, which forces that $A_{m+1}[j_1] = C'A_{m+1}...$, where $|C'| = r_C$. Then in A_{m+2} we see a block of the form $A_{m+1}A_{m+1}[j_1]$, which in the middle has the block $A_m C' A_m$ (A_{m+1} begins and ends with A_m). The first A_m in this concatenation ends with A_k; hence this concatenation ends with $CC''A_m$, where $|CC''| = L_k$. It is easy to see that this forces the existence of $u^C \in \overline{\mathrm{Orb}}(w^P)$ satisfying

$$u_i^C = a_i^m \text{ for } i = 1, 2, ..., L_m, \text{ where } A_m = a_1^m a_2^m ... a_{L_m}^m, \tag{2.4}$$

and

$$u_{-L_k+i}^C = c_i \text{ for } i = 1, 2, ..., |C|, \text{ where } C = c_1 c_2 ... c_{|C|}\ . \tag{2.5}$$

We can repeat this argument for the block D to obtain a sequence u^D satisfying (2.4) and (2.5) with D instead of C. By (2.4), $\sigma u^C,\ \sigma u^D \in \mathcal{AS}_0(X^P)$, and by (2.5), C appears at the 0th place in $\sigma^{L_k} u^C$ and D appears at the 0th place in $\sigma^{L_k} u^D$. This means that $\mathcal{AS}_0(X^P)$ is orbitally dense in the Cartesian square $O_\sigma(w^P) \times O_\sigma(w^P)$. Thus Remark 1.4 gives us the conclusion. □

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