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NEW INTEGRAL IDENTITIES FOR ORTHOGONAL POLYNOMIALS ON THE REAL LINE

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ABSTRACT. Let μ be a positive measure on the real line, with associated orthogonal polynomials $\{p_n\}$ and leading coefficients $\{\gamma_n\}$. Let $h \in L_1(\mathbb{R})$. We prove that for $n \geq 1$ and all polynomials P of degree $\leq 2n-2$,

$$\int_{-\infty}^{\infty} \frac{P(t)}{p_n^2(t)} h\left(\frac{p_{n-1}}{p_n}(t)\right) dt = \frac{\gamma_{n-1}}{\gamma_n} \left(\int_{-\infty}^{\infty} h(t)\, dt\right) \left(\int P(t)\ d\mu(t)\right).$$

As a consequence, we establish weak convergence of the measures on the left-hand side.

1. Introduction

Let μ be a positive measure on the real line with infinitely many points in its support, and let $\int x^j d\mu(x)$ be finite for $j = 0, 1, 2, \ldots$. Then we may define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + \cdots, \quad \gamma_n > 0,$$

satisfying

$$\int_{-\infty}^{\infty} p_n p_m d\mu = \delta_{mn}.$$

Let

$$L_n(x, y) = \frac{\gamma_{n-1}}{\gamma_n} \left(p_n(x) p_{n-1}(y) - p_{n-1}(x) p_n(y)\right) \tag{1.1}$$

and for non-real a,

$$E_{n,a}(z) = \sqrt{\frac{2\pi}{|L_n(a, \bar{a})|}} L_n(\bar{a}, z). \tag{1.2}$$

In a recent paper [6], we used the theory of de Branges spaces [1] to show that for Im $a > 0$, and all polynomials P of degree $\leq 2n-2$, we have

$$\int_{-\infty}^{\infty} \frac{P(t)}{|E_{n,a}(t)|^2} dt = \int P(t)\ d\mu(t). \tag{1.3}$$

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This may be regarded as an analogue of Geronimus' formula for the unit circle, where instead of $E_{n,a}$, we have a multiple of the orthonormal polynomial on the unit circle in the denominator [3, Thm. V.2.2, p. 198], [8, pp. 95, 955]. There is an earlier real line analogue, due to Barry Simon [9, Theorem 2.1, p. 5], namely

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{P(t)}{\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2 p_n^2(t)+p_{n-1}^2(t)}dt=\int P(t)\ d\mu(t).$$

Simon calls this a real line orthogonal polynomial analogue of *Carmona's formula* and refers also to earlier work of Krutikov and Remling [5] and Carmona [2]. The latter is the special case of (1.3) with $(p_{n-1}/p_n)(\bar{a}) = \pm i\gamma_{n-1}/\gamma_n$. In a subsequent paper, we gave a self-contained proof of (1.3), and deduced results on weak convergence, discrepancy, and Gauss quadrature.

In this paper, we first establish the following alternative form of (1.3):

Proposition 1.1. *Let μ be a positive measure on the real line with infinitely many points in its support, and with $\int x^j d\mu(x)$ finite for $j = 0, 1, 2, \ldots$. Let $z \in \mathbb{C}\backslash\mathbb{R}$. Then for all polynomials P of degree $\leq 2n-2$,*

$$\frac{1}{\pi}\left|\operatorname{Im} z\right|\int_{-\infty}^{\infty}\frac{P(t)}{\left|zp_n(t)-p_{n-1}(t)\right|^2}dt=\frac{\gamma_{n-1}}{\gamma_n}\int P(t)\,d\mu(t) \tag{1.4}$$

and

$$\frac{1}{\pi}\left|\operatorname{Im} z\right|\int_{-\infty}^{\infty}\frac{P(t)}{\left|p_n(t)-zp_{n-1}(t)\right|^2}dt=\frac{\gamma_{n-1}}{\gamma_n}\int P(t)\,d\mu(t). \tag{1.5}$$

The factor involving z inside the integral above is essentially the Poisson kernel for the upper-half plane. By using limiting properties of Poisson integrals, we deduce our main result, a new integral identity for orthogonal polynomials:

Theorem 1.2. *Let μ be a positive measure on the real line with infinitely many points in its support, and with $\int x^j d\mu(x)$ finite for $j = 0, 1, 2, \ldots$. Let $\{p_n\}$ and $\{\gamma_n\}$ denote, respectively, the orthogonal polynomials, and leading coefficients corresponding to μ. Let $h \in L_1(\mathbb{R})$. Then for all polynomials P of degree $\leq 2n-2$,*

$$\int_{-\infty}^{\infty}\frac{P(t)}{p_n(t)^2}h\left(\frac{p_{n-1}(t)}{p_n(t)}\right)dt=\frac{\gamma_{n-1}}{\gamma_n}\left(\int_{-\infty}^{\infty}h(t)\,dt\right)\left(\int P(t)\,d\mu(t)\right) \tag{1.6}$$

and

$$\int_{-\infty}^{\infty}\frac{P(t)}{p_{n-1}(t)^2}h\left(\frac{p_n(t)}{p_{n-1}(t)}\right)dt=\frac{\gamma_{n-1}}{\gamma_n}\left(\int_{-\infty}^{\infty}h(t)\,dt\right)\left(\int P(t)\,d\mu(t)\right). \tag{1.7}$$

Note that if we choose $P = p_{n-1}^2$ in (1.7), we obtain, if the denominator integral is not 0,

$$\frac{\gamma_{n-1}}{\gamma_n}=\frac{\int_{-\infty}^{\infty}h\left(\frac{p_n(t)}{p_{n-1}(t)}\right)dt}{\int_{-\infty}^{\infty}h(t)\,dt}.$$

It might be possible to derive this special case in an alternative way, i.e., from the partial fraction expansion of $\frac{p_{n-1}}{p_n}(x)$ and known formulae for the distribution function, meas $\left\{x : \frac{p_{n-1}}{p_n}(x) > t\right\}$. We may replace $h(t)\,dt$ in (1.6) and (1.7) by a signed measure $d\nu(t)$ of finite total mass, provided one appropriately defines

$d\nu\left(\frac{p_n(t)}{p_{n-1}(t)}\right)$ over each interval in which $\frac{p_n(t)}{p_{n-1}(t)}$ is monotone. If we choose $h(x) = \frac{\log x^{-2}}{1-x^2}$, in Theorem 1.2, we obtain an entropy-type integral:

Corollary 1.3. *With the notation of Theorem* 1.2*,*

$$\frac{2}{\pi^2}\int_{-\infty}^{\infty} P(t)\,\frac{\ln|p_{n-1}(t)| - \ln|p_n(t)|}{p_{n-1}(t)^2 - p_n(t)^2}dt = \frac{\gamma_{n-1}}{\gamma_n}\int P(t)\,d\mu(t)\,. \tag{1.8}$$

We also obtain a weak convergence-type result: recall that μ is said to be *determinate* if the moment problem

$$\int x^j d\nu(x) = \int x^j d\mu(x)\,, \quad j = 0, 1, 2, \ldots,$$

has the unique solution $\nu = \mu$ from the class of positive measures. We also say that a function f has *polynomial growth at* ∞ if for some $L > 0$ and for large enough $|x|$,

$$|f(x)| \leq |x|^L\,.$$

Theorem 1.4. *Assume the hypotheses of Theorem* 1.2*, and in addition assume that* μ *is determinate. Then for all functions* $f : \mathbb{R} \to \mathbb{R}$ *having polynomial growth at* ∞*, and such that they are Riemann-Stieltjes integrable with respect to* μ*, we have*

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{-1}\int_{-\infty}^{\infty}\frac{f(t)}{p_n(t)^2}h\left(\frac{p_{n-1}(t)}{p_n(t)}\right)dt \\ = \left(\int_{-\infty}^{\infty} h(t)\,dt\right)\left(\int f(t)\;\; d\mu(t)\right)\end{aligned} \tag{1.9}$$

and

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{-1}\int_{-\infty}^{\infty}\frac{f(t)}{p_{n-1}(t)^2}h\left(\frac{p_n(t)}{p_{n-1}(t)}\right)dt \\ = \left(\int_{-\infty}^{\infty} h(t)\,dt\right)\left(\int f(t)\;\; d\mu(t)\right).\end{aligned} \tag{1.10}$$

Of course, if f is continuous on the real line, it will be locally Riemann-Stieltjes integrable with respect to μ. Simon [9] proved weak convergence involving his Carmona-type formula.

2. Proof of the results

Proof of Proposition 1.1. Fix $z \in \mathbb{C}\backslash\mathbb{R}$. Choose $a \in \mathbb{C}$ such that

$$p_{n-1}(\bar{a}) = z p_n(\bar{a})\,.$$

There are n choices for a, counting multiplicity. Then from (1.1), we see that

$$L_n(\bar{a}, t) = -\frac{\gamma_{n-1}}{\gamma_n}p_n(\bar{a})\,(z p_n(t) - p_{n-1}(t))$$

and

$$L_n(a, \bar{a}) = 2i\frac{\gamma_{n-1}}{\gamma_n}\text{Im }(z)\,|p_n(a)|^2\,.$$

Hence

$$\begin{aligned} |E_{n,a}(t)|^2 &= \frac{2\pi}{|L_n(a,\bar{a})|} |L_n(\bar{a},t)|^2 \\ &= \frac{\pi}{|\mathrm{Im}\ z|} \frac{\gamma_{n-1}}{\gamma_n} |zp_n(t) - p_{n-1}(t)|^2 . \end{aligned}$$

Substituting into (1.3) gives (1.4), while replacing z by $\frac{1}{z}$ in (1.4) gives (1.5). □

Proof of (1.6) *of Theorem* 1.2.

Step 1: A Poisson integral identity. Let $z = x + iy$, where $y > 0$. We can recast (1.4) as

$$\int_{-\infty}^{\infty} P(t) \frac{1}{\pi} \frac{y}{(p_n(t)x - p_{n-1}(t))^2 + y^2 p_n^2(t)} dt = \frac{\gamma_{n-1}}{\gamma_n} \int P(t)\ d\mu(t) . \tag{2.1}$$

Let $h \in L_1(\mathbb{R})$. We multiply (2.1) by $h(x)$, integrate over the real line, and interchange integrals, obtaining

$$\begin{aligned} &\int_{-\infty}^{\infty} P(t) \left[\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y h(x)}{(p_n(t)x - p_{n-1}(t))^2 + y^2 p_n^2(t)} dx \right] dt \\ &= \frac{\gamma_{n-1}}{\gamma_n} \left(\int_{-\infty}^{\infty} h(t)\, dt \right) \left(\int P(t)\ d\mu(t) \right). \end{aligned} \tag{2.2}$$

This is justified if the integral on the left converges absolutely, namely,

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \frac{|P(t)|\,|h(x)|}{(p_n(t)x - p_{n-1}(t))^2 + y^2 p_n^2(t)} dx \right] dt < \infty. \tag{2.3}$$

To prove this, choose A such that all zeros of p_n lie in $(-A, A)$. Let

$$c = \inf_{t,x\in\mathbb{R}} \left[(p_n(t)x - p_{n-1}(t))^2 + y^2 p_n^2(t) \right].$$

This is positive, as p_{n-1} and p_n do not have common zeros. Then we can bound the left-hand side in (2.3) above by

$$\int_{|t|\geq A} \frac{|P(t)|}{y^2 p_n^2(t)} \left(\int_{-\infty}^{\infty} |h(x)|\, dx \right) dt + \int_{|t|\leq A} |P(t)| \left(\int_{-\infty}^{\infty} |h(x)|\, dx \right) dt/c < \infty.$$

Thus (2.3) is valid. Recall that if $h \in L_1(\mathbb{R})$, its Poisson integral for the upper-half plane is

$$\mathcal{P}[h](\alpha + i\beta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\beta}{(x-\alpha)^2 + \beta^2} h(x)\, dx.$$

We can recast (2.2) as

$$\int_{-\infty}^{\infty} \frac{P(t)}{p_n^2(t)} \mathcal{P}[h] \left(\frac{p_{n-1}(t)}{p_n(t)} + iy \right) dt = \frac{\gamma_{n-1}}{\gamma_n} \left(\int_{-\infty}^{\infty} h(t)\, dt \right) \left(\int P(t)\, d\mu(t) \right). \tag{2.4}$$

Step 2: The case where h is bounded and has compact support. Firstly, as h is bounded, we have the elementary bound

$$\left| \mathcal{P}[h] \left(\frac{p_{n-1}(t)}{p_n(t)} + iy \right) \right| \leq \|h\|_{L_\infty(\mathbb{R})} ,$$

valid for all y and t. Next, if $\frac{p_{n-1}(t)}{p_n(t)}$ is a Lebesgue point of h, we have the classic result

$$\lim_{y\to 0+} \mathcal{P}[h]\left(\frac{p_{n-1}(t)}{p_n(t)} + iy\right) = h\left(\frac{p_{n-1}(t)}{p_n(t)}\right). \tag{2.5}$$

Now, if u is not a Lebesgue point of h (and such points have measure 0), the equation $\frac{p_{n-1}(t)}{p_n(t)} = u$ has at most n solutions for t, and locally these vary differentiably with u. It follows that (2.5) holds for a.e. t.

Let $\varepsilon > 0$ and $\mathcal{E}_\varepsilon$ denote the union of n closed intervals of radius ε, centered on the zeros of p_n. Since $P(t)/p_n^2(t) = O\left(t^{-2}\right)$ at ∞, we may use Lebesgue's Dominated Convergence Theorem to deduce that

$$\begin{aligned}\lim_{y\to 0+} &\int_{\mathbb{R}\setminus\mathcal{E}_\varepsilon} \frac{P(t)}{p_n^2(t)} \mathcal{P}[h]\left(\frac{p_{n-1}(t)}{p_n(t)} + iy\right) dt \\ &= \int_{\mathbb{R}\setminus\mathcal{E}_\varepsilon} \frac{P(t)}{p_n^2(t)} h\left(\frac{p_{n-1}(t)}{p_n(t)}\right) dt.\end{aligned} \tag{2.6}$$

It remains to estimate

$$I_{\varepsilon,y} = \int_{\mathcal{E}_\varepsilon} \frac{P(t)}{p_n^2(t)} \mathcal{P}[h]\left(\frac{p_{n-1}(t)}{p_n(t)} + iy\right) dt$$

and

$$I_{\varepsilon,0} = \int_{\mathcal{E}_\varepsilon} \frac{P(t)}{p_n^2(t)} h\left(\frac{p_{n-1}(t)}{p_n(t)}\right) dt.$$

As p_{n-1} and p_n have no common zeros, if $\varepsilon > 0$ is small enough,

$$\inf_{\mathcal{E}_\varepsilon} |p_{n-1}| > 0.$$

Moreover, as h has compact support, we may choose $\varepsilon > 0$ so small that for x in the support of h and $t \in \mathcal{E}_\varepsilon$, we have

$$|p_n(t)x - p_{n-1}(t)| \geq \frac{1}{2}|p_{n-1}(t)|.$$

Then for $0 < y \leq 1$

$$\begin{aligned}|I_{\varepsilon,y}| &= \left|\frac{y}{\pi}\int_{\mathcal{E}_\varepsilon}\left[\int_{-\infty}^{\infty} \frac{P(t)h(x)}{(p_n(t)x - p_{n-1}(t))^2 + y^2 p_n^2(t)} dx\right] dt\right| \\ &\leq \frac{1}{\pi}\int_{\mathcal{E}_\varepsilon}\left[\int_{-\infty}^{\infty} \frac{|P(t)||h(x)|}{\left(\frac{1}{2}|p_{n-1}(t)|\right)^2} dx\right] dt \\ &\leq \frac{4}{\pi}\sup_{t\in\mathcal{E}_\varepsilon}\left|\frac{P(t)}{p_{n-1}^2(t)}\right| \left(\int_{-\infty}^{\infty}|h(x)|dx\right)\int_{\mathcal{E}_\varepsilon} 1\ dt.\end{aligned}$$

This is a bound independent of y and decreases to 0 as ε decreases to 0. Finally, if $\varepsilon > 0$ is small enough, $h\left(\frac{p_{n-1}(t)}{p_n(t)}\right) = 0$ for $t \in \mathcal{E}_\varepsilon$ (recall that h has compact support), so for such an ε,

$$I_{\varepsilon,0} = 0.$$

Combining the above, we obtain

$$\lim_{y\to 0+}\int_{-\infty}^{\infty} \frac{P(t)}{p_n^2(t)} \mathcal{P}[h]\left(\frac{p_{n-1}(t)}{p_n(t)} + iy\right) dt = \int_{-\infty}^{\infty} \frac{P(t)}{p_n^2(t)} h\left(\frac{p_{n-1}(t)}{p_n(t)}\right) dt, \tag{2.7}$$

and hence, from (2.4),

$$\int_{-\infty}^{\infty} \frac{P(t)}{p_n^2(t)} h\left(\frac{p_{n-1}(t)}{p_n(t)}\right) dt = \frac{\gamma_{n-1}}{\gamma_n} \left(\int_{-\infty}^{\infty} h(t)\, dt\right) \left(\int P(t)\ d\mu(t)\right). \tag{2.8}$$

Thus we have (1.6) for the case where h is bounded and has compact support.

Step 3: The case where h is bounded but has non-compact support. Let

$$h_m = h\chi_{[-m,m]}, \quad m \geq 1.$$

We have (1.6) for h_m; that is,

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{P(t)}{p_n(t)^2} h_m\left(\frac{p_{n-1}(t)}{p_n(t)}\right) dt = \frac{\gamma_{n-1}}{\gamma_n} \left(\int_{-\infty}^{\infty} h_m\right) \int P d\mu. \tag{2.9}$$

Now for each t with $p_n(t) \neq 0$ and all large enough m,

$$h_m\left(\frac{p_{n-1}(t)}{p_n(t)}\right) = h\left(\frac{p_{n-1}(t)}{p_n(t)}\right).$$

Next,

$$\left|\frac{P(t)}{p_n(t)^2} h_m\left(\frac{p_{n-1}(t)}{p_n(t)}\right)\right| \leq \left|\frac{P(t)}{p_n(t)^2} h\left(\frac{p_{n-1}(t)}{p_n(t)}\right)\right|.$$

This upper bound is independent of m and moreover is integrable over $(-\infty, \infty)$, since it is $O\left(t^{-2}\right)$ at ∞ and has an integrable singularity at each zero of p_n. To see the latter, we proceed as follows. Let x_{jn} be a zero of p_n. We can write, in $(x_{jn}, x_{jn} + \varepsilon]$, with small enough $\varepsilon > 0$,

$$\frac{p_{n-1}(t)}{p_n(t)} = \frac{g(t)}{t - x_{jn}},$$

where g is non-vanishing and continuously differentiable. If $\varepsilon > 0$ is small enough, we have for some appropriate constant C and $t \in (x_{jn}, x_{jn} + \varepsilon]$:

$$\begin{aligned}
\left|\frac{P(t)}{p_n(t)^2} h\left(\frac{p_{n-1}(t)}{p_n(t)}\right)\right| &\leq C \frac{1}{(t - x_{jn})^2} \left|h\left(\frac{g(t)}{t - x_{jn}}\right)\right| \\
&\leq C \left|\frac{g'(t)(t - x_{jn}) - g(t)}{(t - x_{jn})^2}\right| \left|h\left(\frac{g(t)}{t - x_{jn}}\right)\right| \\
&= C \left|\frac{d}{dt}\left(\frac{g(t)}{t - x_{jn}}\right)\right| \left|h\left(\frac{g(t)}{t - x_{jn}}\right)\right|.
\end{aligned}$$

In the second to last line, we use the fact that if ε is small enough, $|g(t)| \gg |g'(t)(t - x_{jn})|$, while $|g|$ is bounded below. Then, if $g(x_{jn}) > 0$, the substitution $s = \frac{g(t)}{t - x_{jn}}$ gives

$$\begin{aligned}
\int_{x_{jn}}^{x_{jn}+\varepsilon} \left|\frac{P(t)}{p_n(t)^2} h\left(\frac{p_{n-1}(t)}{p_n(t)}\right)\right| dt &\leq C \int_{x_{jn}}^{x_{jn}+\varepsilon} \left|h\left(\frac{g(t)}{t - x_{jn}}\right)\right| \left|\frac{d}{dt}\left(\frac{g(t)}{t - x_{jn}}\right)\right| dt \\
&= C \int_{\frac{g(x_{jn}+\varepsilon)}{\varepsilon}}^{\infty} |h(s)|\, ds \leq C \int_{-\infty}^{\infty} |h(s)|\, ds.
\end{aligned}$$

If $g(x_{jn}) < 0$, we proceed similarly. Thus, indeed, the function $\left|\frac{P(t)}{p_n(t)^2} h\left(\frac{p_{n-1}(t)}{p_n(t)}\right)\right|$ provides an integrable bound independent of m. Then Lebesgue's Dominated Convergence Theorem allows us to let $m \to \infty$ in (2.9) to obtain (1.6) for the case where h is bounded but has non-compact support.

Step 4: The case where h is unbounded. Let us define

$$H_m(t) = \begin{cases} h(t), & \text{if } |h(t)| \leq m, \\ 0, & \text{otherwise.} \end{cases}$$

We have that (1.6) holds for $h = H_m$. Next, for each t with $p_n(t) \neq 0$, $h\left(\frac{p_{n-1}(t)}{p_n(t)}\right)$ finite, and all large enough m,

$$H_m\left(\frac{p_{n-1}(t)}{p_n(t)}\right) = h\left(\frac{p_{n-1}(t)}{p_n(t)}\right).$$

Moreover, $\left|\frac{P(t)}{p_n(t)^2} H_m\left(\frac{p_{n-1}(t)}{p_n(t)}\right)\right|$ admits the same integrable bound as in Step 3. Then Lebesgue's Dominated Convergence Theorem gives the result. □

Proof of (1.7) *of Theorem* 1.2. For the given h, define a new function $\tilde{h}$ by

$$\tilde{h}(x) = x^{-2} h\left(x^{-1}\right).$$

A substitution shows that also $\tilde{h} \in L_1(\mathbb{R})$ and

$$\frac{1}{p_n^2(t)} \tilde{h}\left(\frac{p_{n-1}(t)}{p_n(t)}\right) = \frac{1}{p_{n-1}^2(t)} h\left(\frac{p_n(t)}{p_{n-1}(t)}\right).$$

So applying (1.6) to $\tilde{h}$ gives (1.7) for h. □

Proof of Corollary 1.3. Choose in (1.6) of Theorem 1.2

$$h(x) = \frac{\log x^{-2}}{1 - x^2},$$

which has $h \in L_1(\mathbb{R})$. Moreover, the fact that h is even and a substitution show that [4, p. 533, 4.231.13]

$$\int_{-\infty}^{\infty} h = 8 \int_0^1 \frac{\log x^{-1}}{1 - x^2} dx = \pi^2. \qquad \square$$

Proof of Theorem 1.4. We may prove the result for non-negative h, because every h satisfying the hypotheses of Theorem 1.2 is the difference of two non-negative functions satisfying the same hypotheses. Let f be Riemann-Stieltjes integrable with respect to μ and of polynomial growth at ∞, and let $\varepsilon > 0$. Since μ is determinate, there exist upper and lower polynomials P_u and P_ℓ such that

$$P_\ell \leq f \leq P_u \quad \text{in } (-\infty, \infty)$$

and

$$\int (P_u - P_\ell) \, d\mu < \varepsilon.$$

See, for example, [3, Theorem 3.3, p. 73]. Then for n so large that $2n-2$ exceeds the degree of P_u and P_ℓ, (1.3) gives

$$\begin{aligned}
&\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{-1}\int_{-\infty}^{\infty}\frac{f}{p_{n-1}^2}h\left(\frac{p_n}{p_{n-1}}\right)-\int f d\mu \\
&=\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{-1}\int_{-\infty}^{\infty}\frac{f-P_\ell}{p_{n-1}^2}h\left(\frac{p_n}{p_{n-1}}\right)-\int\left(f-P_\ell\right)d\mu \\
&\leq\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{-1}\int_{-\infty}^{\infty}\frac{P_u-P_\ell}{p_{n-1}^2}h\left(\frac{p_n}{p_{n-1}}\right)-0 \\
&=\int\left(P_u-P_\ell\right)d\mu<\varepsilon.
\end{aligned}$$

Similarly, for large enough n,

$$\begin{aligned}
&\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{-1}\int_{-\infty}^{\infty}\frac{f}{p_{n-1}^2}h\left(\frac{p_n}{p_{n-1}}\right)-\int f d\mu \\
&=\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{-1}\int_{-\infty}^{\infty}\frac{f-P_u}{p_{n-1}^2}h\left(\frac{p_n}{p_{n-1}}\right)-\int\left(f-P_u\right)d\mu \\
&\geq\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{-1}\int_{-\infty}^{\infty}\frac{P_\ell-P_u}{p_{n-1}^2}h\left(\frac{p_n}{p_{n-1}}\right)-0 \\
&=\int\left(P_\ell-P_u\right)d\mu>-\varepsilon. \qquad \square
\end{aligned}$$

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