

## BOUNDED TURNING CIRCLES ARE WEAK-QUASICIRCLES

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ABSTRACT. We show that a metric Jordan curve  $\Gamma$  is *bounded turning* if and only if there exists a *weak-quasisymmetric* homeomorphism  $\varphi: \mathbb{S}^1 \rightarrow \Gamma$ .

### 1. INTRODUCTION

A metric Jordan curve  $\Gamma$  is *bounded turning* (or  $C$ -bounded turning) if there is a constant  $C \geq 1$  such that for each pair of points  $x, y \in \Gamma$ , the arc of smaller diameter  $\Gamma[x, y] \subset \Gamma$  between  $x, y$  satisfies

$$(1.1) \quad \text{diam } \Gamma[x, y] \leq C|x - y|.$$

Here and in the following, we denote metrics by the *Polish notation*, i.e., by  $|x - y|$ . A homeomorphism of metric spaces  $\varphi: X \rightarrow Y$  is called a *weak-quasisymmetry* (or  $H$ -weak-quasisymmetry) if there is a constant  $H \geq 1$  such that

$$(1.2) \quad |x - y| \leq |x - z| \quad \Rightarrow \quad |f(x) - f(y)| \leq H|f(x) - f(z)|,$$

for all  $x, y, z \in X$ . In the present paper, we prove the following theorem.

**Theorem 1.1.** *A metric Jordan curve  $\Gamma$  is bounded turning if and only if there exists a weak-quasisymmetric homeomorphism  $\varphi: \mathbb{S}^1 \rightarrow \Gamma$ .*

The same proof shows the following.

**Corollary 1.2.** *A metric Jordan arc  $A$  is bounded turning if and only if there is a weak-quasisymmetric homeomorphism  $\varphi: [0, 1] \rightarrow A$ .*

**1.1. Background.** The following notion is closely related to weak-quasisymmetry. A homeomorphism  $\varphi: X \rightarrow Y$  of metric spaces is called a *quasisymmetry* if there exists a homeomorphism  $\eta: [0, \infty) \rightarrow [0, \infty)$  such that

$$(1.3) \quad |x - y| \leq t|x - z| \quad \Rightarrow \quad |\varphi(x) - \varphi(y)| \leq \eta(t)|\varphi(x) - \varphi(z)|,$$

for all points  $x, y, z \in X$  and  $t \in [0, \infty)$ . General background on (weak-)quasisymmetries can be found in [Hei01].

Every quasisymmetry is a weak-quasisymmetry (pick  $H = \eta(1)$ ). While the reverse does not hold in general, it is true in many practically relevant situations.

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Recall that a metric space is *doubling* if there is a constant  $N$  such that every ball of radius  $r$  can be covered by at most  $N$  balls of radius  $r/2$ . Note that every Jordan curve  $\Gamma \subset \mathbb{R}^n$  is doubling.

**Theorem 1.3** ([Hei01, Theorem 10.19]). *If  $X$  is connected and both  $X, Y$  are doubling, then every weak-quasisymmetry  $\varphi: X \rightarrow Y$  is quasisymmetric.*

Definition (1.3) for quasisymmetry appears in [TV80]. In earlier work (for example in [AB56], [Ahl63]) quasisymmetry is defined by (1.2); it is however only applied to maps where the two notions agree by the theorem cited above.

A *quasicircle* is the image of the unit circle  $\mathbb{S}^1$  by a quasisymmetric map. Ahlfors has given in [Ahl63] the following geometric characterization for planar quasicircles. For a Jordan curve  $\Gamma \subset \mathbb{C}$  it holds that

$$\Gamma \text{ is a quasicircle} \Leftrightarrow \Gamma \text{ is bounded turning.}$$

Tukia and Väisälä generalize this characterization to all metric Jordan curves in [TV80], namely for a metric Jordan curve  $\Gamma$  it holds that

$$\Gamma \text{ is a quasicircle} \Leftrightarrow \Gamma \text{ is bounded turning and doubling.}$$

If we call the weak-quasisymmetric image of the unit circle  $\mathbb{S}^1$  a *weak-quasicircle*, then Theorem 1.1 may be expressed as follows. For a Jordan curve  $\Gamma$  it holds that

$$\Gamma \text{ is a weak-quasicircle} \Leftrightarrow \Gamma \text{ is bounded turning.}$$

It is easy to see that the quasisymmetric image of a doubling space is doubling (see [Hei01, Theorem 10.18]). Thus one recovers from Theorem 1.1 together with Theorem 1.3 the Tukia-Väisälä characterization of quasicircles.

The first example of a bounded turning circle that is not a quasicircle was given by Tukia-Väisälä in [TV80, Example 4.12]. A simple catalog  $\mathcal{S}$  of bounded turning circles that includes a bi-Lipschitz copy of any bounded turning circle is given in [HM]. A curve  $S \in \mathcal{S}$  from this catalog is doubling, i.e., a quasicircle, if and only if a simple condition is satisfied.

**1.2. Organization of the paper.** The “if”-part of Theorem 1.1 is trivial. Namely, let  $\varphi: \mathbb{S}^1 \rightarrow \Gamma$  be  $H$ -weak-quasisymmetric. Consider arbitrary points  $a, b \in \mathbb{S}^1$ , and let  $[a, b] \subset \mathbb{S}^1 = [0, 1]/\{0 \sim 1\}$  be the arc between  $a$  and  $b$  of smaller diameter. Then for points  $x, y \in [a, b]$  it holds that

$$(1.4) \quad |\varphi(x) - \varphi(y)| \leq |\varphi(x) - \varphi(a)| + |\varphi(a) - \varphi(y)| \leq 2H|\varphi(a) - \varphi(b)|$$

or, assuming  $a \leq x \leq y \leq b$ ,

$$|\varphi(x) - \varphi(y)| \leq H|\varphi(x) - \varphi(b)| \leq H^2|\varphi(a) - \varphi(b)|.$$

Therefore  $\text{diam } \varphi([a, b]) \leq C|\varphi(a) - \varphi(b)|$ , where  $C = \min\{2H, H^2\}$ . Thus  $\Gamma$  is  $C$ -bounded turning.

The rest of this paper concerns the construction of a weak-quasisymmetry  $\varphi: \mathbb{S}^1 \rightarrow \Gamma$  for a given bounded turning circle  $\Gamma$ . In Section 2 we show that we can restrict our attention to the case when  $\Gamma$  is 1-bounded turning. Also, an elementary lemma about dividing arcs into subarcs of equal diameter is proved.

In Section 3 we divide  $\Gamma$  into arcs  $\Gamma_1^n, \dots, \Gamma_{N^n}^n$  (for each  $n \in \mathbb{N}$ ). Two arcs  $\Gamma_i^n, \Gamma_j^n$  have roughly the same diameter. Each arc  $\Gamma_i^{n+1}$  is contained in a (unique) arc  $\Gamma_j^n$ ; thus the sets  $\mathbf{\Gamma}^n = \{\Gamma_j^n \mid j = 1, \dots, N^n\}$  form *subdivisions* of  $\Gamma$ .

In Section 4 we divide the unit circle  $S^1$  into intervals  $I_1^n, \dots, I_{N^n}^n$ . Neighboring intervals  $I_j^n, I_{j+1}^n$  have roughly the same diameter. Furthermore, the combinatorics of the subdivisions of  $\Gamma$  and  $S^1$  is the same, namely  $\Gamma_i^{n+1} \subset \Gamma_j^n \Leftrightarrow I_i^{n+1} \subset I_j^n$ .

The map  $\varphi: S^1 \rightarrow \Gamma$  is defined in Section 5 by mapping endpoints of intervals  $I_j^n$  to endpoints of corresponding arcs  $\Gamma_j^n$ .

Section 6 and Section 7 are preparations to prove the weak-quasisymmetry of  $\varphi$ . Namely, we show that the diameter of any interval in  $S^1$  can be estimated in terms of the subdivision-intervals  $I_j^n$ . Then we show that if  $I_i^n, I_j^m$  are the largest subdivision-intervals contained in adjacent intervals of the same length, then  $|m-n|$  is bounded.

Section 8 finishes the proof of Theorem 1.1.

**1.3. Notation.** The unit circle is denoted by  $S^1$ , which we identify with  $[0, 1]/\{0 \sim 1\}$ . The unit circle is thus equipped with the orientation inherited from the real line. We always assume that  $S^1$  is equipped with the arc-length metric denoted by  $\lambda(s, t)$ ; i.e., if  $0 \leq s \leq t \leq 1$ , then

$$(1.5) \quad \lambda(s, t) = \min\{|t - s|, |s + (1 - t)|\}.$$

The diameter with respect to this metric of an interval  $I \subset S^1 = [0, 1]/\{0 \sim 1\}$  is denoted by  $|I|$ . Note that  $|I|$  equals the Lebesgue measure of  $I$  in the case when  $|I| \leq |S^1 \setminus I|$ .

## 2. PRELIMINARIES

We first show that we can restrict our attention to 1-bounded turning circles. More precisely, we show that any bounded turning circle is bi-Lipschitz equivalent to a 1-bounded turning circle.

Then we prove that any arc can be divided into subarcs of equal diameter.

**2.1. Diameter distance.** Given any metric Jordan curve or Jordan arc  $\Gamma$  we define the *diameter distance* on  $\Gamma$  by

$$(2.1) \quad \text{dd}(x, y) := \text{diam } \Gamma[x, y],$$

for all  $x, y \in \Gamma$ , where  $\Gamma[x, y] \subset \Gamma$  is the arc of smaller diameter between  $x, y$ . We record some properties of  $\text{dd}$ .

**Lemma 2.1.**

- (1)  $\text{dd}$  is a metric on  $\Gamma$ .
- (2)  $\Gamma$  is  $C$ -bounded turning if and only if  $\text{id}: \Gamma \rightarrow (\Gamma, \text{dd})$  is  $C$ -bi-Lipschitz.
- (3) For any arc  $A \subset \Gamma$  it holds that

$$\text{diam}_{\text{dd}} A = \text{diam } A.$$

Here  $\text{diam}_{\text{dd}}$  denotes the diameter with respect to  $\text{dd}$ .

- (4)  $(\Gamma, \text{dd})$  is 1-bounded turning.

*Proof.* (1) is elementary.

To prove (3), first observe that for all  $x, y \in A$ ,  $|x - y| \leq \text{dd}(x, y)$ , so  $\text{diam } A \leq \text{diam}_{\text{dd}} A$ . Next, for all  $x, y \in A$ ,  $\text{dd}(x, y) \leq \text{diam } A$ , so  $\text{diam}_{\text{dd}} A \leq \text{diam } A$ .

In the following,  $\Gamma[x, y] \subset \Gamma$  will always denote the arc of smaller diameter between points  $x, y \in \Gamma$ . Property (4) follows directly from (3), since  $\text{dd}(x, y) = \text{diam } \Gamma[x, y] = \text{diam}_{\text{dd}} \Gamma[x, y]$  for all  $x, y \in \Gamma$ .

It remains to establish (2). If  $\Gamma$  is  $C$ -bounded turning, then for all  $x, y \in \Gamma$ ,

$$\mathbf{dd}(x, y) = \text{diam}(\Gamma[x, y]) \leq C|x - y| \leq C \mathbf{dd}(x, y).$$

Thus the identity map  $\text{id}: \Gamma \rightarrow (\Gamma, \mathbf{dd})$  is  $C$ -bi-Lipschitz. Conversely, if this map is  $C$ -bi-Lipschitz, then for all  $x, y \in \Gamma$ ,

$$\text{diam}(\Gamma[x, y]) = \text{diam}_{\mathbf{dd}}(\Gamma[x, y]) = \mathbf{dd}(x, y) \leq C|x - y|.$$

Therefore  $(\Gamma, |\cdot|)$  is  $C$ -bounded turning. □

It is elementary that postcomposing an  $H$ -weak-quasisymmetry with an  $L$ -bi-Lipschitz map yields an  $HL^2$ -weak-quasisymmetry.

Assume we have constructed for a given bounded turning circle  $\Gamma$  a weak-quasisymmetry  $\varphi: \mathbb{S}^1 \rightarrow (\Gamma, \mathbf{dd})$ . Then the composition  $\mathbb{S}^1 \xrightarrow{\varphi} (\Gamma, \mathbf{dd}) \xrightarrow{\text{id}} \Gamma$  is the desired weak-quasisymmetric parametrization of  $\Gamma$ . Thus to prove Theorem 1.1 it is enough to construct a weak-quasisymmetry  $\varphi: \mathbb{S}^1 \rightarrow \Gamma$  for any 1-bounded turning circle  $\Gamma$ .

**2.2. Dividing arcs.** Here we prove that any metric Jordan arc can be divided into any given number of subarcs, each having exactly the same diameter.

The problem of finding points on a metric Jordan arc such that consecutive points are at the same distance is a non-trivial problem. In 1930 Menger gave a proof [Men30, p. 487]; that is short, simple, and natural, but wrong. It was proved for arcs in Euclidean space in [AB35] and in the general case (indeed in more generality) in [Sch40, Theorem 3]; see also [Väi82].

For the case at hand, i.e., for bounded turning arcs, it suffices to find subarcs that have equal diameter. We give the following elementary proof for this problem.

**Lemma 2.2.** *Let  $A$  be a metric Jordan arc and  $N \geq 2$  an integer. Then we can divide  $A$  into  $N$  subarcs of equal diameter.*

*Proof.* We may assume that  $A$  is the unit interval  $[0, 1]$  equipped with some metric  $d$ . We claim that there are points  $0 = s_0 < s_1 < \dots < s_{N-1} < s_N = 1$  such that

$$\text{diam}[s_0, s_1] = \text{diam}[s_1, s_2] = \dots = \text{diam}[s_{N-1}, s_N],$$

where  $\text{diam}$  denotes diameter with respect to the metric  $d$ . When  $N = 2$  this follows by applying the intermediate value theorem to the function  $[0, 1] \ni s \mapsto \text{diam}[0, s] - \text{diam}[s, 1]$ .

According to Lemma 2.1 (3), we may measure the diameter with respect to the diameter distance. Thus, using Lemma 2.1 (4), we may assume that  $A$  is 1-bounded turning, i.e., that for any  $[s, t] \subset [0, 1]$ ,

$$(2.2) \quad d(s, t) = \text{diam}[s, t].$$

Next we modify  $d$  to get a metric  $d_\epsilon$  that is *strictly increasing* in the sense that

$$(2.3) \quad [s, t] \subsetneq [s', t'] \subset [0, 1] \implies d_\epsilon(s, t) < d_\epsilon(s', t').$$

The crucial point here is the *strict* inequality, which need not hold in general. To this end, fix  $\epsilon > 0$ , and for all  $s, t \in [0, 1]$  set

$$d_\epsilon(s, t) := d(s, t) + \epsilon|t - s|.$$

Then from (2.2) it follows that

$$\text{diam}_\epsilon[s, t] = \text{diam}[s, t] + \epsilon|t - s| = d_\epsilon(s, t),$$

where  $\text{diam}_\epsilon$  denotes diameter with respect to  $d_\epsilon$ . This immediately implies (2.3).

We now show that  $[0, 1]$  can be divided into  $N$  subintervals of equal  $d_\epsilon$ -diameter. Consider the compact set  $S := \{\mathbf{s} = (s_1, \dots, s_{N-1}) \mid 0 \leq s_1 \leq \dots \leq s_{N-1} \leq 1\}$ . Set  $s_0 := 0, s_N := 1$ . The function  $\varphi: S \rightarrow \mathbb{R}$  defined by

$$\varphi(\mathbf{s}) := \max_{0 \leq i \leq N-1} \text{diam}_\epsilon[s_i, s_{i+1}] - \min_{0 \leq j \leq N-1} \text{diam}_\epsilon[s_j, s_{j+1}]$$

assumes a minimum on  $S$ . If this minimum is zero, we are done. Otherwise, there are adjacent intervals  $[s_{i-1}, s_i], [s_i, s_{i+1}]$  that have different  $d_\epsilon$ -diameter. Using the intermediate value theorem as before, we can find  $s'_i \in [s_{i-1}, s_{i+1}]$  such that  $\text{diam}_\epsilon[s_{i-1}, s'_i] = \text{diam}_\epsilon[s'_i, s_{i+1}]$ . Then from (2.3) it follows that

$$\min_{0 \leq j < N} \text{diam}_\epsilon[s_j, s_{j+1}] < \text{diam}_\epsilon[s_{i-1}, s'_i] = \text{diam}_\epsilon[s'_i, s_{i+1}] < \max_{0 \leq i < N} \text{diam}_\epsilon[s_i, s_{i+1}].$$

Applying this procedure to all subintervals of maximal  $d_\epsilon$ -diameter we obtain a strictly smaller minimum for the function  $\varphi$ , which is impossible. Thus the minimum must be zero, and so we can subdivide  $[0, 1]$  into  $N$  subintervals of equal  $d_\epsilon$ -diameter.

Now consider a sequence  $\epsilon_n \searrow 0$ , as  $n \rightarrow \infty$ . Let  $s_1^n < \dots < s_{N-1}^n$  be the points that divide  $[0, 1]$  into  $N$  subintervals of equal diameter with respect to  $d_{\epsilon_n}$ . We can assume that for all  $1 \leq j < N$ , all points  $s_j^n$  converge to  $s_j$  as  $n \rightarrow \infty$ . It follows that for all  $1 \leq i, j < N$ ,

$$\text{diam}[s_i, s_{i+1}] = \lim_{n \rightarrow \infty} \text{diam}_{\epsilon_n}[s_i^n, s_{i+1}^n] = \lim_{n \rightarrow \infty} \text{diam}_{\epsilon_n}[s_j^n, s_{j+1}^n] = \text{diam}[s_j, s_{j+1}],$$

as desired. □

The previous lemma is also true for metric Jordan curves  $\Gamma$ . In this case we are free to choose any point in  $\Gamma$  to be an endpoint of one of the subarcs.

### 3. DIVIDING $\Gamma$

Consider a 1-bounded turning metric Jordan curve  $\Gamma$ . We fix a point  $a_0 \in \Gamma$  and an orientation of  $\Gamma$ .

For each  $n \in \mathbb{N}$  we will divide  $\Gamma$  into arcs  $\Gamma_1^n, \dots, \Gamma_{N^n}^n$ , labeled consecutively on  $\Gamma$ , such that  $a_0$  is the common endpoint of  $\Gamma_1^n, \Gamma_{N^n}^n$ . The set of these arcs is denoted by  $\mathbf{\Gamma}^n$ . Here and in the following the upper index  $n$  will denote the order of the subdivision. In particular  $N^1, N^2, \dots, N^n, \dots$  will be some (increasing) sequence of positive integers, not a geometric sequence.

**Lemma 3.1.** *There are divisions  $\mathbf{\Gamma}^n$  of  $\Gamma$  as above with the following properties:*

- (1)  $\mathbf{\Gamma}^{n+1}$  is a subdivision of  $\mathbf{\Gamma}^n$ . This means that every  $\Gamma^{n+1} \in \mathbf{\Gamma}^{n+1}$  is contained in a (unique)  $\Gamma^n \in \mathbf{\Gamma}^n$ .
- (2) The diameters of the arcs of the  $n$ -th subdivision are comparable; more precisely,

$$\frac{1}{2} \leq \frac{\text{diam } \Gamma}{\text{diam } \Gamma'} \leq 2,$$

for all  $\Gamma, \Gamma' \in \mathbf{\Gamma}^n$ .

- (3) The diameters of the  $n$ -th and the  $(n + 1)$ -th subdivision are comparable; more precisely,

$$\frac{1}{16} \text{diam } \Gamma^n \leq \text{diam } \Gamma^{n+1} \leq \frac{1}{4} \text{diam } \Gamma^n,$$

for all  $\Gamma^{n+1} \in \mathbf{\Gamma}^{n+1}$  and  $\Gamma^n \in \mathbf{\Gamma}^n$ .

The last property implies that each arc  $\Gamma^n \in \mathbf{\Gamma}^n$  is subdivided into at least four arcs  $\Gamma^{n+1} \in \mathbf{\Gamma}^{n+1}$ .

Before we construct these divisions of  $\Gamma$ , i.e., prove the previous lemma, we need some preparation.

**Lemma 3.2.** *Let  $A$  be a 1-bounded turning arc, and let  $0 < \delta \leq \text{diam } A$ . For each  $n$  we divide  $A$  into  $n$  arcs  $A_1, \dots, A_n$  of equal diameter (see Lemma 2.2). Let  $n$  be the smallest integer such that  $\text{diam } A_1 = \text{diam } A_2 = \dots = \text{diam } A_n \leq \delta$ . Then  $\text{diam } A_j \geq \delta/2$  for all  $j = 1, \dots, n$ .*

*Proof.* Let  $n$  be as in the statement. If  $n = 1$ , then  $\delta = \text{diam } A$ , and there is nothing to prove.

Assume now that  $n \geq 2$ . Assume that the statement is false. Then the subarcs of equal diameter  $A_1, \dots, A_n$  have common diameter  $\text{diam } A_j < \delta/2$ .

*Claim.* Suppose  $A$  is subdivided into  $k$  subarcs  $A'_1, \dots, A'_k$  of equal diameter greater than  $\delta$ . Then  $2k + 1 \leq n$ .

Assuming the  $A_i$  and the  $A'_j$  are ordered in the same order along  $A$ , we see that one needs at least  $A_1, A_2, A_3$  to cover  $A'_1$ . Similarly, at least the first five arcs  $A_1, \dots, A_5$  are needed to cover  $A'_1 \cup A'_2$ . Inducting over the arcs  $A'_1, \dots, A'_k$  proves the claim.

We obtain a contradiction when we set  $k = n - 1$ . □

*Proof of Lemma 3.1.* We start by dividing  $\Gamma$  into arcs  $\Gamma^1_1, \dots, \Gamma^1_{N^1}$  of equal diameter, such that  $\text{diam } \Gamma/8 \leq \text{diam } \Gamma^1_j \leq \text{diam } \Gamma/4$  for all  $j = 1, \dots, N^1$  using Lemma 2.2 and Lemma 3.2, for some  $N^1 \in \mathbb{N}$ . Here  $a_0$  is the common endpoint of  $\Gamma^1_1$  and  $\Gamma^1_{N^1}$ .

Assume  $\Gamma$  has been divided into arcs  $\Gamma^n_1, \dots, \Gamma^n_{N^n}$  satisfying Lemma 3.1. In particular,  $1/2 \leq \text{diam } \Gamma^n_i / \text{diam } \Gamma^n_j \leq 2$  for all  $i, j \in \{1, \dots, N^n\}$ . Set  $\delta = \frac{1}{4} \min_j \text{diam } \Gamma^n_j$ . Using Lemma 2.2 and Lemma 3.2 we divide each arc  $\Gamma^n = \Gamma^n_i$  into arcs  $\Gamma^{n+1}_1, \dots, \Gamma^{n+1}_N$  (here  $\Gamma^{n+1}_j = \Gamma^{n+1}_{i,j}$  and  $N = N^n_i$ ) of equal diameter such that

$$\delta/2 \leq \text{diam } \Gamma^{n+1}_1 = \dots = \text{diam } \Gamma^{n+1}_N \leq \delta.$$

Let  $\Gamma^{n+1}_1, \dots, \Gamma^{n+1}_{N^{n+1}}$  be the set of all these arcs, labeled along  $\Gamma$ , such that  $a_0$  is the common point of  $\Gamma^{n+1}_1, \Gamma^{n+1}_{N^{n+1}}$ . It is clear that these arcs satisfy the properties of Lemma 3.1.

Thus the arcs  $\Gamma^n_1, \dots, \Gamma^n_{N^n}$  have been constructed for all  $n$ . □

#### 4. DIVIDING THE UNIT CIRCLE

For each  $n \in \mathbb{N}$  we divide the unit circle  $S^1 = [0, 1]/\{0 \sim 1\}$  into intervals  $I^n_1, \dots, I^n_{N^n}$ , labeled consecutively on  $S^1$ . The common endpoint of  $I^n_1$  and  $I^n_{N^n}$  is 0. The set of these intervals is denoted by  $\mathbf{I}^n$ .

**Lemma 4.1.** *There are divisions  $\mathbf{I}^n$  of the unit circle  $S^1$  as above satisfying the following:*

- (1)  $\mathbf{I}^{n+1}$  is a subdivision of  $\mathbf{I}^n$ . This means that every  $I^{n+1} \in \mathbf{I}^{n+1}$  is contained in a (unique) interval  $I^n \in \mathbf{I}^n$ .

Two adjacent intervals  $I, I' \in \mathbf{I}^n$  are called neighbors (i.e.,  $I = I^n_j, I' = I^n_{j+1}$ ). Note that neighbors are always elements of the same subdivision  $\mathbf{I}^n$ .

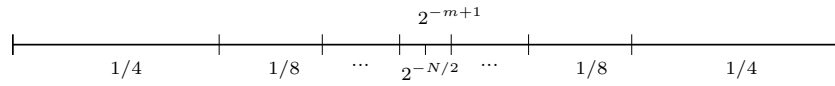


FIGURE 1. Subdividing an interval.

- (2) The diameter of neighboring intervals are comparable. More precisely, they agree or differ by the factor 2,

$$|I|/|I'| \in \{1/2, 1, 2\},$$

for all neighbors  $I, I'$ .

- (3) If  $I_i^{n+1} \subset I_j^n$ , then  $|I_i^{n+1}| \leq |I_j^n|/4$  for all  $i = 1, \dots, N^{n+1}$ ,  $j = 1, \dots, N^n$ .

- (4) The subdivisions  $\mathbf{I}^n$  have the same combinatorics as the subdivisions  $\mathbf{\Gamma}^n$ . Namely,

$$I_i^{n+1} \subset I_j^n \Leftrightarrow \Gamma_i^{n+1} \subset \Gamma_j^n$$

for all  $i = 1, \dots, N^{n+1}$ ,  $j = 1, \dots, N^n$ .

*Proof.* Let  $I = I_i^n$  be given. Assume that the corresponding arc  $\Gamma^n = \Gamma_i^n$  is divided into  $N = N_i^n$  arcs  $\Gamma_j^{n+1}$ . Note that by construction  $N_i^n \geq 4$ .

Let  $c$  be the *midpoint* of the interval  $I$  (i.e.,  $c = \frac{1}{2}(a + b)$  if  $I = [a, b]$ ). It divides  $I$  into the *left* and *right half* of  $I$ .

To simplify the discussion we assume that  $|I| = 1$ . For the general case, if we write in the following “length of a subinterval is  $1/4$ ”, it has to be replaced by “length of a subinterval is  $1/4 \cdot |I|$ ”, and so on.

*Case 1.*  $N$  is even.

Starting from the left endpoint of  $I$ , we divide the left half of  $I$  into intervals of length  $1/4, 1/8, \dots, 2^{-N/2}$  (times the length of  $I$ ). There is one remaining interval of length  $2^{-N/2}$ , which is the last interval of the left half of  $I$ . The right half of the interval is divided in a symmetric fashion, meaning that starting from the right endpoint we divide the right half into intervals of length  $1/4, 1/8, \dots, 2^{-N/2+1}, 2^{-N/2}, 2^{-N/2}$ . See the bottom of Figure 1.

*Case 2.*  $N = 2m - 1$  is odd.

We divide  $I$  into  $N + 1 = 2m$  subintervals as in Case 1. We then take the union of the two middle subintervals, i.e., the two subintervals containing the midpoint  $c$ . Thus  $I$  is divided into  $N$  subintervals of lengths

$$1/4, 1/8, \dots, 2^{-m+1}, 2^{-m}, 2^{-m+1}, 2^{-m}, 2^{-m+1}, \dots, 1/8, 1/4.$$

See the top of Figure 1.

This finishes the division of  $I$ , thus of all  $I_i^n$ , into intervals. Thus all  $I_j^n$  have been constructed for all  $n \in \mathbb{N}$ . It is clear that they satisfy the properties of Lemma 4.1.

In Case 1 there are two subintervals of  $I$  containing the midpoint of  $I$ ; in Case 2 there is a single subinterval of  $I$ . Such a subinterval is called a *middle* subinterval of  $I$ . □

### 5. THE WEAK QUASISYMMETRY

Let  $s_0^n, \dots, s_{N^n-1}^n$  be the endpoints of the intervals  $I_j^n$  ordered increasingly on  $S^1 = [0, 1]/\{0 \sim 1\}$ ,  $s_0^n = 0$  for all  $n \in \mathbb{N}$ . Let  $a_0^n, \dots, a_{N^n-1}^n$  be the endpoints of

the arcs  $\Gamma_j^n$ . Then we define  $\varphi(s_j^n) = a_j^n$ . From Lemma 3.1 (1) and Lemma 4.1 (4) it follows that  $\varphi$  is well defined; i.e., if  $s_i^n = s_j^m$ , then  $\varphi(s_i^n) = a_i^n = a_j^m = \varphi(s_j^m)$ .

We show uniform continuity of  $\varphi$  on the set  $\mathbf{s} = \{s_j^n \mid n \in \mathbb{N}, j = 0, \dots, N^n - 1\}$ . Let  $\delta_n := \min_j |I_j^n|$ . Then if  $\lambda(s, t) \leq \delta_n/2$  for two points  $s, t \in \mathbf{s}$  (recall from (1.5) that  $\lambda$  is the metric on  $S^1$ ), then  $s, t$  are contained in adjacent intervals  $I_j^n, I_{j+1}^n$ . Thus  $\varphi(s), \varphi(t)$  are contained in adjacent arcs  $\Gamma_j^n, \Gamma_{j+1}^n$ . Thus

$$|\varphi(s) - \varphi(t)| \leq \text{diam } \Gamma_j^n + \text{diam } \Gamma_{j+1}^n \leq 2 \cdot 4^{-n} \text{diam } \Gamma$$

by Lemma 3.1 (3), showing uniform continuity of  $\varphi$  on  $\mathbf{s}$ . Since this set is dense in  $S^1$ ,  $\varphi$  extends continuously to  $S^1$ . The surjectivity is clear, since the set  $\{a_j^n \mid n \in \mathbb{N}, j = 0, \dots, N^n - 1\}$  is dense in  $\Gamma$ . Injectivity follows from the fact that disjoint sets  $I_i^n, I_j^n$  are mapped to disjoint arcs  $\Gamma_i^n, \Gamma_j^n$ . Thus  $\varphi: S^1 \rightarrow \Gamma$  is a homeomorphism.

### 6. ESTIMATING INTERVALS

Given an interval  $[x, y] \subset S^1$  we define

$$(6.1) \quad \delta([x, y]) := \max\{|I_j^n| \mid I_j^n \subset [x, y]\}.$$

Here the maximum is taken over  $n \in \mathbb{N}$  and all intervals  $I_j^n \in \mathbf{I}^n$  as defined in Section 4.

**Lemma 6.1.** *Let  $[x, y] \subset S^1$  be any interval. Then*

$$\delta([x, y]) \leq |[x, y]| \leq 12 \delta([x, y]).$$

*Furthermore, if the maximum in equation (6.1) is attained for an interval  $I = I_j^n \in \mathbf{I}^n$ , then there are two intersecting (possibly identical) intervals  $\hat{I}, \hat{J} \in \mathbf{I}^{n-1}$  such that*

$$I \subset [x, y] \subset \hat{I} \cup \hat{J}.$$

*Proof.* Let  $I = I_j^n \subset [x, y]$  be one interval where the maximum from (6.1) is attained, i.e.,  $|I| = \delta([x, y])$ . The left inequality, i.e.,  $|I| = \delta([x, y]) \leq |[x, y]|$ , is obvious. Let  $\hat{I} \supset I$  be the parent of  $I$ , i.e., the unique interval  $\hat{I} \in \mathbf{I}^{n-1}$  containing  $I$ . Assume that  $\hat{I}$  was subdivided into  $N$  intervals  $I^n \in \mathbf{I}^n$ . We consider several cases.

*Case 1.*  $|I| = |\hat{I}|/4$ .

This can happen only in the following instances: either  $I$  is the left- or rightmost interval in  $\hat{I}$  (i.e.,  $I, \hat{I}$  share a boundary point) or  $N$  is equal to 4 or 5 and  $I$  contains the midpoint of  $\hat{I}$ .

If  $[x, y] \subset \hat{I}$ , we are done, since then  $|[x, y]| \leq |\hat{I}| = 4|I| = 4\delta([x, y])$ . We set  $\hat{J} := \hat{I}$ .

Assume that  $[x, y] \not\subset \hat{I}$ . This means that one endpoint of  $\hat{I}$ , without loss of generality the left endpoint, is an interior point of  $[x, y]$ . From the maximality of  $I$  it follows that  $y \in \hat{I}$ . Consider the left neighbor  $\hat{J} \in \mathbf{I}^{n-1}$  of  $\hat{I}$ . Note that  $|\hat{J}| \geq \frac{1}{2}|\hat{I}| = 2|I|$ . Thus  $\hat{J} \not\subset [x, y]$  by the maximality of  $I$ . Thus  $[x, y] \subset \hat{J} \cup \hat{I}$ . It holds that  $|\hat{I}| = 4|I|$  and  $|\hat{J}| \leq 2|\hat{I}| = 8|I|$  so that

$$|[x, y]| \leq 12 \delta([x, y]).$$



Case 2.  $N \geq 6$  is even and  $I = I_j^n$  is a middle subinterval of  $\hat{I}$  (i.e., contains the midpoint of  $I$ ).

Then either both  $I_{j-2}^n, I_{j+3}^n$  or both  $I_{j-3}^n, I_{j+2}^n$  have diameter strictly larger than  $I$ . We can assume without loss of generality the former case. This means that  $I$  is in the left half of  $\hat{I}$  and that

$$I_{j-2}^n \cup I_{j-1}^n \cup I_j^n \cup I_{j+1}^n \cup I_{j+2}^n \cup I_{j+3}^n$$

cover  $[x, y]$ . Note that the total length of these sets is  $8|I|$ . Thus  $\delta([x, y]) \leq |[x, y]| \leq 8\delta([x, y])$ .

Case 3.  $N \geq 7$  is odd and  $I_j^n$  is the middle subinterval of  $\hat{I}$ .

Similar to the preceding case,  $I_{j-3}^n, I_{j+3}^n$  have twice the length of  $I$ ; thus they are not contained in  $[x, y]$  and

$$[x, y] \subset I_{j-3}^n \cup \dots \cup I_{j+3}^n.$$

Note that the total length of these intervals is  $8|I|$ . This finishes the claim in this case.

Case 4. Remaining case.

One of the neighbors of  $I = I_j^n$ , without loss of generality the left neighbor  $I_{j-1}^n$ , has twice the length of  $I$ .

Furthermore, there is a subinterval  $I_{j+k}^n \in \mathbf{I}^n$  of  $\hat{I}$  that has the same length as  $I$ . It is symmetric to  $I$  with respect to the midpoint of  $\hat{I}$ . Then  $I_{j-1}^n, I_{j+k+1}^n$  have twice the length of  $I$ , thus are not contained in  $[x, y]$ . Thus

$$[x, y] \subset I_{j-1}^n \cup I_j^n \cup \dots \cup I_{j+k}^n \cup I_{j+k+1}^n.$$

The total length of the right-hand side is  $8|I|$ , finishing the claim.

Note that in Case 2–Case 4 the subintervals that cover  $[x, y]$  are all contained in the parent  $\hat{I}$ ; we then set  $\hat{J} := \hat{I}$ . □

### 7. ESTIMATING ORDER

Now consider two adjacent intervals (in  $S^1$ ) of the same length, i.e.,  $[x - t, x]$ ,  $[x, x + t]$  for some  $x \in S^1$  and  $0 < t \leq 1/2$ . Consider the largest subdivision intervals contained in  $[x - t, x]$ ,  $[x, x + t]$ , meaning we consider intervals  $J^m \in \mathbf{I}^m, I^n \in \mathbf{I}^n$  such that

$$\begin{aligned} J^m &\subset [x - t, x], & I^n &\subset [x, x + t] \quad \text{and} \\ |J^m| &= \delta([x - t, x]), & |I^n| &= \delta([x, x + t]). \end{aligned}$$

We want to show that  $n, m$  differ by at most a constant  $k_0$  (in fact  $k_0 = 4$ ). Before giving the detailed argument, let us quickly describe the idea. From Lemma 6.1 it follows that  $|I^n|, |J^m|$  are comparable. Without loss of generality, we can assume that  $n \leq m$ . Let  $J^n \in \mathbf{I}^n$  be the (unique)  $n$ -th order subdivision-interval containing  $J^m$ . If  $m - n$  is large, then  $|J^n|$  is large compared to  $|J^m|$ , and thus large compared to  $|I^n|$ . Then  $J^n, I^n$  have to be far apart. This is impossible.

**Lemma 7.1.** *In the setting as above it holds that  $|m - n| \leq 4$ .*

*Proof.* As in the outline given above we assume that  $n \leq m$ , and let  $J^n \in \mathbf{I}^n$  be the subdivision-interval containing  $J^m$ . If  $m - n = k_0$ , then  $|J^m| \leq 4^{-k_0}|J^n|$  by Lemma 4.1 (3).

*Claim.* Consider two intervals  $I, I' \in \mathbf{I}^n$  such that  $|I'|/|I| \geq 2^{i+1}$  for some  $i \geq 1$ . Then  $\text{dist}(I, I') \geq 2^i |I|$ .

This is clear, since the interval between  $I, I'$  has to contain one of size  $2^i |I|$  by Lemma 4.1 (2).

From Lemma 6.1 it follows that  $|[x - t, x + t]| \leq 24|I^n|$ . Thus it follows from the previous claim that  $|J^n|/|I^n| \leq 2^5$ . Indeed  $|J^n|/|I^n| \geq 2^6$  implies by the claim that  $\text{dist}(J^n, I^n) \geq 2^5 |I^n| = 32|I^n|$ , which is impossible. Thus by Lemma 6.1

$$\frac{1}{12}|I^n| \leq \frac{1}{12}|[x, x + t]| = \frac{1}{12}|[x - t, x]| \leq |J^m| \leq 4^{-k_0}|J^n| \leq 4^{-k_0}2^5|I^n|.$$

We obtain a contradiction if we choose  $k_0$  such that  $4^{-k_0}2^5 < 1/12$  or  $k_0 \geq 5$ . This finishes the proof. □

### 8. PROOF OF THE THEOREM

After these preparations, we are ready to prove the main theorem.

*Proof of Theorem 1.1.* Recall from Section 2.1 that it is enough to prove the theorem in the case when  $\Gamma$  is 1-bounded turning. This means that for any two points  $x, y \in \Gamma$ , the arc of smaller diameter  $\Gamma[x, y] \subset \Gamma$  between  $x, y$  satisfies  $\text{diam } \Gamma[x, y] = |x - y|$ .

Thus it is enough to show that the arcs  $\varphi([x - t, x]), \varphi([x, x + t])$  have comparable diameter for all  $x \in \mathbf{S}^1, 0 < t \leq 1/2$ . Let  $I_- \in \mathbf{I}^m, I_+ \in \mathbf{I}^n$  be the largest intervals contained in  $[x - t, x], [x, x + t]$ ; i.e.,

$$\begin{aligned} I_- &\subset [x - t, x], & I_+ &\subset [x, x + t] \quad \text{and} \\ |I_-| &= \delta([x - t, x]), & |I_+| &= \delta([x, x + t]). \end{aligned}$$

Let  $\hat{I}_-, \hat{J}_- \in \mathbf{I}^{m-1}$  be the intervals that cover  $[x - t, x]$  according to Lemma 6.1. Then

$$\begin{aligned} |\varphi(x - t) - \varphi(x)| &= \text{diam } \varphi([x - t, x]) \leq \text{diam } \varphi(\hat{I}_- \cup \hat{J}_-) \\ &\leq 32 \text{diam } \varphi(I_-) \quad \text{by Lemma 3.1 (3)} \\ &\leq 32 \cdot 2 \cdot 16^4 \text{diam } \varphi(I_+) \end{aligned}$$

by Lemma 7.1, Lemma 3.1 (3), and Lemma 3.1 (2),

$$\begin{aligned} &\leq 32 \cdot 2 \cdot 16^4 \text{diam } \varphi([x, x + t]) \\ &= 32 \cdot 2 \cdot 16^4 |\varphi(x) - \varphi(x + t)|. \end{aligned}$$

This finishes the proof. □

### 9. CONCLUDING REMARKS

It is natural to ask how small the involved constants can be chosen. In particular, how small can the constant  $H \geq 1$  of the weak-quasisymmetric parametrization  $\varphi: \mathbf{S}^1 \rightarrow \Gamma$  for a given  $C$ -bounded turning circle be chosen? Recall from (1.4) that the image of the unit circle by an  $H$ -weak-quasisymmetry is  $C$ -bounded turning, where  $C = \min\{2H, H^2\}$ . Thus it is natural to ask if any  $C$ -bounded turning circle admits an  $H$ -weak-quasisymmetric parametrization, where  $H = \max\{C/2, \sqrt{C}\}$ . As a starting point one may ask if any 1-bounded turning circle admits a 1-weak-quasisymmetric parametrization.

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