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EXTENDING THE KNOPS-STUART-TAHERI TECHNIQUE TO C^1 WEAK LOCAL MINIMIZERS IN NONLINEAR ELASTICITY

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Abstract. We prove that any C^1 weak local minimizer of a certain class of elastic stored-energy functionals $I(u) = \int_\Omega f(\nabla u)\,dx$ subject to a linear boundary displacement $u_0(x) = \xi x$ on a star-shaped domain Ω with C^1 boundary is necessarily affine provided f is strictly quasiconvex at ξ. This is done without assuming that the local minimizer satisfies the Euler-Lagrange equations, and therefore extends in a certain sense the results of Knops and Stuart, and those of Taheri, to a class of functionals whose integrands take the value $+\infty$ in an essential way.

1. Introduction

This short paper advances arguments to be found in [22] concerning the relative energies of C^1 weak local minimizers of energy functionals of the form

$$I(u) = \int_\Omega f(\nabla u(x))\,dx. \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^n$ is a star-shaped domain with a C^1 boundary, $u : \Omega \to \mathbb{R}^m$ belongs to an appropriate Sobolev space, and $f : \mathbb{R}^{n\times m} \to \mathbb{R}\cup\{\infty\}$ belongs to a particular class of quasiconvex functions that are sufficiently smooth where finite. Previous works on this topic, most notably [13] and [22], established the uniqueness of sufficiently smooth solutions of the Euler-Lagrange equations associated with the functional (1.1) and subject to a linear boundary displacement. Formally, these are solutions of the system

$$\operatorname{div} Df(\nabla u) = 0, \tag{1.2}$$

where as usual $Df(A)$ is the $m \times n$ matrix whose (i,j) entry is $\frac{\partial f(A)}{\partial A_{ij}}$.

The technique referred to in the title, first used by Knops and Stuart in nonlinear elastostatics [13] and later developed by Taheri in [22], can be distilled into two steps, the ultimate goal of which is to compare two energies $I(u)$ and $I(v)$, say, where u and v agree on $\partial\Omega$ and at least one of them is a stationary point in some appropriate sense. The first step is to write the energies as integrals over the boundary $\partial\Omega$. The second hinges on the observation that if u and v agree on $\partial\Omega$ and are sufficiently smooth, then $\nabla u(x) - \nabla v(x)$ is a matrix of rank one provided

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$x \in \partial\Omega$. Thus one can use rank-one convexity of f to order $\int_{\partial\Omega} f(\nabla u(x))$ and $\int_{\partial\Omega} f(\nabla v(x))$, and hence, by step 1, to order $I(u)$ and $I(v)$. (See (2.1) and (1.5) below for the definition of rank-one convexity and quasiconvexity, respectively.)

In the intervening period the results contained in [13] applying to nonlinear elasticity were rederived by Sivaloganathan [21] using an interesting invariant integral method. Both [21] and [13] rely crucially on the smoothness of the solution to (1.2) to circumvent potential difficulties associated with the so-called stored-energy functions commonly used in nonlinear elasticity theory. In the case $m = n = 3$, for example, the corresponding f are polyconvex and take the form

$$f(A) = g(A, \operatorname{cof} A, \det A), \tag{1.3}$$

where g is convex on $\mathbb{R}^{3\times 3}_+ \times \mathbb{R}^{3\times 3}_+ \times \mathbb{R}_+$, and $f(A) = +\infty$ if $\det A \leq 0$. This class of functions was introduced and subsequently developed by Ball in [1], [2], and studied by others, including but not limited to [20], [6], [7], and [18]. See [3] for an overview.

The results of this paper apply to stored-energy functions for which additional regularity results, such as those of [6], are available. Introduced by Ball in [1], these f take the special form

$$f(A) = F(A) + h(\det A), \tag{1.4}$$

where $h : \mathbb{R}^+ \to \mathbb{R} \cup \{\infty\}$ is convex and satisfies $h(s) = +\infty$ for all $s \leq 0$, and where $F : \mathbb{R}^{n\times n} \to \mathbb{R}$ is C^1, quasiconvex and satisfies for some $q \geq n$ and all $n \times n$ matrices A the inequality

$$c|A|^q \leq F(A) \leq C(1 + |A|^q)$$

with constants $c, C > 0$. We recall that a function $F : \mathbb{R}^{m\times n} \to \mathbb{R}$ is quasiconvex if

$$\int_\Omega f(A + \nabla\varphi)\, dx \geq \int_\Omega f(A)\, dx \tag{1.5}$$

for all $m \times n$ matrices A and all Lipschitz functions φ vanishing on $\partial\Omega$, and strictly quasiconvex if (1.5) holds with strict inequality whenever $\varphi \neq 0$. See [9] for further details.

Taheri's approach [22] applies to C^1 integrands f satisfying a p-growth condition

$$|f(A)| \leq c(1 + |A|^p), \tag{1.6}$$

where $1 \leq p < \infty$, c is a constant and A is any $m\times n$ real matrix. Although condition (1.6) is clearly not satisfied by integrands such as (1.4), [22] nevertheless contains an innovation which can be exploited in the context of stored-energy functions. Taheri observes that the conservation law [13, Proposition 2.1] relied on by Knops and Stuart can be replaced by a weaker conservation law, the so-called energy-momentum equations:

$$\operatorname{div}\left(f(\nabla u)\mathbf{1} - \nabla u^T Df(\nabla u)\right) = 0. \tag{1.7}$$

Here, $f(\nabla u)\mathbf{1} - \nabla u^T Df(\nabla u)$ is Eshelby's energy-momentum tensor; it is classically derived by applying Noether's theorem to the variational symmetry $x \mapsto x + a$, $a \in \mathbb{R}^n$. It is well-known that (1.7) can be derived rigorously not only for weak local minimizers of functionals whose integrands f satisfy (1.6) but also for stored-energy functions such as (1.4). See [4] or [6] for details.

The Euler-Lagrange equation (1.2), however, may not automatically hold for general forms of the stored energy including functions of the form (1.4), even while

(1.7) holds. See [8] for an example; see also [12], [19] and [11]. Indeed, it forms part of the hypotheses of the main results in [13], [21] and [22]. But in this paper we note that the full Euler-Lagrange equations are not needed in order to apply Taheri's argument [22]. In fact, it is sufficient that the weak local minimizer is only a 'subsolution' of the Euler-Lagrange equations in a small neighbourhood of the boundary. This point is clarified in Section 3.2 below, but to give an initial idea let us suppose for now that u is a smooth solution of the Euler-Lagrange equation (1.2). A straightforward approximation argument can be used to check that

$$\int_\Omega Df(\nabla u)\cdot\nabla u\,dx = \int_{\partial\Omega} Df(\nabla u(y))\cdot u(y)\otimes\nu(y)\,d\mathcal{H}^{n-1}(y),$$

where ν is the outward pointing normal to $\partial\Omega$. By 'subsolution' we mean, roughly speaking, that

$$\int_\Omega Df(\nabla u)\cdot\nabla u\,dx \leq \int_{\partial\Omega} Df(\nabla u(y))\cdot u(y)\otimes\nu(y)\,d\mathcal{H}^{n-1}(y). \tag{1.8}$$

We therefore introduce in Section 3 a functional $K(u)$ with the property that $K(u) < \infty$ implies that a suitable version of (1.8) holds. In particular, we do not assume that u is a solution of the Euler-Lagrange system (1.2). $K(u)$ is effectively a limiting measure of the 'twist' of the function u near the boundary of the domain: we return to this point below. To conclude the summary, inequality (1.8) then allows us to compare the bulk energies

$$I(u^{\text{hom}}) \geq I(u),$$

where u^{hom} is the one-homogeneous extension of $u|_{\partial\Omega}$ and u is the C^1 weak local minimizer. For less regular u a weaker statement can be deduced; its limitations can most profitably be viewed in the context of [14].

The paper is organized as follows. In Section 3 we motivate and discuss the functional K referred to above. The main result of Section 3 is Lemma 3.3, yielding an inequality such as (1.8) subsequently used in Section 4 to compare the energies $I(u^{\text{hom}})$ and $I(u)$. The results apply to general boundary data up to the end of Section 4.1; in Section 4.2 the boundary data is assumed to be linear and admissible in the sense outlined in Section 2 below. The paper concludes with a brief discussion of how these methods might be adapted to weak local minimizers that are not necessarily C^1.

2. Notation and preliminaries

We denote the $m\times n$ real matrices by $\mathbb{R}^{m\times n}$, and unless stated otherwise we sum over repeated indices. We denote those $n\times n$ real matrices with positive determinant by $\mathbb{R}^{n\times n}_+$, and the identity matrix by $\mathbf{1}$. Throughout B is the unit ball in $\mathbb{R}^2$, and B_t the ball centred at 0 with radius t. We say that a function $f:\mathbb{R}^{m\times n}\to\mathbb{R}\cup\{\infty\}$ is rank-one convex if

$$f(\lambda\xi_1+(1-\lambda)\xi_2) \leq \lambda f(\xi_1)+(1-\lambda)f(\xi_2) \tag{2.1}$$

for all $\xi_1,\xi_2\in\mathbb{R}^{m\times n}$ such that $\operatorname{rank}(\xi_1-\xi_2)=1$ and all $\lambda\in[0,1]$. When f is everywhere real-valued this condition is implied by quasiconvexity; for extended real-valued f the implication need not hold. See [9, Chapter 5] for a proof of the former, and [5] for an example of the latter.

Other standard notation includes $||\cdot||_{k,p;\Omega}$ for the norm on the Sobolev space $W^{k,p}(\Omega)$, $||\cdot||_{p;\Omega}$ for the norm on $L^p(\Omega)$, and $\rightharpoonup$ to represent weak convergence in both of these spaces. $\mathcal{H}^k$ represents k-dimensional Hausdorff measure. The tensor product of two vectors $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ is written $a \otimes b$; it is the $m \times n$ matrix whose (i,j) entry is $a_i b_j$. The inner product of two matrices $X, Y \in \mathbb{R}^{m\times n}$ is $X \cdot Y = \mathrm{tr}\,(X^T Y)$. This obviously holds for vectors too.

The functional I will henceforth be

$$I(u) = \int_\Omega f(\nabla u)\, dx,$$

where f is defined in (1.4). In addition, we assume that there are constants t_0, $s > 0$, $c_2 > c_1 > 0$ such that

$$c_1 t^{-s-j} \leq (-1)^j \frac{d^j h(t)}{dt^j} \leq c_2 t^{-s-j} \tag{2.2}$$

for $j = 0, 1, 2$ and all $t \in (0, t_0)$. This assumption allows us to apply the results of [6] later in the paper.

Since the set Ω is assumed to be star-shaped with a C^1 boundary we can write

$$\Omega = \{x \in \mathbb{R}^n : \ |x| < d(\theta(x))\},$$

where $\theta(x) = \frac{x}{|x|}$ for nonzero x, and $d : \mathbb{S}^{n-1} \to \mathbb{R}$ is C^1. In this notation the normal $N(\theta(x))$ to $\partial\Omega$ at $x \in \partial\Omega$ is

$$N(\theta(x)) = \frac{1}{\alpha(\theta)} \left(\theta - (\mathbf{1} - \theta \otimes \theta)\frac{\nabla d}{d}\right),$$

where α is chosen so that $|N| = 1$.

Let

$$\mathcal{A}_{u_0} = \{v \in W^{1,n}(\Omega, \mathbb{R}^n) : \ I(v) < \infty, \ \mathrm{tr}\, v = \mathrm{tr}\, u_0\},$$

where $\mathrm{tr}\, u_0$ is the trace of a fixed function for which $I(u_0) < \infty$.

Definition 2.1. We shall say that $u \in \mathcal{A}_{u_0}$ is a weak local minimizer of I in $\mathcal{A}_{u_0}$ if there exists $\delta > 0$ such that any $v \in \mathcal{A}_{u_0}$ satisfying $||v - u||_{1,\infty;\Omega} \leq \delta$ necessarily satisfies $I(v) \geq I(u)$.

3. Weak local minimizers with positive twist near the boundary

It is clear from the definition of the functional I that any admissible function u necessarily satisfies $\det \nabla u > 0$ almost everywhere. Our strategy, by analogy with [22], will be to compare $I(u^{\mathrm{hom}})$ with $I(u)$, where u is a C^1 weak local minimizer of I and u^{hom} is the one-homogeneous extension of the restriction of u to $\partial\Omega$. (See below for details.) In particular, were $\det \nabla u^{\mathrm{hom}} > 0$ to fail on a set of positive Lebesgue measure, then the desired inequality

$$I(u^{\mathrm{hom}}) \geq I(u)$$

would be trivial. Using the functional K described below we are able to restrict attention to those admissible u for which $\det \nabla u^{\mathrm{hom}} > 0$ holds $\mathcal{H}^{n-1}$-almost everywhere on $\partial\Omega$; properties of one-homogeneous functions then imply that $\det \nabla u^{\mathrm{hom}} > 0$ holds $\mathcal{L}^n$-almost everywhere in Ω.

3.1. **One-homogeneous extensions and the functional K.** Let $u \in \mathcal{A}_{u_0}$, let $t \in (0,1]$ and define $u_t(x) = u(t\theta d(\theta))$ for $x \in \Omega$ such that $|x| = td(\theta(x))$. Thus u_t is the restriction of u to the boundary of the set

$$\Omega_t = \{x \in \Omega : \ |x| < td(\theta)\}.$$

We define the one-homogeneous extension u_t^{hom} of u_t by

$$u_t^{\mathrm{hom}}(x) = \frac{|x|}{td(\theta)} u(t\theta d(\theta))$$

for each $x \in \Omega$. Then $\nabla u_t(x)$ exists for almost every $x \in \Omega_t$, and in this case it follows that

$$\nabla u_t^{\mathrm{hom}}(x) = \nabla u(t\theta d(\theta)) + \left(\frac{u(t\theta d(\theta))}{td(\theta)} - \nabla u(t\theta d(\theta))\theta\right) \otimes \alpha N.$$

Hence

$$\det \nabla u^{\mathrm{hom}}(x) = \mathrm{cof}\, \nabla u(\theta d(\theta)) \cdot \left(u(t\theta d(\theta)) \otimes \frac{\alpha N}{td(\theta)}\right). \tag{3.1}$$

Since $\det \nabla_t^{\mathrm{hom}}$ clearly depends only on $\theta(x)$, it follows that $\det \nabla u_t^{\mathrm{hom}} > 0$ $\mathcal{L}^n$-almost everywhere if and only if

$$\mathrm{cof}\, \nabla u(t\theta d(\theta)) \cdot \left(u(t\theta d(\theta)) \otimes \frac{\alpha N}{td(\theta)}\right) > 0 \quad \mathcal{H}^{n-1}\text{-a.e.} \tag{3.2}$$

Remark 3.1. When Ω is the unit ball B in $\mathbb{R}^2$ and when u is sufficiently smooth, condition (3.2) with $t = 1$ is equivalent to the condition that $u^{\mathrm{hom}}(\partial B)$ is the boundary of a star-shaped region. The definition of u^{hom} then implies that $u^{\mathrm{hom}}(B)$ is star-shaped. Alternatively, maps u with $\det \nabla u^{\mathrm{hom}} > 0$ $\mathcal{H}^1$-a.e. may be interpreted as having a 'positive twist' at the boundary ∂B. To see this we appeal to a result of Littlewood [15, Theorem 253]. Indeed, setting

$$w(e^{i\alpha}) = u_1(\cos\alpha, \sin\alpha) + iu_2(\cos\alpha, \sin\alpha),$$

writing $w = R(\alpha)e^{i\Phi(\alpha)}$, and using $N(\theta(x)) = \theta(x) = x$ when $x \in \partial B$, $d(\theta(x)) = 1$ for all $x \in B$, it follows from

$$\mathrm{cof}\, \nabla u(\theta) \cdot (u(\theta) \otimes \theta) = \mathrm{Re}\, (\overline{iw}\partial_\alpha w)$$

that

$$\det \nabla u^{\mathrm{hom}} = R^2 \partial_\alpha \Phi. \tag{3.3}$$

Now, [15, Theorem 253] states that the positivity $\mathcal{H}^1$-a.e. of

$$\mathrm{Re}\left(\frac{zw'(z)}{w(z)}\right)$$

with $z = e^{i\alpha}$ is necessary and sufficient for

$$\{w(e^{i\alpha}) : \alpha \in [0, 2\pi]\}$$

to be star-shaped. A short calculation shows that

$$\mathrm{Re}\left(\frac{zw'(z)}{w(z)}\right) = \partial_\alpha \Phi,$$

which has the same sign as the term $R^2\partial_\alpha\Phi$ appearing in (3.3). Therefore (3.2) holds if and only if $u^{\mathrm{hom}}(B)$ is star-shaped.

Remark 3.2. Littlewood's proof can be adapted to show that general two-dimensional star-shaped domains for which (3.2) holds are such that $u^{\text{hom}}(\Omega)$ is also star-shaped. Whether the same is true for star-shaped Ω and sufficiently smooth maps $u : \Omega \to \mathbb{R}^n$, $n \geq 3$, is an interesting question. We note that u may be required to satisfy certain smoothness and invertibility hypotheses in order to infer $u(B) = u^{\text{hom}}(B)$ from the fact that $u^{\text{hom}} = u$ on ∂B. See [16] for results of this kind.

Now for smooth enough u the assumption of (3.2) at the boundary $\partial\Omega$ would suffice for our purposes; but for less regular competitors we need to strengthen (3.2) to hold 'asymptotically close to $\partial\Omega$'. To make this precise, let $s \geq 3$ be an integer, let $t \in [\frac{1}{2}, 1]$ and define

$$e_t^{(s)}(x) = \chi_{B_t \setminus B_{t-\frac{1}{s}}}(x) \frac{\alpha N}{d}.$$

Let $\sigma : \mathbb{R}^+ \to \mathbb{R} \cup \{\infty\}$ be smooth, convex and such that

$$\lim_{y \to 0+,\infty} \sigma(y) = +\infty.$$

Definition 3.1. Let $v \in \mathcal{A}_{u_0}$ and define

$$K(v) = \text{ess}\liminf_{t \to 1} \liminf_{s \to \infty} \int_\Omega \sigma(\text{cof}\, \nabla v(x) \cdot v(x) \otimes e_t^{(s)}(x))\, dx. \tag{3.4}$$

3.2. **Consequences of $K(v) < \infty$.** The goal of this section is to derive a version of inequality (1.8) for a sequence of sets Ω_{t_n} where $t_n \to 1$. Thus we aim to prove that

$$\int_{\Omega_{t_n}} Df(\nabla u) \cdot \nabla u\, dx \leq \int_{\partial\Omega_{t_n}} Df(\nabla u) \cdot u \otimes \frac{\alpha N}{d}\, d\mathcal{H}^{n-1} \tag{3.5}$$

for a sequence $t_n \to 1^-$. First we note that the weak energy-momentum equations associated with the functional I still have a key role to play.

Proposition 3.1. *Let u be a weak local minimizer of I in $\mathcal{A}$. Then the weak energy-momentum equations hold:*

$$\int_\Omega (f(\nabla u)\mathbf{1} - \nabla u^T Df(\nabla u)) \cdot \nabla\varphi\, dx = 0 \quad \forall \varphi \in C_c^\infty(\Omega; \mathbb{R}^n). \tag{3.6}$$

If u is in addition $C^1(\Omega)$, then

$$\frac{1}{\det \nabla u} \in L^p(\Omega') \quad \forall\ p \in (1, \infty) \tag{3.7}$$

and each $\Omega' \Subset \Omega$.

Proof. The energy-momentum equations are usually derived by considering so-called inner variations of the form

$$u_\delta(x) := u(x + \delta\varphi(x)),$$

where φ is a fixed but arbitrary test function. Provided δ is sufficiently small, it is easily checked that u_δ is both admissible and $W^{1,\infty}$-close to u. Consequently the limit

$$\lim_{\delta \to 0} \frac{I(u_\delta) - I(u)}{\delta}$$

is zero whenever it exists. One can now follow [7, Theorem A.1] or [3, Theorem 2.4(ii)] to deduce (3.6).

Statement (3.7) follows by first noting that $|\nabla u|^q \in L^p(\Omega)$ for all $p \in (1,\infty)$ and each $\Omega' \Subset \Omega$ whenever $u \in C^1(\Omega)$ and then by applying [6, Lemma 2.4], which states that

$$||(\det \nabla u)^{-s}||_{L^p(\Omega')} \leq C(1 + I(u) + |||\nabla u|^q||_{L^p(\Omega)}).$$

Here, q is the exponent which controls the growth of F in the definition of the stored-energy function W.

□

We remark that $u_\delta - u$ has compact support in Ω, and hence

$$K(u_\delta) = K(u)$$

for all small enough δ. In particular, $K(u_\delta) < \infty$ for all sufficiently small δ whenever $K(u) < \infty$.

Lemma 3.3. *Let u be a C^1 weak local minimizer of I. Let $t < 1$ be such that*

$$\liminf_{s\to\infty} \int_\Omega \sigma(\operatorname{cof} \nabla u \cdot u \otimes e_t^{(s)})\, dx < \infty. \tag{3.8}$$

Then

$$\int_{\Omega_t} Df(\nabla u) \cdot \nabla u\, dx \leq \int_{\partial\Omega_t} Df(\nabla u) \cdot u \otimes \frac{\alpha N}{d}\, d\mathcal{H}^{n-1}.$$

Remark 3.4. Condition (3.8) necessarily holds for t in a set of positive measure whenever $K(u) < \infty$. Without loss of generality, therefore, we may assume that (3.14) below and (3.8) hold simultaneously.

Proof. Let

$$\eta_t^{(s)}(x) = \begin{cases} 1 & \text{if } 0 \leq \frac{|x|}{d} \leq t - \frac{1}{s}, \\ s\left(t - \frac{|x|}{d}\right) & \text{if } t - \frac{1}{s} \leq \frac{|x|}{d} \leq t, \\ 0 & \text{if } \frac{|x|}{d} \geq t \end{cases}$$

and note that

$$\nabla \eta_t^{(s)} = -s e_t^{(s)}. \tag{3.9}$$

Let $u^\epsilon(x) = (1 + \epsilon \eta_t^{(s)})u(x)$. Then

$$\det \nabla u^\epsilon = (1 + \epsilon\eta_t^{(s)})^n \det \nabla u - \epsilon s(1 + \epsilon \eta_t^{(s)})^{n-1} \operatorname{cof} \nabla u \cdot u \otimes e_t^{(s)}. \tag{3.10}$$

In view of (3.8), we may assume that

$$\int_\Omega \sigma(\operatorname{cof} \nabla u \cdot u \otimes e_t^{(s)})\, dx < \infty$$

for infinitely many s; therefore, for each such s,

$$\operatorname{cof} \nabla u \cdot u \otimes e_t^{(s)} > 0$$

for almost every x. In particular, provided $\epsilon < 0$,

$$-\epsilon s(1 + \epsilon\eta_t^{(s)})^{n-1} \operatorname{cof} \nabla u \cdot u \otimes e_t^{(s)} > 0 \ \text{ a.e.},$$

from which it follows that

$$\det \nabla u^\epsilon > \frac{1}{2} \det \nabla u \text{ a.e.}$$

Since u is a weak local minimizer of I it follows that

$$\limsup_{\epsilon\to 0-} \frac{I(u^\epsilon)-I(u)}{\epsilon} \le 0. \tag{3.11}$$

The rest of the proof consists in calculating this difference quotient. Now $f(\nabla u)$ is the sum of $F(\nabla u)$ and $h(\det \nabla u)$. The calculation of the quotient

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_\Omega (F(\nabla u^\epsilon)-F(\nabla u))\,dx = \int_\Omega DF(\nabla u)\cdot(\eta_t^{(s)}\nabla u + u\otimes\nabla\eta_t^{(s)})\,dx \tag{3.12}$$

is straightforward. We focus on calculating

$$\lim_{\epsilon\to 0-}\int_B \frac{h(\det\nabla u^\epsilon)-h(\det\nabla u)}{\epsilon}\,dx$$

by writing

$$\int_\Omega \frac{h(\det\nabla u^\epsilon)-h(\det\nabla u)}{\epsilon}\,dx = I + II,$$

where

$$\begin{aligned} I &= \int_\Omega \frac{1}{\epsilon}\int_0^\epsilon h'(\det\nabla u^\lambda)(n\eta\det\nabla u + \operatorname{cof}\nabla u\cdot u\otimes\nabla\eta)\,d\lambda\,dx, \\ II &= \int_\Omega \frac{1}{\epsilon}\int_0^\epsilon n\lambda\eta h'(\det\nabla u^\lambda)((1+\lambda\eta)^{n-1}-1)\big[n\eta\det\nabla u \\ &\qquad\qquad + \operatorname{cof}\nabla u\cdot u\otimes\nabla\eta\big]\,d\lambda\,dx. \end{aligned}$$

We have suppressed the dependence of η on s and t, and $\det\nabla u^\lambda$ is exactly (3.10) with λ in place of ϵ. In each case the integrand is dominated by

$$C(|h'(\det\nabla u)||\det\nabla u| + |h'(\det\nabla u)||\nabla u||\nabla\eta|), \tag{3.13}$$

where C is a constant independent of ϵ and λ. The first term $|h'(\det\nabla)||\det\nabla u|$ in (3.13) is $L^1(\Omega)$ by the inequality $y|h'(y)|\le C(1+y+h(y))$, which holds for all positive y and which follows from the growth hypotheses on h expressed in (2.2). The second is in $L^1(\Omega)$ by applying (3.7) with $p=s+1$. Note that this reasoning also shows that $Df(\nabla u)\in L^1(\Omega)$. By dominated convergence, $\lim_{\epsilon\to 0} II = 0$ and

$$\lim_{\epsilon\to 0} I = \int_\Omega h'(\det\nabla u)(n\eta\det\nabla u + \operatorname{cof}\nabla u\cdot u\otimes\nabla\eta)\,dx.$$

The latter may be rewritten as

$$\int_\Omega Dh(\det\nabla u)\cdot(\eta\nabla u + u\otimes\nabla\eta)\,dx.$$

Thus, in view of (3.12),

$$\lim_{\epsilon\to 0-}\frac{I(u^\epsilon)-I(u)}{\epsilon} = \int_\Omega Df(\nabla u)\cdot(\eta_t^{(s)}\nabla u + u\otimes\nabla\eta_t^{(s)})\,dx.$$

Finally, and bearing in mind (3.9) and (3.11), let $s\to\infty$ to obtain

$$\int_{\Omega_t} Df(\nabla u)\cdot\nabla u\,dx \le \int_{\partial\Omega_t} Df(\nabla u)\cdot u\otimes\frac{\alpha N}{d}\,d\mathcal{H}^{n-1}.$$

Here we used the observation made above that $Df(\nabla u) \in L^1(\Omega)$ together with the fact that

$$\lim_{s\to\infty} s\int_{\Omega_t\setminus\Omega_{t-\frac{1}{s}}} Df(\nabla u)\cdot u\otimes\frac{\alpha N}{d}\,dx = \int_{\partial\Omega_t} Df(\nabla u)\cdot u\otimes\frac{\alpha N}{d}\,d\mathcal{H}^{n-1} \tag{3.14}$$

for a.e. t; see [10] for details of the latter. This concludes the proof of Lemma 3.3. □

4. Uniqueness subject to linear boundary conditions

4.1. **Comparing $I(u^{\mathrm{hom}})$ and $I(u)$.** Assume for now that u is a weak local minimizer of I in $\mathcal{A}_{u_0}$ and is such that $K(u) < \infty$. Recall that for each $t \in (0,1]$,

$$u_t^{\mathrm{hom}}(x) = \frac{|x|}{td}u(t\theta d)$$

and that

$$\nabla u_t^{\mathrm{hom}}(x) = \nabla u(t\theta d) + \left(\frac{u(t\theta d)}{td} - \nabla u(t\theta d)\theta\right)\otimes\alpha N.$$

The fact that the right-hand side is a function of the angular variable θ only suggests that a suitable version of the coarea formula can be used to evaluate $\int_{\Omega_t} f(\nabla u_t^{\mathrm{hom}})\,dx$. One can apply [22, Equation 2.1], or else use a variant of [10, Proposition 3.4.4], to obtain

$$n\int_{\Omega_t} f(\nabla u_t^{\mathrm{hom}})\,dx = t\int_{\partial\Omega_t} f\left(\nabla u(t\theta d) + \left(\frac{u(t\theta d)}{td} - \nabla u(t\theta d)\theta\right)\otimes\alpha N\right)\,d\mathcal{H}^{n-1}. \tag{4.1}$$

Now $f(A)$ is the sum of the everywhere finite quasiconvex function $F(A)$ and the function $h(\det A)$. The former is rank-one convex on $\mathbb{R}^{n\times n}$ by standard results (see, for example, [9, Theorem 5.3 (i)]). The latter is rank-one convex on the half-lines

$$\left\{C_\lambda := \nabla u(t\theta d) + \lambda t\left(\frac{u(t\theta d)}{td} - \nabla u(t\theta d)\theta\right)\otimes\alpha N : \quad \lambda\geq 0\right\}.$$

This can be verified directly by noting that $\det C_\lambda = \lambda \mathrm{cof}\,\nabla u(t\theta d)\cdot u(t\theta d)\otimes\frac{\alpha N}{d}$, which by (3.2) implies that $\det C_\lambda > 0$ for $\mathcal{H}^{n-1}$-a.e. θ and all $\lambda > 0$. Since h is convex on $(0,\infty)$, it follows in particular that

$$h(\det C_1) \geq h(\det C_0) + Dh(\det C_0)\cdot t\left(\frac{u(t\theta d)}{td} - \nabla u(t\theta d)\theta\right)\otimes\alpha N.$$

The rank-one convexity of F implies that exactly the same inequality holds with $F(A)$ in place of $h(\det A)$. Hence, from (4.1),

$$\begin{aligned} n\int_{\Omega_t} f(\nabla u_t^{\mathrm{hom}})\,dx \geq\ & t\int_{\partial\Omega_t} f(\nabla u(t\theta d)) \\ & + Df(\nabla u(t\theta d))\cdot\left(\frac{u(t\theta d)}{td} - \nabla u(t\theta d)\theta\right)\otimes\alpha N\,d\mathcal{H}^{n-1}. \end{aligned} \tag{4.2}$$

Following Taheri's [22] argument, we set $\phi(x) = \eta_t^{(s)}(x)x$ in the energy-momentum equations

$$\int_\Omega \left(f(\nabla u)\mathbf{1} - \nabla u^T Df(\nabla u)\right)\cdot\nabla\phi\,dx = 0.$$

This gives

$$\begin{aligned} 0 &= \int_\Omega n f(\nabla u)\eta_t^{(s)}\,dx - s\int_\Omega f(\nabla u)x\cdot e_t^{(s)}\,dx \\ &\quad + s\int_\Omega \nabla u^T Df(\nabla u)\cdot x\otimes e_t^{(s)}\,dx - \int_\Omega \nabla u\cdot Df(\nabla u)\eta_t^{(s)}\,dx. \end{aligned}$$

Sending $s\to\infty$, applying the result from [4] that $\nabla u^T Df(\nabla u)\in L^1(\Omega)$, and rearranging give

$$\begin{aligned} n\int_{\Omega_t} f(\nabla u)\,dx &= \int_{\Omega_t}\nabla u\cdot Df(\nabla u)\,dx \\ &\quad + t\int_{\partial\Omega_t} f(\nabla u) - \nabla u^T Df(\nabla u)\cdot\theta\otimes\alpha N\,d\mathcal{H}^{n-1} \end{aligned}$$

for a.e. t. Since $K(u)<\infty$, we may assume without loss of generality that condition (3.8) holds for a sequence of t to which the above reasoning also applies. Without relabelling these t, we apply Lemma 3.3 to deduce

$$\int_{\Omega_t} Df(\nabla u)\cdot\nabla u\,dx \le \int_{\partial\Omega_t} Df(\nabla u)\cdot u\otimes\frac{\alpha N}{d}\,d\mathcal{H}^{n-1}. \tag{4.3}$$

Therefore

$$\begin{aligned} n\int_{\Omega_t} f(\nabla u)\,dx &\le \int_{\partial\Omega_t} Df(\nabla u)\cdot u\otimes\frac{\alpha N}{d}\,d\mathcal{H}^{n-1} \\ &\quad + t\int_{\partial\Omega_t} f(\nabla u) - \nabla u^T Df(\nabla u)\cdot\theta\otimes\alpha N\,d\mathcal{H}^{n-1}. \end{aligned} \tag{4.4}$$

When compared with the right-hand side of (4.2), inequality (4.4) implies that

$$\int_{\Omega_t} f(\nabla u)\,dx \le \int_{\Omega_t} f(\nabla u_t^{\mathrm{hom}})\,dx.$$

The above reasoning proves:

Proposition 4.1. *Let $u\in\mathcal{A}_{u_0}$ be a C^1 weak local minimizer of I such that $K(u)<\infty$. Then*

$$\int_{\Omega_{t_n}} f(\nabla u)\,dx \le \int_{\Omega_{t_n}} f(\nabla u_{t_n}^{\mathrm{hom}})\,dx \tag{4.5}$$

for a sequence $t_n\to 1$.

Remark 4.1. The calculation shown above is clearly inspired by that given in [22]. But there are two key differences, the main one being that the full Euler-Lagrange equation is not assumed to hold for the weak local minimizer u. Instead, we rely on Lemma 3.3 for the inequality (4.3). Also, the p-growth assumption made in [22] easily supplies the inclusion $Df(\nabla u)\in L^1(\Omega)$ for all $u\in W^{1,p}$. Our route is more circuitous: it relies on estimates in [6] derived from the energy-momentum equations and which only apply to solutions of these equations.

4.2. **Uniqueness of C^1 weak local minimizers.** We now apply the foregoing analysis to the case $u_0(y) = \xi y$, where ξ is a constant $n \times n$ matrix. It is straightforward to check that any $u \in C^1(\bar{\Omega}) \cap \mathcal{A}_{u_0}$ is such that

$$K(u) < \infty \text{ if } \det \xi > 0.$$

Since u is C^1, and in view of the boundary condition, it is the case that

$$\int_{\Omega_{t_n}} f(\nabla u_{t_n}^{\mathrm{hom}})\,dx \to \int_{\Omega} f(\xi)\,dx$$

as $n \to \infty$ for any sequence $t_n \to 1^-$. If, in addition, u is a weak local minimizer, then Proposition 4.1 applies, giving

$$\int_{\Omega_{t_n}} f(\nabla u)\,dx \le \int_{\Omega_{t_n}} f(\nabla u_{t_n}^{\mathrm{hom}})\,dx$$

for each n, and hence on letting $n \to \infty$,

$$\int_{\Omega} f(\nabla u)\,dx \le \int_{\Omega} f(\xi)\,dx. \tag{4.6}$$

We now assume that f is strictly quasiconvex at ξ, implying in particular that

$$\int_{\Omega} f(\xi)\,dx \le \int_{\Omega} f(\nabla u)\,dx \tag{4.7}$$

with equality if and only if $u(x) = \xi x$ on Ω. Putting (4.6) and (4.7) together yields:

Proposition 4.2. *Let $u \in C^1(\bar{\Omega})$ be a weak local minimizer of I in $\mathcal{A}_{u_0}$, where $u_0(y) = \xi y$, $\det \xi > 0$, and f defined in* (1.4) *is strictly quasiconvex at ξ. Then $u(x) = \xi x$ for all x in Ω.*

4.3. **Concluding remarks.** We briefly address the question of whether (3.4) is the only or right choice for the auxiliary functional K. Clearly, the K defined by (3.4) suffices in the situation that u is $C^1(\bar{\Omega})$. Thus the following remarks apply primarily to weak local minimizers that are not *a priori* assumed to be C^1.

(i) Ideally, any replacement for K (again denoted K) would be sequentially lower semicontinuous with respect to weak convergence in $W^{1,n}$, say. One could then (locally) minimize $I + K$, and the conclusion $K(u) < \infty$ would be automatic rather than imposed.

(ii) Potentially, one could allow the set

$$E := \{x \in \Omega : \ \operatorname{cof} \nabla u \cdot u \otimes \alpha N \le 0\}$$

to approach $\partial\Omega$ in a less restrictive manner than is prescribed by the condition $K(u) < \infty$, where K is as per (3.4). Indeed, if $K(u)$ is finite, then for t in a set of positive measure,

$$\operatorname{cof} \nabla u \cdot u \otimes \alpha N > 0 \text{ a.e. } x \in \Omega_t \setminus \Omega_{t-\frac{1}{s}}$$

for at least one $s = s(t)$. Moreover, one can take t for which this holds arbitrarily close to 1. So E is trapped in a specific sequence of sets which approach $\partial\Omega$. But it is possible to imagine a set E for which $K(u) = +\infty$ but which might nevertheless admit an analysis similar to that given in Sections 3 and 4 above. This will be investigated in a future paper.

(iii) K should not depend on values of u in the interior of the domain. Energy functionals for elastic materials typically depend only on the gradient of the deformation in the interior. The K proposed in (3.4) does this to an extent; any modifications with (i) and (ii) above in mind should preserve this property. It would not do, for example, to require that for fixed $l < 1$,

$$\hat{K}(v) := \int_{\Omega \setminus \Omega_l} \sigma(\operatorname{cof} \nabla u \cdot u \otimes \alpha N)\, dx$$

be finite. Although $\hat{K}$ would be sequentially weakly lower semicontinuous (by [5, Proposition A.3], for example), its value would still depend on $u|_{\Omega \setminus \Omega_l}$.

(iv) Dropping the assumption that u is C^1 is problematic for the reasons pointed out in [22]. See [14, Section 7] for examples of nowhere C^1 weak local minimizers based on the construction of [17]. The assumption $K(v) < \infty$ would appear to limit possible oscillations of ∇u in the direction tangent to $\partial\Omega$, say, but there is still room for bad behaviour in the directions normal to $\partial\Omega$. Any modification of (3.4) should take these difficulties into account.

References

1. J. M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. Arch. Rational Mech. Anal. 63 (1977), no. 4, 337–403. MR0475169 (57:14788)
2. J. M. Ball. Discontinuous equilibrium solutions and cavitation in nonlinear elasticity. Philos. Trans. Roy. Soc. London Ser. A 306 (1982), no. 1496, 557–611. MR703623 (84i:73041)
3. J. M. Ball. Some open problems in elasticity. Geometry, mechanics, and dynamics, 3–59, Springer, New York, 2002. MR1919825 (2003g:74001)
4. J. M. Ball. Minimizers and the Euler - Lagrange equations. Trends and applications of pure mathematics to mechanics (Palaiseau, 1983), 1–4, Lecture Notes in Phys., 195, Springer, Berlin, 1984. MR755716 (86e:49022)
5. J. M. Ball, F. Murat. $W^{1,p}$-quasiconvexity and variational problems for multiple integrals. J. Funct. Anal. 58 (1984), no. 3, 225–253. MR759098 (87g:49011a)
6. P. Bauman, D. Phillips, N. C. Owen. Maximal smoothness of solutions to certain Euler-Lagrange equations from nonlinear elasticity. Proc. Roy. Soc. Edinburgh Sect. A 119 (1991), no. 3-4, 241–263. MR1135972 (92m:73042)
7. P. Bauman, D. Phillips, N. C. Owen. Maximum principles and a priori estimates for a class of problems from nonlinear elasticity. Ann. Inst. H. Poincaré Anal. Non Linéaire 8 (1991), no. 2, 119–157. MR1096601 (92c:35040)
8. J. J. Bevan. On one-homogeneous solutions to elliptic systems with spatial variable dependence in two dimensions. Proc. Roy. Soc. Edinburgh 140A (2010), 447-475.
9. B. Dacorogna. Direct methods in the calculus of variations, second edition. Applied Mathematical Sciences, 78. Springer-Verlag, Berlin, 2008. MR2361288 (2008m:49003)
10. L. C. Evans and R. F. Gariepy. Measure theory and fine properties of functions. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992. MR1158660 (93f:28001)
11. J. L. Ericksen. Special topics in elastostatics. In: Advances in Applied Mechanics, 17, 189-224. Academic Press, 1977.
12. I. M. Gelfand and S. V. Fomin. Calculus of Variations, Prentice-Hall, 1963. MR0160139 (28:3353)
13. R. J. Knops, C. A. Stuart. Quasiconvexity and uniqueness of equilibrium solutions in nonlinear elasticity. Arch. Rational Mech. Anal. 86 (1984), no. 3, 233-249. MR751508 (85j:73012)
14. J. Kristensen, A. Taheri. Partial regularity of strong local minimizers in the multi-dimensional calculus of variations. Arch. Ration. Mech. Anal. 170 (2003), no. 1, 63–89. MR2012647 (2004i:49073)
15. J. E. Littlewood. Lectures on the Theory of Functions. Oxford University Press, 1944. MR0012121 (6:261f:)

16. S. Müller, S. J. Spector, Q. Tang. Invertibility and a topological property of Sobolev maps. SIAM J. Math. Anal. 27 (1996), no. 4, 959-976. MR1393418 (97e:46041)
17. S. Müller, V. Šverák. Convex integration for Lipschitz mappings and counterexamples to regularity. Ann. of Math. (2) 157 (2003), no. 3, 715–742. MR1983780 (2005i:35028)
18. S. Müller, Qi Tang, B. S. Yan. On a new class of elastic deformations not allowing for cavitation. Ann. Inst. H. Poincaré Anal. Non Linéaire 11 (1994), no. 2, 217–243. MR1267368 (95a:73025)
19. P. J. Olver. Applications of Lie Groups to Differential Equations, Springer-Verlag, 1986. MR836734 (88f:58161)
20. J. Sivaloganathan. Uniqueness of regular and singular equilibria for spherically symmetric problems of nonlinear elasticity. Arch. Rational Mech. Anal. 96 (1986), 97-136. MR853969 (88a:73035)
21. J. Sivaloganathan. Implications of rank one convexity. Ann. Inst. Henri Poincaré 5 (1988), no. 2, 99–118. MR954467 (89g:49030)
22. A. Taheri. Quasiconvexity and uniqueness of stationary points in the multi-dimensional calculus of variations. Proc. Amer. Math. Soc. 131 (2003), 3101-3107. MR1993219 (2004f:49041)

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