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HAMILTON'S GRADIENT ESTIMATES AND LIOUVILLE THEOREMS FOR FAST DIFFUSION EQUATIONS ON NONCOMPACT RIEMANNIAN MANIFOLDS

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Abstract. Let M be a complete noncompact Riemannian manifold of dimension n. In this paper, we derive a local gradient estimate for positive solutions of fast diffusion equations

$$\partial_t u = \Delta u^\alpha, \quad 1 - \frac{2}{n} < \alpha < 1$$

on $M \times (-\infty, 0]$. We also obtain a theorem of Liouville type for positive solutions of the fast diffusion equation.

1. Introduction

In this paper we study the fast diffusion equation (FDE for short)

$$\partial_t u = \Delta u^\alpha, \tag{1.1}$$

where $\alpha \in (0, 1)$. FDE arises in the study of fast diffusions, in particular in diffusion in plasma ([3]), in thin liquid film dynamics driven by Van der Waals forces ([7], [8]), and in models of gas-kinetics ([4]). It also arises in geometry: the case $\alpha = \frac{n-2}{n+2}$ in dimensions $n > 3$ describes the evolution of a conformal metric by the Yamabe flow ([18]); the case $\alpha = 0$, $n = 2$ describes the Ricci flow on surfaces ([10], [6], [16]). Precisely, we can find the relationship from $\partial_t u = \Delta(\frac{1}{\alpha} u^\alpha) = \mathrm{div}(u^\alpha \frac{\nabla u}{u})$ and $\mathrm{div}(\frac{\nabla u}{u}) = \mathrm{div}(\nabla \log u) = \Delta \log u$. We refer the reader to the book by Daskalopoulos-Kenig ([5]) and the references therein for more about FDE.

As a nonlinear problem, the mathematical theory of FDE is based on a priori estimates. In 1979, Aronson and Bénilan obtained a celebrated second-order differential inequality of the form ([1])

$$\sum_i \frac{\partial}{\partial x^i}(\alpha u^{\alpha-2} \frac{\partial u}{\partial x^i}) \geq -\frac{\kappa}{t}, \quad \kappa := \frac{n}{n(\alpha - 1) + 2},$$

which applies to all positive solutions of (1.1) defined on the whole Euclidean space on the condition that $\alpha > 1 - \frac{2}{n}$.

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There are few results about FDE on manifolds. In 2008, Lu, Ni, Vázquez and Villani studied the FDE on manifolds ([13]) and got a local Aronson-Bénilan estimate. We do not state their result here. What we will do in this paper is to get Hamilton's gradient estimates. First, let us recall what Hamilton's result is:

Theorem (Hamilton [9]). *Let $\boldsymbol{M}$ be a closed Riemannian manifold of dimension $n \geq 2$ with Ricci$(\boldsymbol{M}) \geq -k$ for some $k \geq 0$. Suppose that u is any positive solution to the heat equation with $u \leq M$ for all $(x,t) \in \boldsymbol{M} \times (0,\infty)$. Then*

$$\frac{|\nabla u|^2}{u^2} \leq (\frac{1}{t} + 2k) \log \frac{M}{u}.$$

Hamilton's estimate tells us that when the temperature is bounded we can compare the temperature of two different points at the same time.

In 2006, P. Souplet and Qi S. Zhang ([14]) generalized Hamilton's estimate to complete noncompact Riemannian manifolds. In 2007, B. Kotschwar ([11]) used Hamilton's estimate and obtained a global gradient estimate for heat kernels on complete noncompact manifolds. In 2010, M. Bailesteanu, X. Cao and A. Pulemotov ([2]) generalized Souplet and Zhang's result to Ricci flow. Also in 2010, on complete noncompact Riemannian manifolds, J. Wu ([15]) obtained a localized Hamilton-type gradient estimate for the positive smooth bounded solutions to the nonlinear diffusion equation $u_t = \Delta u - \nabla\phi \cdot \nabla u - au \log u - bu$, where ϕ is a C^2 function and $a \neq 0$ and b are two real constants. We would like to remark here that this equation was also studied by Y. Yang in ([17]), where they derived a parabolic gradient estimate.

In this paper, we consider the positive solution for FDE (1.1). Like what they did for the heat equation, we derive a similar Hamilton's estimate for FDE. Inspired by the inequality of Aronson and Bénilan, we let $\alpha > 1 - \frac{2}{n}$ throughout this paper. Note that the pressure $\tilde{v} := \frac{\alpha}{\alpha-1} u^{\alpha-1} < 0$,

$$\partial_t \tilde{v} = (\alpha - 1)\tilde{v}\Delta\tilde{v} + |\nabla\tilde{v}|^2.$$

Conveniently, we let $v = -\tilde{v}$. Then $v > 0$ and satisfies

$$\partial_t v = (1-\alpha) v \Delta v - |\nabla v|^2. \tag{1.2}$$

Our main result is the following:

Theorem 1 (Gradient estimates). *Let $\boldsymbol{M}$ be a Riemannian manifold of dimension $n \geq 2$ with Ricci$(\boldsymbol{M}) \geq -k$ for some $k \geq 0$. Suppose that v is any positive solution to the equation (1.2) in $Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset \boldsymbol{M} \times (-\infty, \infty)$. Suppose also that $v \leq M$ in $Q_{R,T}$. Then there exists a constant $C = C(\alpha, \boldsymbol{M})$ such that*

$$\frac{|\nabla v|}{v^{1/2}} \leq C M^{1/2} (\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k})$$

in $Q_{\frac{R}{2}, \frac{T}{2}}$.

As an application, we get the following Liouville type theorem:

Corollary 1.1 (Liouville type theorem)**.** *Let* $\boldsymbol{M}$ *be a complete, noncompact manifold of dimension* n *with nonnegative Ricci curvature. Let* u *be a positive ancient solution to the equation* (1.1) *with* $1-\frac{2}{n}<\alpha<1$ *such that* $\frac{1}{u(x,t)}=o([d(x)+\sqrt{|t|}]^{\frac{2}{1-\alpha}})$. *Then* u *is a constant.*

2. Proof of Theorem 1

Let $w \equiv \frac{|\nabla v|^2}{v^\beta}$, $\beta>0$ to be determined.

We will derive an equation for w. First notice that

$$\begin{aligned}
w_t &= \frac{2v_i v_{it}}{v^\beta} - \beta\frac{v_i^2 v_t}{v^{\beta+1}} \\
&= \frac{2v_i((1-\alpha)v\Delta v - |\nabla v|^2)_i}{v^\beta} - \beta\frac{v_i^2((1-\alpha)v\Delta v-|\nabla v|^2)}{v^{\beta+1}} \\
&= 2(1-\alpha)\frac{v_i^2 v_{jj}}{v^\beta} + 2(1-\alpha)\frac{v_i v_{jji}}{v^{\beta-1}} - 4\frac{v_i v_{ij} v_j}{v^\beta} \\
&\qquad - \beta(1-\alpha)\frac{v_i^2 v_{jj}}{v^\beta} + \beta\frac{v_i^2 v_j^2}{v^{\beta+1}},
\end{aligned} \tag{2.1}$$

$$w_j = \frac{2v_i v_{ij}}{v^\beta} - \beta\frac{v_i^2 v_j}{v^{\beta+1}},$$

$$w_{jj} = \frac{2v_{ij}^2}{v^\beta} + \frac{2v_i v_{ijj}}{v^\beta} - 4\beta\frac{v_i v_{ij} v_j}{v^{\beta+1}} - \beta\frac{v_i^2 v_{jj}}{v^{\beta+1}} + \beta(\beta+1)\frac{v_i^2 v_j^2}{v^{\beta+2}}. \tag{2.2}$$

By (2.1) and (2.2),

$$\begin{aligned}
&(1-\alpha)v\Delta w - w_t \\
&= 2(1-\alpha)\frac{v_{ij}^2}{v^{\beta-1}} + 2(1-\alpha)\frac{v_i v_{ijj}}{v^{\beta-1}} - 4\beta(1-\alpha)\frac{v_i v_{ij} v_j}{v^\beta} \\
&\quad - \beta(1-\alpha)\frac{v_i^2 v_{jj}}{v^\beta} + \beta(\beta+1)(1-\alpha)\frac{v_i^2 v_j^2}{v^{\beta+1}} \\
&\quad - 2(1-\alpha)\frac{v_i^2 v_{jj}}{v^\beta} - 2(1-\alpha)\frac{v_i v_{jji}}{v^{\beta-1}} + 4\frac{v_i v_{ij} v_j}{v^\beta} \\
&\quad + \beta(1-\alpha)\frac{v_i^2 v_{jj}}{v^\beta} - \beta\frac{v_i^2 v_j^2}{v^{\beta+1}} \\
&= 2(1-\alpha)\frac{v_{ij}^2}{v^{\beta-1}} - 2(1-\alpha)\frac{v_i^2 v_{jj}}{v^\beta} + 2(1-\alpha)\frac{R_{ij} v_i v_j}{v^{\beta-1}} \\
&\quad + 4(1-\beta(1-\alpha))\frac{v_i v_{ij} v_j}{v^\beta} + (\beta(\beta+1)(1-\alpha)-\beta)\frac{v_i^2 v_j^2}{v^{\beta+1}}.
\end{aligned}$$

Here, in the second equality, we use the Ricci formula: $v_{ijj}-v_{jji}=R_{ij}v_j$.

Notice that

$$\nabla w\cdot\nabla v = w_j v_j = \frac{2v_i v_{ij} v_j}{v^\beta} - \beta\frac{v_i^2 v_j^2}{v^{\beta+1}}. \tag{2.3}$$

Using (2.3), we obtain

$$
\begin{aligned}
&(1-\alpha)v\Delta w - w_t \\
=& 2(1-\alpha)\frac{v_{ij}^2}{v^{\beta-1}} - 2(1-\alpha)\frac{v_i^2 v_{jj}}{v^{\beta}} + 2(1-\alpha)\frac{R_{ij}v_i v_j}{v^{\beta-1}} \\
&+ 4(1-\beta(1-\alpha))\frac{v_i v_{ij} v_j}{v^{\beta}} + (\beta(\beta+1)(1-\alpha)-\beta)\frac{v_i^2 v_j^2}{v^{\beta+1}} \\
\geq& -\frac{n(1-\alpha)}{2}\frac{|\nabla v|^4}{v^{\beta+1}} - 2(1-\alpha)k\frac{|\nabla v|^2}{v^{\beta-1}} + 2(1-\beta(1-\alpha))\nabla w\cdot\nabla v \\
&+ 2\beta(1-\beta(1-\alpha))\frac{v_i^2 v_j^2}{v^{\beta+1}} + (\beta(\beta+1)(1-\alpha)-\beta)\frac{v_i^2 v_j^2}{v^{\beta+1}} \\
=& 2(1-\beta(1-\alpha))\nabla w\cdot\nabla v - 2(1-\alpha)k\frac{|\nabla v|^2}{v^{\beta-1}} \\
&+ (\beta(\beta+1)(1-\alpha)-\beta-\frac{n(1-\alpha)}{2} + 2\beta(1-\beta(1-\alpha)))\frac{|\nabla v|^4}{v^{\beta+1}}.
\end{aligned}
$$

For the purpose of obtaining the gradient estimates as in [14], we need to have the coefficient of w^2 be positive. Fortunately, we can do this by choosing a suitable β. In fact,

$$
\begin{aligned}
&\beta(\beta+1)(1-\alpha)-\beta-\frac{n(1-\alpha)}{2}+2\beta(1-\beta(1-\alpha)) \\
=& -(1-\alpha)(\beta^2-\frac{2-\alpha}{1-\alpha}\beta+\frac{n}{2}).
\end{aligned}
$$

It is easily found that the discriminant is $(-\frac{2-\alpha}{1-\alpha})^2-2n$, which is positive when $\alpha\in(1-\frac{2}{n},1)$. So we can choose a suitable β to make sure the term will be positive.

Rearranging, we have

$$
\begin{aligned}
&(1-\alpha)v\Delta w - w_t \\
=& 2(1-\beta(1-\alpha))\nabla w\cdot\nabla v - 2(1-\alpha)kvw - (1-\alpha)(\beta^2-\frac{2-\alpha}{1-\alpha}\beta+\frac{n}{2})v^{\beta-1}w^2.
\end{aligned}
$$

From here, we will use the well-known cutoff function of Li and Yau to derive the desired bounds. We caution the reader that the calculation is not the same as that in [12] due to the difference of the first order.

Let $\psi=\psi(x,t)$ be a smooth cutoff function supported in $Q_{R,T}$, satisfying the following properties:

(1) $\psi=\psi(d(x,x_0),t)\equiv\psi(r,t)$; $\psi(r,t)=1$ in $Q_{R/2,T/2}$, $0\leq\psi\leq 1$.
(2) ψ is decreasing as a radial function in the spatial variables.
(3) $|\partial_r\psi|/\psi^a\leq C_a/R$, $|\partial_r^2\psi|/\psi^a\leq C_a/R^2$ when $0<a<1$.
(4) $|\partial_t\psi|/\psi^{1/2}\leq C/T$.

By straightforward calculation, one has

$$
\begin{aligned}
&(1-\alpha)v\Delta(\psi w) + b\cdot\nabla(\psi w) - 2(1-\alpha)v\frac{\nabla\psi}{\psi}\cdot\nabla(\psi w) - (\psi w)_t \\
\geq& -(1-\alpha)(\beta^2-\frac{2-\alpha}{1-\alpha}\beta+\frac{n}{2})v^{\beta-1}\psi w^2 + (b\cdot\nabla\psi)w \\
&- 2(1-\alpha)v\frac{|\nabla\psi|^2}{\psi}w + (1-\alpha)v(\Delta\psi)w - \psi_t w - 2(1-\alpha)kv\psi w,
\end{aligned}
$$

where $b = -2(1-\beta(1-\alpha))\nabla v$.

Suppose that the maximum of ψw is reached at (x_1, t_1). By [12], we can assume, without loss of generality, that x_1 is not on the cut-locus of $\mathbf{M}$. Then at (x_1, t_1), one has $\Delta(\psi w) \leq 0$, $(\psi w)_t \geq 0$ and $\nabla(\psi w) = 0$. Therefore,

$$\begin{aligned}
&-(1-\alpha)(\beta^2 - \frac{2-\alpha}{1-\alpha}\beta + \frac{n}{2})v^{\beta-1}\psi w^2(x_1,t_1)\\
&\leq -[(b\cdot\nabla\psi)w - 2(1-\alpha)v\frac{|\nabla\psi|^2}{\psi}w\\
&\qquad + (1-\alpha)v(\Delta\psi)w - \psi_t w - 2(1-\alpha)kv\psi w](x_1,t_1).
\end{aligned} \tag{2.4}$$

Denote $-(1-\alpha)(\beta^2 - \frac{2-\alpha}{1-\alpha}\beta + \frac{n}{2}) = \frac{2}{\gamma}$. Then $\gamma > 0$ only depends on α, β.

Rearranging, we have

$$\begin{aligned}
&2\psi w^2(x_1,t_1)\\
&\leq [-\gamma(b\cdot\nabla\psi)v^{1-\beta}w + 2(1-\alpha)\gamma v^{2-\beta}\frac{|\nabla\psi|^2}{\psi}w\\
&\quad - (1-\alpha)\gamma v^{2-\beta}(\Delta\psi)w + \gamma v^{1-\beta}\psi_t w + 2(1-\alpha)\gamma k v^{2-\beta}\psi w](x_1,t_1).
\end{aligned} \tag{2.5}$$

We need to find an upper bound for each term on the right-hand side of (2.5). For the first term,

$$\begin{aligned}
-\gamma(b\cdot\nabla\psi)v^{1-\beta}w &\leq \gamma(b\cdot\nabla\psi)v^{1-\beta}w\\
&= C|\nabla v||\nabla\psi|v^{1-\beta}w\\
&\leq CM^{1-\beta/2}w^{3/2}|\nabla\psi|\\
&\leq \frac{1}{4}\psi w^2 + C(\frac{M^{1-\beta/2}|\nabla\psi|}{\psi^{3/4}})^4\\
&\leq \frac{1}{4}\psi w^2 + CM^{4-2\beta}\frac{1}{R^4}.
\end{aligned} \tag{2.6}$$

Here we used the fact that $0 < v \leq M$.

For the second term on the right-hand side of (2.5), we proceed as follows:

$$\begin{aligned}
2(1-\alpha)\gamma v^{2-\beta}\frac{|\nabla\psi|^2}{\psi}w &\leq CM^{2-\beta}\frac{|\nabla\psi|^2}{\psi^{\frac{3}{2}}}\psi^{\frac{1}{2}}w\\
&\leq \frac{1}{4}\psi w^2 + CM^{4-2\beta}(\frac{|\nabla\psi|^2}{\psi^{\frac{3}{2}}})^2\\
&\leq \frac{1}{4}\psi w^2 + CM^{4-2\beta}\frac{1}{R^4}.
\end{aligned} \tag{2.7}$$

Furthermore, by the properties of ψ and the assumption on the Ricci curvature, one has

$$\begin{aligned}
&-(1-\alpha)\gamma v^{2-\beta}(\Delta\psi)w\\
&=-(1-\alpha)\gamma(\partial_r^2\psi+(n-1)\frac{\partial_r\psi}{r}+\partial_r\psi\partial_r\log\sqrt{g})v^{2-\beta}w\\
&\leq CM^{2-\beta}(|\partial_r^2\psi|+(n-1)\frac{|\partial_r\psi|}{r}+\sqrt{k}|\partial_r\psi|)w\\
&\leq CM^{2-\beta}(\frac{|\partial_r^2\psi|}{\psi^{\frac{1}{2}}}+2(n-1)\frac{|\partial_r\psi|}{R\psi^{\frac{1}{2}}}+\sqrt{k}\frac{|\partial_r\psi|}{\psi^{\frac{1}{2}}})\psi^{\frac{1}{2}}w\\
&\leq \frac{1}{4}\psi w^2+CM^{4-2\beta}((\frac{|\partial_r^2\psi|}{\psi^{\frac{1}{2}}})^2+(\frac{|\partial_r\psi|}{R\psi^{\frac{1}{2}}})^2+(\sqrt{k}\frac{|\partial_r\psi|}{\psi^{\frac{1}{2}}})^2)\\
&\leq \frac{1}{4}\psi w^2+CM^{4-2\beta}(\frac{1}{R^4}+k\frac{1}{R^2}). \qquad (2.8)
\end{aligned}$$

Now we estimate $\gamma v^{1-\beta}\psi_t w$ as

$$\begin{aligned}
\gamma v^{1-\beta}\psi_t w &\leq \gamma v^{1-\beta}|\psi_t|w\\
&\leq \gamma\frac{|\psi_t|}{\psi^{1/2}}\psi^{1/2}wM^{1-\beta}\\
&\leq \frac{1}{4}\psi w^2+CM^{2-2\beta}\frac{1}{T^2}. \qquad (2.9)
\end{aligned}$$

Here we suppose $\beta\leq 1$.

Finally, for the last term, we have

$$\begin{aligned}
2(1-\alpha)\gamma k v^{2-\beta}\psi w &\leq C\psi^{1/2}wkM^{2-\beta}\\
&\leq \frac{1}{4}\psi w^2+CM^{4-2\beta}k^2. \qquad (2.10)
\end{aligned}$$

Substituting (2.6)–(2.10) into the right-hand side of (2.5), we deduce that

$$2\psi w^2(x_1,t_1)\leq \frac{5}{4}\psi w^2(x_1,t_1)+CM^{4-2\beta}(\frac{1}{R^4}+\frac{1}{T^2}+k^2).$$

Therefore,

$$\psi w^2(x_1,t_1)\leq CM^{4-2\beta}(\frac{1}{R^4}+\frac{1}{T^2}+k^2).$$

What we get shows, for all $(x,t)\in Q_{R,T}$, that

$$\begin{aligned}
\psi^2(x,t)w^2(x,t) &\leq \psi^2(x_1,t_1)w^2(x_1,t_1)\\
&\leq \psi(x_1,t_1)w^2(x_1,t_1)\\
&\leq CM^{4-2\beta}(\frac{1}{R^4}+\frac{1}{T^2}+k^2).
\end{aligned}$$

Notice that $\psi(x,t)=1$ in $Q_{R/2,T/2}$, $w=\frac{|\nabla v|^2}{v^\beta}$. We have

$$\frac{|\nabla v(x,t)|}{v^{\beta/2}(x,t)} \le CM^{1-\beta/2}(\frac{1}{R}+\frac{1}{\sqrt{T}}+\sqrt{k}),$$

where $C=C(\alpha,\beta,\mathbf{M})$.

Then we choose $\beta=1$. This ends the proof of Theorem 1.

3. Simple proof of Corollary 1.1

From Theorem 1, we know that, when v is a positive ancient solution to the equation (1.2) such that $v(x,t)=o([d(x,x_0)+\sqrt{|t|}]^2)$, then v is a constant.

Notice that $v=\frac{\alpha}{1-\alpha}u^{\alpha-1}=\frac{\alpha}{1-\alpha}(\frac{1}{u})^{1-\alpha}$, so when u is a positive ancient solution to the equation (1.1) such that $\frac{1}{u(x,t)}=o([d(x,x_0)+\sqrt{|t|}]^{\frac{2}{1-\alpha}})$, then u is a constant. This ends the proof of Corollary 1.1.

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