

HOMOMORPHISMS BETWEEN FUNDAMENTAL GROUPS OF KÄHLER MANIFOLDS

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ABSTRACT. We construct a group homomorphism, which can be realized as the induced homomorphism of fundamental groups from a holomorphic map between compact Kähler manifolds, but cannot be realized by a holomorphic map between smooth projective varieties. It is also proved that there exists no such example between abelian groups.

1. INTRODUCTION

The fundamental groups of compact Kähler manifolds and smooth complex projective varieties have been studied for more than twenty years. Arapura had a survey paper [A1] on this subject, and a more detailed introduction was given in the book [AB]. A natural and unsolved question is whether the class of fundamental groups of compact Kähler manifolds and the class of fundamental groups of smooth complex projective varieties coincide. In [V], Voisin gave examples of compact Kähler manifolds of dimension four or higher, which do not have the homotopy type of any smooth complex projective variety. In this paper, we use Voisin's method to answer a weaker question. If one considers not only the fundamental groups, but also the homomorphisms between them, we can give a negative answer.

First, we need some definitions. A group is called *Kähler* (or *projective*) if it is isomorphic to the fundamental group of a compact Kähler manifold (or smooth complex projective variety). A homomorphism between Kähler (or projective) groups is called *Kähler* (or *projective*) if it is the homomorphism of fundamental groups induced by a holomorphic map between compact Kähler manifolds (or smooth complex projective varieties). The main result of this paper is the following.

Theorem 1.1. *There exists a Kähler homomorphism $f : A \rightarrow B$ which is not projective. However, if f is Kähler and if A and B are both abelian groups, then f must be projective.*

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2. BETWEEN ABELIAN GROUPS

In this section, we prove that for a homomorphism in the category of abelian groups, the condition of being Kähler is the same as being projective.

Let me start with some basic properties about Kähler groups and Kähler homomorphisms.

Lemma 2.1. *The abelianization of any Kähler group has even rank.*

Proof. By Hodge theory, the first Betti number of a compact Kähler manifold must be even, and for any manifold X ,

$$H_1(X, \mathbb{C}) \cong \frac{\pi_1(X)}{[\pi_1(X), \pi_1(X)]} \otimes \mathbb{C}.$$

□

Lemma 2.2. *The kernel, cokernel and image of a Kähler homomorphism must have even ranks. More precisely, suppose $f : A \rightarrow B$ is Kähler and that*

$$\bar{f} : \frac{A}{[A, A]} \rightarrow \frac{B}{[B, B]}$$

is the abelianization of f . Then $\text{Ker}(\bar{f})$, $\text{Coker}(\bar{f})$ and $\text{Im}(\bar{f})$ all have even rank.

Proof. This is because a holomorphic map between Kähler manifolds is compatible with the Hodge structure. In fact, suppose $\varphi : X \rightarrow Y$ is such a holomorphic map. Then

$$\varphi^* : H^1(Y, \mathbb{C}) \rightarrow H^1(X, \mathbb{C})$$

preserves the Hodge decomposition. By Poincaré duality, $\varphi^* : H^1(Y, \mathbb{C}) \rightarrow H^1(X, \mathbb{C})$ is dual to $\varphi_* : H_1(X, \mathbb{C}) \rightarrow H_1(Y, \mathbb{C})$. Therefore by the isomorphism in Lemma 2.1, the result follows. □

Lemma 2.3. *Every finite group is projective, hence Kähler.*

This is observed by Serre; a proof can also be found in [S], page 227. There is a similar result for Kähler homomorphisms:

Lemma 2.4. *Suppose $f : A \rightarrow B$ is a Kähler homomorphism between Kähler groups, and C is a finite group. If $h : A \rightarrow B \times C$ satisfies $p_1 \circ h = f$, where p_1 is the projection to the first component, then h is Kähler. In particular, any homomorphism between Kähler groups is Kähler if the target group is finite.*

Proof. Suppose A , B and C are the fundamental groups of compact Kähler manifolds X , Y and Z , and f is represented by a holomorphic map $\varphi : X \rightarrow Y$; i.e., $\varphi_* : \pi_1(X) \rightarrow \pi_1(Y)$ is isomorphic to f .

Let \tilde{X} , \tilde{Y} and \tilde{Z} be the universal covers of X , Y and Z . Then A , B and C act on \tilde{X} , \tilde{Y} and \tilde{Z} respectively. Let $h_i = p_i \circ h$, for $i = 1, 2$; then $h_1 = f$. Denote by $G(h_2) = \{(x, h_2(x)) \in A \times C \mid x \in A\}$ the graph of h_2 . It is a subgroup of $A \times C$ with finite index, so we can take the quotient of $\tilde{X} \times \tilde{Z}$ by this subgroup, which has a further quotient:

$$\tilde{X} \times \tilde{Z} / G(h_2) \xrightarrow{\pi} \tilde{X} \times \tilde{Z} / (A \times C) = X \times Z.$$

Since π is a finite covering map, $\tilde{X} \times \tilde{Z} / G(h_2)$ is Kähler and compact. Composing π with (f, id) , we get

$$\tilde{X} \times \tilde{Z} / G(h_2) \xrightarrow{\pi} X \times Z \xrightarrow{(f, id)} Y \times Z.$$

The corresponding homomorphisms of fundamental groups are

$$G(h_2) \rightarrow A \times C \rightarrow B \times C,$$

where $G(h_2)$ is isomorphic to A , and the composition is isomorphic to $h : A \rightarrow B \times C$. Hence h is Kähler. □

Notice that a finitely generated abelian group is Kähler if and only if it is projective if and only if it has even rank. In fact, given any finitely generated abelian group of even rank, it is a direct sum of a free part and a torsion part. Therefore, according to Lemma 2.3, we can take the fiber product of an abelian variety and a smooth projective variety, and the fundamental group of the product is isomorphic to the given abelian group. We will prove that for a homomorphism between abelian groups, the obstruction of being Kähler is the same.

Theorem 2.5. *Let $f : A \rightarrow B$ be a homomorphism of abelian Kähler groups. Then the following three conditions are equivalent:*

(i) *f has even rank, i.e., one of $Ker(f), Coker(f), Im(f)$ has even rank, hence all of them.*

(ii) *f is Kähler.*

(iii) *f is projective.*

Proof. (iii) \Rightarrow (ii) is trivial. (ii) \Rightarrow (i) follows from Lemma 2.2.

For (i) \Rightarrow (iii), suppose $f : A \rightarrow B$ is a homomorphism of abelian groups satisfying the properties of (i).

We first prove the case that A and B are both free of rank $2r$ and $2s$. Then f is represented by a matrix, which we can assume to be a Smith normal form by a proper choice of basis. Hence f factors into three steps: a projection to a direct summand, an injection with finite cokernel, and an injection as a direct summand. Therefore, we will define a homomorphism $\phi : X \rightarrow Y$ between some abelian varieties as a composition of three steps, a projection to a direct summand, a finite covering map, and an injection as a direct summand. The induced map of the fundamental groups $\phi_* : \pi(X) \rightarrow \pi(Y)$ will be isomorphic to f .

Suppose the Smith normal form of f is represented by the $2r \times 2s$ matrix

$$\begin{pmatrix} a_1 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & a_{2l} & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix}.$$

We choose X to be the r -dimensional torus

$$\mathbb{C}^r / (a_1\mathbb{Z} + a_2i\mathbb{Z}) + \dots + (a_{2l-1}\mathbb{Z} + a_{2l}i\mathbb{Z}) + (\mathbb{Z} + i\mathbb{Z})^{r-l}.$$

The first step is projection to the torus of the first l components

$$\mathbb{C}^l / (a_1\mathbb{Z} + a_2i\mathbb{Z}) + \dots + (a_{2l-1}\mathbb{Z} + a_{2l}i\mathbb{Z}).$$

In the second step, we take the covering map onto the torus $\mathbb{C}^l / (\mathbb{Z} + i\mathbb{Z})^l$ as a further quotient. Then we take the embedding to $\mathbb{C}^s / (\mathbb{Z} + i\mathbb{Z})^s$, which will be our Y . All the tori we chose are abelian varieties, since they are products of elliptic curves.

The case that only B is free follows from the previous case and Lemma 2.3, since a homomorphism from a finite group to a free abelian group must be zero, and a finitely generated abelian group is a direct sum of its free part and its torsion part.

The general case follows from the previous case and the previous lemma. \square

3. NONPROJECTIVE EXAMPLE

In this section, we construct a Kähler homomorphism which is not projective.

In [V], Voisin proved the following:

Lemma 3.1. *Suppose an $n(\geq 2)$ -dimensional complex torus T has an endomorphism ϕ . Then ϕ_* acts on $H_1(T, \mathbb{Z})$. Let p be the characteristic polynomial of ϕ_* . If p is irreducible, has no real roots and the Galois group of its splitting field acts as the symmetric group on the roots, then the torus T is not an abelian variety.*

Voisin also gave the method to construct such a torus. A concrete example for $n = 2$ can be found in [H].

Suppose (T, ϕ) satisfies the assumption of the lemma, and suppose W is a compact Kähler manifold with $\pi_1(W)$ isomorphic to the symmetric group S_3 . Denote the universal cover of W by \tilde{W} . Let S_3 act on $(T \times T \times T) \times \tilde{W}$ by permutation on the first part and by deck transformation on the second. Since the action is fixed point free, we get a Kähler manifold as the quotient $(T \times T \times T) \times \tilde{W}/S_3$. Fixing any point P on \tilde{W} , we define a sequence of maps:

$$T \times T \xrightarrow{\mu'} T \times T \times T \xrightarrow{i} T \times T \times T \times \tilde{W} \xrightarrow{\pi} T \times T \times T \times \tilde{W}/S_3,$$

where we let the first map $\mu' : (a, b) \mapsto (a + b, \phi(a) + b, b)$, and i is mapping to the fiber over $P \in \tilde{W}$, and π is the quotient map.

Let μ be the composition $\pi \circ i \circ \mu'$. Then the induced map μ_* of fundamental groups is Kähler. But it cannot be projective.

Proposition 3.2. *μ_* is not projective.*

Proof. Suppose $\nu : X \rightarrow Y$ is a holomorphic map between compact Kähler manifolds, which induces the same homomorphism of fundamental groups as μ . We will show that Y cannot be projective.

Since

$$\pi_1(Y) \cong \pi_1((T \times T \times T) \times \tilde{W}/S_3) \cong (\mathbb{Z}^{2n} \times \mathbb{Z}^{2n} \times \mathbb{Z}^{2n}) \rtimes S_3$$

we can take \tilde{Y} as the 6-fold covering of Y corresponding to the surjection $\mathbb{Z}^{2n} \times \mathbb{Z}^{2n} \times \mathbb{Z}^{2n} \rtimes S_3 \rightarrow S_3$. S_3 will act on \tilde{Y} . Notice the composition of μ_* and that the surjection $\mathbb{Z}^{2n} \times \mathbb{Z}^{2n} \times \mathbb{Z}^{2n} \rtimes S_3 \rightarrow S_3$ is trivial. Hence $\nu : X \rightarrow Y$ factors through $X \xrightarrow{\nu'} \tilde{Y} \rightarrow Y$. The choice of ν' is not unique, but we will specify which one we want later.

Consider the induced map of Albanese: $Alb(\nu') : Alb(X) \rightarrow Alb(\tilde{Y})$, where $Alb(\tilde{Y})$ has an S_3 action on it induced by the action on \tilde{Y} . Let Σ be the fixed locus of $Alb(\tilde{Y})$ by this S_3 action. Then Σ is an n -dimensional subtorus. Also let Γ_1, Γ_2 and Γ_3 be the fixed loci by the three order 2 elements $s_1 = (23), s_2 = (31), s_3 = (12)$ of S_3 respectively. Therefore Γ_i corresponds to the permutation $s_i \in S_3$ fixing the i -th T . Therefore, s_3 gives an isomorphism between Γ_1 and Γ_2 . Let Γ_4 be the image of $Alb(\nu')$.

Since, as maps of the fundamental groups, μ_* and ν_* are isomorphic, lifting to the 6-fold covering, $\mu'_* \cong (i \circ \mu')_*$ must be isomorphic to ν'_* . Therefore as real Lie groups, we can identify $T \times T \times T$ with $Alb(\tilde{Y})$ and hence parametrize $Alb(\tilde{Y})$ by elements in T . For instance, $\Sigma = \{(b, b, b)\}$, $\Gamma_1 = \{(a + b, b, b)\}$, $\Gamma_2 = \{(b, a + b, b)\}$. We choose ν' in such a way that according to the above identification, $Alb(\nu')$ is the same as $Alb(\mu')$. Then $\Gamma_4 = \{(a + b, \phi(a) + b, b)\}$. Therefore we can take quotients of those subtori of $Alb(\tilde{Y})$ by Σ . Denote the quotients with a bar on top. Now we get n -dimensional subtori $\overline{\Gamma}_i$ in the $2n$ -dimensional torus $\overline{Alb} \stackrel{\text{def}}{=} \overline{Alb(\tilde{Y})}$. Obviously, $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = 0$, and we have a decomposition $\overline{Alb} = \overline{\Gamma}_1 \oplus \overline{\Gamma}_2$. Projection to each component restricting to $\overline{\Gamma}_4$ gives two homomorphisms of tori, $\overline{\Gamma}_4 \xrightarrow{pr_i} \overline{\Gamma}_i$, $i = 1, 2$, and pr_1 is an isomorphism.

Since s_3 permutes Γ_1 and Γ_2 and fixes Σ , it induces an isomorphism between $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$, which we also call s_3 by abuse of notation. Now, consider the composition

$$\overline{\Gamma}_1 \xrightarrow{pr_1^{-1}} \overline{\Gamma}_4 \xrightarrow{pr_2} \overline{\Gamma}_2 \xrightarrow{s_3} \overline{\Gamma}_1.$$

It is the same as ϕ via identifying $\overline{\Gamma}_1$ with T as real tori. Here $\overline{\Gamma}_4$ plays the role of the graph.

By the previous lemma, $\overline{\Gamma}_1$ is not an abelian variety. Consequently, none of Γ_1 , $Alb(\tilde{Y})$, \tilde{Y} or Y can be a projective variety. Therefore μ_* is not a projective homomorphism. □

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