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RELATIVELY POINTWISE RECURRENT GRAPH MAP

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ABSTRACT. Let f be a self-continuous map of a graph G. Let $P(f)$ and $R(f)$ denote the sets of periodic points and recurrent points respectively. We say that the map f is *relatively recurrent* if $\overline{R(f)} = G$. In this paper, it is shown that f is relatively recurrent if and only if one of the following statements holds:

(a) G is a circle and f is a homeomorphism topologically conjugate to an irrational rotation of the unit circle $\mathbb{S}^1$;
(b) $\overline{P(f)} = G$.

Part (b) extends a result of Blokh.

1. INTRODUCTION

A *topological dynamical system* is a pair (X, f), where X is a compact metric space and f is a continuous map from X to itself. Let $\mathbb{N}$ be the set of positive integers. Let f^0 be the identity map of X. Define, inductively, $f^n = f \circ f^{n-1}$ for any non-zero positive integer n. For $x \in X$, $\{f^n(x) : n \in \mathbb{N}\}$ is called the *orbit of* x and is denoted by $O(x, f)$. Here x is *periodic* if $f^n(x) = x$ for some non-zero positive integer n. Also, x is called a *recurrent point* of f if for any neighborhood U of x and any $m \in \mathbb{N}$ there exists $n > m$ such that $f^n(x) \in U$. It is easy to see that if x is recurrent, then every iterate of x is also recurrent. The converse is false. Here x is called an *almost periodic point* of f if for any neighborhood U of x there exists $N \in \mathbb{N}$ such that $\{f^{n+i}(x) : i = 0, 1, ..., N\} \cap U \neq \emptyset$ for all $n \in \mathbb{N}$. Also, x is a *non-wandering point* of f provided that for any open set U containing x there exist $y \in U$ and $n \in \mathbb{N}$ such that $f^n(y) \in U$. Let $P(f)$, $AP(f)$, $R(f)$ and $\Omega(f)$ denote the set of periodic points, almost periodic points, recurrent points and non-wandering points respectively. Notice that $\Omega(f)$ is closed and for the general topological system (X, f) there are no further relations except for $P(f) \subset AP(f) \subset R(f) \subset \Omega(f)$. But for one-dimensional systems one can say more. For a dendrite map Illanes [8] proved that $\overline{P(f)} = \overline{R(f)}$ if and only if the dendrite does not contain any copy of the Gehman dendrite. For a graph map Mai and Shao [9] showed that $\overline{R(f)} = \overline{P(f)} \cup R(f)$. In Lemma 3.1, for a graph map, we will show that $\overline{AP(f)} = \overline{R(f)}$.

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It is interesting to study maps such that $P(f)$ or $R(f)$ satisfies some additional properties. Montgomery [12], Weaver [15], Epstein [4], and others studied homeomorphisms such that $P(f)$ is the whole space X (X is a connected manifold or a continuum embedded in a 2-manifold). For an interval map, Nitecki [13] showed that if $P(f)$ is closed, then $P(f) = \Omega(f)$. For a graph map Mai [11] proved that $R(f)$ is the whole space X if and only if one of the following statements holds:

(1) X is a circle, and f is a homeomorphism topologically conjugate to an irrational rotation of the unit circle;
(2) f is a periodic homeomorphism.

In this paper we will study a relatively pointwise recurrent graph map. Our main result is the following theorem:

Main Theorem. *Let G be a graph and let $f : G \to G$ be a continuous map. Then f is relatively pointwise recurrent if and only if one of the following two statements holds:*

(1) *G is a circle, and f is a homeomorphism topologically conjugate to an irrational rotation of the unit circle $\mathbb{S}^1$;*
(2) $\overline{P(f)} = G$.

A set $W \subset X$ is called a *minimal set* of f if it is nonempty, closed, invariant ($f(W) \subset W$) and no proper subset of W satisfies these three properties, which is equivalent to the fact that the orbit of every element of W is dense.

In a topological dynamical system there are two well-known theorems which exhibit the close relationship between almost periodic points and minimal sets; see Birkhoff [2]. In fact, if X is a locally compact metric space, then the set of almost periodic points $AP(f)$ is the union of all minimal sets of f.

If X has no isolated point, then f is a *transitive map* if it has a dense orbit. If every orbit of f is dense in X, the map f is called *minimal*. For a transitive graph map Blokh [3] proved that $P(f)$ is dense. In this paper we show that for a graph map if $R(f)$ is dense, then $P(f)$ is also dense, which extends this result of Blokh [3] (see Corollary 3.4).

2. Relatively pointwise recurrent map

A map f of a compact metric space (X, d) is *recurrent* if it admits iterates arbitrarily close to the identity, i.e. if there exists a sequence $n_k \to +\infty$ such that

$$d(f^{n_k}, Id) \to 0 \quad as \quad k \to +\infty.$$

We say that f is *pointwise recurrent* if $R(f) = X$.

We continue, motivated by a desire to understand the mechanics of recurrent maps, by a desire to extend some known result and by the following:

In [6] Gottschalk proved that if X is a compact connected metric space, f is a homeomorphism, and if $\overline{R(f)} = X$, then every recurrent cut point of X is periodic.

We start by the following definition.

Definition. f is called a *relatively pointwise recurrent map* if $\overline{R(f)} = X$.

For example a transitive map is a relatively pointwise recurrent map.

Let X be a closed domain of finite volume of the n-Euclidian space $\mathbb{R}^n$ or the n-torus $\mathbb{T}^n$ and let f be an invertible volume-preserving self-map of X. Then f is a relatively pointwise recurrent map [7, Theorem 6.1.9].

We have the following implications:

recurrent $\Rightarrow^1$ pointwise recurrent $\Rightarrow^2$ relatively pointwise recurrent.

The following examples show that none of these implications can be reversed.

Examples 2.1. (1) An irrational rotation of $\mathbb{R}^2$ is a pointwise recurrent non-recurrent map.

(2) In $\mathbb{R}^2$ we consider the points $A_n(0, \frac{1}{n})$ and $B(n, \frac{1}{n})$ for every integer $n > 0$. Put $X = (\bigcup_{n>0} [A_n, B_n]) \cup [0, +\infty[\times\{0\}$. Define the map f on X by:

- $f(x, \frac{1}{n}) = (\varphi(x), \frac{1}{n})$, where

$$\varphi(x) = \begin{cases} x+1 & \text{if} \quad x \leq n-1, \\ x+1-n & \text{if} \quad n-1 < x \leq n. \end{cases}$$

- $f(x, 0) = (x+1, 0)$.

f is a relatively pointwise recurrent non-pointwise recurrent map.

In this example one can choose X to be connected.

Theorem 2.2. *If a continuous map of a topological space X to itself is either* (1) *recurrent,* (2) *pointwise recurrent, or* (3) *relatively pointwise recurrent, then so is f^k, for each integer k.*

Proof. (1) and (2) follow from [14] and [6] respectively.

We have $R(f) \subset R(f^k)$, and so if $\overline{R(f)} = X$, then $\overline{R(f^k)} = X$, which implies (3). $\square$

Proposition 2.3. *If f is relatively pointwise recurrent, then f is surjective.*

Proof. If $y \in R(f)$, then there exists a point $x \in X$ such that $f(x) = y$.

Let y be an element of $X - R(f)$. Then there exists a sequence y_n of $R(f)$ which converges to y. For all n there exits x_n such that $f(x_n) = y_n$. Since X is compact, the sequence x_n has a limit point x and so $f(x) = y$. Therefore f is surjective. $\square$

Proposition 2.4. *If f is relatively pointwise recurrent, then $\Omega(f) = X$.*

Proof. Let x be an element of X. If x is a recurrent point, it is also non-wandering. If x is a non-recurrent point, then every neighborhood of x contains a recurrent point, and so it is in $\Omega(f)$. $\square$

The referee noticed that the converse of Proposition 2.4 also holds; see, for example, [5, Theorem 1.27].

Proposition 2.5. *If f is a relatively pointwise recurrent non-recurrent map, then f is not equicontinuous.*

Proof. Since f is a relatively pointwise recurrent non-recurrent map, then $R(f)$ is not closed and so is not equal to $\Omega(f)$. From [10, Proposition 2.1] it follows that f is not equicontinuous. $\square$

The following proposition can be derived from [10, Proposition 2.1].

Proposition 2.6. *If f is an equicontinuous relatively pointwise recurrent map, then f is a pointwise recurrent map.*

3. Proof of main theorem

Before going into the proof of the main theorem we recall the definition of a graph. A *(finite) graph* G is a compact connected Hausdorf space which contains a finite non-empty set V (the set of *vertices*), such that every connected component of $G \setminus V$ is homeomorphic to an open interval of the real line. These connected components are called *edges*. Since any graph can be embedded in $\mathbb{R}^3$, in what follows we will consider each graph endowed with the topology induced by the topology of $\mathbb{R}^3$. A *graph map* is a continuous map from a graph G to itself.

To prove the main theorem we need the following lemmas:

Lemma 3.1. *Let f be a graph map. Then $\overline{R(f)} = \overline{AP(f)}$.*

Proof. We recall that $\overline{AP(f)}$ is the closure of the union of all minimal sets of f and $\overline{AP(f)} \subset \overline{R(f)}$.

In [1, Theorem 1] the authors showed that a minimal set of a graph map is a finite set or a Cantor set or a union of (finitely many) pairwise disjoint circles. One can deduce that each component of $G - (\overline{AP(f)} \cup V)$ is an open arc of G.

Let $]a, b[$ be a component of $G - (\overline{AP(f)} \cup V)$ such that $a \in \overline{AP(f)}$. We suppose that $]a, b[\cap R(f) \neq \emptyset$. Let x be an element of $]a, b[\cap R(f)$. Since x is recurrent, there exists an increasing sequence (n_q) such that $(f^{n_q}(x))$ converges to x. There exist three integers $i < j < k$ such that one of the following two statement holds:

(1) $a < x < f^{n_k}(x) < f^{n_j}(x) < f^{n_i}(x)$.
(2) $a < f^{n_i}(x) < f^{n_j}(x) < f^{n_k}(x) < x$.

By applying [9, Lemma 2.2] we obtain:

(1) $\Rightarrow f^{n_k}(a) \in]a, b[$.
(2) $\Rightarrow f^{n_j - n_i}(a) \in]a, b[$.

In both cases the interval $]a, b[$ will intersect $\overline{AP(f)}$, which is impossible. □

Lemma 3.2. *Let f be a relatively pointwise recurrent map of a graph G. If W is a proper minimal set of f, then W is not a union of (finitely many) pairwise disjoint circles.*

Proof. In [1, Theorem 1] the authors showed that a minimal set of a graph map is a finite set or a Cantor set or a union of (finitely many) pairwise disjoint circles.

Suppose that W is a union of (finitely many) pairwise disjoint circles. Then G is not a circle (because $W \neq G$). From the fact that G is connected it follows that W contains a branching point w. Let A be an arc of G such that $A \cap W = \{w\}$. Since $\overline{R(f)} = G$, there exists in A a sequence (w_n) of recurrent points which converges to w. From the fact that $O(w, f)$ is dense in W it follows that there exists an integer p such that $f^p(w)$ is not a branching point and so by continuity of f^p, there exists an integer N such that $f^p(w_n) \in W$ for all $n > N$. The recurrence of w_n implies that there exists an integer $q > p$ such that $f^q(w_n) \in A - \{w\}$, which implies that $f^{q-p}(f^p(w_n)) \in A - \{w\}$, which contradicts the fact that W is invariant. □

Lemma 3.3. *Let f be a relatively pointwise recurrent map of a graph G. If f is not a minimal map, then $\overline{P(f)} = \overline{AP(f)}$.*

Proof. We always have the inclusion $\overline{P(f)} \subset \overline{AP(f)}$.

By applying [1, Theorem 1] and Lemma 3.2 it follows that every minimal set of f is a periodic orbit or a Cantor set. Let W be a minimal set which is a

Cantor set. Let w be an element of W and let w' be an element of $G - V(G)$ such that the open arc (w, w') does not meet W. If $(w, w') \cap P(f) = \emptyset$, then from the fact that $(w, w') \cap R(f) \neq \emptyset$ and by applying [9, Lemma 2.2] it follows that $(w, w') \cap \overline{O(w, f)} \neq \emptyset$, which is impossible. Thus $w \in \overline{P(f)}$. Since $P(f)$ is invariant, $\overline{O(w, f)} \subset \overline{P(f)}$ and so $W \subset \overline{P(f)}$. Therefore $\overline{AP(f)} = \overline{P(f)}$. □

Proof of the main theorem. The two statements imply that f is a relatively pointwise recurrent map.

Conversely, (1) if f is a minimal map, then first by [11, Theorem 3.2] it follows that G is a circle, and second by [11, Lemma 3.1] it follows that f is a homeomorphism. From Proposition 2.4 it follows that f is a pointwise non-wandering circle map without periodic points. Thus it is topologically conjugate to an irrational rotation.

(2) If f is not a minimal map, then from Lemma 3.1 and Lemma 3.3 it follows that $\overline{P(f)} = G$. □

The fact that the set of periodic points of a transitive graph map is dense was proved by Blokh in [3]. The following corollary extends this result.

Corollary 3.4. *Let f be a relatively pointwise recurrent map of a graph G. If f is not a minimal map, then $\overline{P(f)} = G$.*

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