PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 139, Number 6, June 2011, Pages 2217–2225
S 0002-9939(2010)10627-5
Article electronically published on November 10, 2010

DENDRITES AS POLISH STRUCTURES

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(Communicated by Julia Knight)

ABSTRACT. It is shown that standard universal dendrites under the action of their group of homeomorphisms give rise to small Polish structures. Moreover, any non-singleton dendrite forming a small Polish structure (or, more generally, having at least one uncountable orbit) under the action of its group of homeomorphisms has $\mathcal{NM}$-rank 1. Finally, dendrites satisfy the existence of nm-independent extensions.

1. Introduction and basic notions

In [K10] the definition of a Polish structure is given as a pair (X, G), where G is a Polish group acting faithfully on the set X in such a way that the stabilisers of singletons are closed.

If (X, G) is a Polish structure and $A \subseteq X$, denote by G_A the pointwise stabiliser of A. A Polish structure (X, G) is *small* if for every $n \geq 1$ there are only countably many orbits of the action of G on X^n. In particular, in an uncountable small Polish structure there are uncountable orbits.

The following is implicitly used in [K10].

Lemma 1. *A Polish structure (X, G) is small if and only if for any $a_1, \ldots, a_n \in X$, the action of $G_{\{a_1,\ldots,a_n\}}$ on X has countably many orbits.*

Proof. Suppose the action of $G_{\{a_1,\ldots,a_n\}}$ on X has uncountably many orbits and let K be a transversal for the orbit equivalence relation. Then all elements of $\{(a_1, \ldots, a_n, k)\}_{k\in K}$ are in different orbits of the action of G on X^{n+1}.

Conversely, suppose the action of G on some X^{n+1} has uncountably many orbits and let n be minimal with this property. If $n = 0$, then there is nothing more to prove, so assume $n > 0$. As the actions of G on X^n and on X have countably many orbits, there are $a_1, \ldots, a_n, b \in X$ such that $G(a_1, \ldots, a_n) \times Gb$ contains uncountably many orbits of the action of G on X^{n+1}. Since each such orbit contains an element of the form $(a_1, \ldots, a_n, c)$, it follows that the action of $G_{\{a_1,\ldots,a_n\}}$ on X has uncountably many orbits. □

Though the definition of a Polish structure (X, G) does not require X to be a topological space, an important class of Polish structures is obtained when X is a compact metric space and G is the group of homeomorphisms of X equipped with

Received by the editors December 8, 2009 and, in revised form, May 5, 2010; June 1, 2010; and June 4, 2010.

2010 *Mathematics Subject Classification.* Primary 03C45, 03E15, 54F15.

Key words and phrases. Small Polish structure, dendrite, universal dendrite.

the compact-open topology and acting on X in the natural way. Among compact metric spaces, dendrites constitute some of the simplest examples: a dendrite is a compact, connected, locally connected metric space that does not contain simple closed curves. Definitions and basic properties about dendrites can be found in [N92]; those most needed in this paper are collected in section 2, for further reference.

This note investigates Polish structures of the form (D, G), where D is a dendrite and G its group of homeomorphisms acting on D in the natural way; when metric considerations will involve the group G, the supremum distance on G is subsumed. A dendrite D will be said to be *small* if the Polish structure (D, G) is small.

Not all dendrites are small: let D be a planar dendrite such that the set of branch points of D is $(]0,1[\cap\mathbb{Q}) \times \{0\}$ with distinct branch points having different orders. Then all points in $[0,1] \times \{0\}$ lie in different orbits, so D is not small. Notice that in this example there are uncountable orbits under homeomorphism, since D contains open free arcs. For another example, let D be a dendrite with a dense set of branch points, all of distinct order (such a dendrite can be obtained as in the construction in [N92, 10.37] of Ważewski's universal dendrite, but taking care that the branch points have pairwise different order). Then D is rigid.

Let (X, G) be a Polish structure, $\vec{a} = (a_1, \ldots, a_n) \in X^n$, and let $A \subseteq B$ be finite subsets of X. According to [K10], we say that $\vec{a}$ is nm-dependent on B over A if $\{g \in G_A \mid g\vec{a} \in G_B\vec{a}\}$ is meagre in G_A; otherwise, $\vec{a}$ is nm-independent from B over A. Using this, define a function $\mathcal{NM}$ from the set of pairs $(\vec{a}, A)$, with $\vec{a}$ in some X^n and A a finite subset of X, to the ordinals satisfying $\mathcal{NM}(\vec{a}, A) \geq \alpha + 1$ if and only if there is a finite B with $A \subseteq B \subseteq X$ such that $\vec{a}$ is nm-dependent on B over A and $\mathcal{NM}(\vec{a}, B) \geq \alpha$. The $\mathcal{NM}$-rank of (X, G) is the supremum of all $\mathcal{NM}(a, \emptyset)$, for a ranging in X. Actually in [K10] this definition is given only in the case of Polish structures admitting nm-independent extensions, to grant some good properties of $\mathcal{NM}$-ranks; the notation employed here differs slightly from the one used there.

In section 3 it will be shown that the so-called standard universal dendrites are small. Section 4 will establish that whenever D is a dendrite with at least one uncountable orbit, then its $\mathcal{NM}$-rank is 1. In section 5 it will be proved that any dendrite D admits nm-independent extensions: this means that for any $\vec{a} \in D^n$ and finite subsets $A, B \subseteq D$ with $A \subseteq B$, there is $\vec{b} \in G_A\vec{a}$ such that $\vec{b}$ is nm-independent from B over A.

2. Review of dendrites

For convenience, this section collects some definitions, properties and notation of dendrites that will be used in the sequel. A reference or a sketchy justification is also provided.

(1) For D a dendrite, denote by $E(D)$ the set of its end points and by $R(D)$ the set of its branch points. This last set is countable for all dendrites ([N92, Theorem 10.23]).
(2) The order of a point x in D will be denoted by $ord(x, D)$. Then, $ord(x, D) \leq \aleph_0$ for any $x \in D$ ([N92, Corollary 10.20.1]).
(3) Since D is arcwise connected and contains no simple closed curves, given $x, y \in D$ with $x \neq y$, there is a unique subarc of D with end points x, y. It will be denoted by A^D_{xy}.

(4) Every sequence of subdendrites of a dendrite pairwise meeting in at most one point has vanishing diameter. Otherwise, one could find a sequence of arcs A_n, pairwise intersecting in at most one point, converging to an arc A. By the condition on the A_n, the diameters of $A \cap A_n$ converge to 0. Let p, q be distinct points in A and let U, V be arcwise connected neighbourhoods of p, q, respectively, with diameters less than $\frac{1}{2}d(p, q)$. So there is n with $U \cap A_n \neq \emptyset \neq V \cap A_n$ and at least one of $A_n \cap A \cap U, A_n \cap A \cap V$ is empty. But then the arc-connectedness of U and V yields at least two arcs joining p and q.

(5) Every point p in a dendrite D has a neighbourhood basis whose members are dendrites whose boundaries in D have finite cardinality. Indeed, by the regularity of D, there is an open neighbourhood basis of p whose members have finite boundaries. By local connectedness, the connected components of open sets are open, so for each of such neighbourhoods consider the closure of the connected component containing p.

(6) If C is a subdendrite of D, denote by $r_C : D \to C$ the first point map for C ([N92, §10.3]).

3. Standard universal dendrites are small

Following [CD94], if J is a non-empty subset of $\{3, 4, \dots, \omega\}$ let D_J be the unique (up to homeomorphism) dendrite such that:

- if $a \in R(D_J)$, then $ord(a, D_J) \in J$;
- for any arc $I \subseteq D_J$ and any $n \in J$ there is $a \in I$ such that $ord(a, D_J) = n$.

The dendrite D_J has the following universality property: if D is any dendrite such that $\forall x \in D\ \exists n \in J\ ord(x, D) \leq n$, then there is a subset of D_J homeomorphic to D. This section (Lemma 3 through Theorem 7) is intended to establish the following result.

Theorem 2. *The Polish structure (D_J, G), where G is the group of homeomorphisms of D_J acting on it in the natural way, is small.*

To begin with, a standard back and forth argument gives the following.

Lemma 3. *Let U, V be arcs, with end points a, b and c, d, respectively. Let $\{U_n\}_{n \in \mathbb{N}}$ and $\{V_n\}_{n \in \mathbb{N}}$ be pairwise disjoint countable dense subsets of $U \setminus \{a, b\}$ and $V \setminus \{c, d\}$, respectively. Then there is a homeomorphism $g : U \to V$ such that:*

- *$g(a) = c, g(b) = d$,*
- *$\forall n \in \mathbb{N}\ g(U_n) = V_n$.*

Lemma 4. *Let a, d be distinct points of D_J and $b, c \in A_{ad}^{D_J} \setminus \{a, d\}$ be such that $ord(b, D_J) = ord(c, D_J)$. Then there is a homeomorphism $\varphi : D_J \to D_J$ such that $\varphi(a) = a, \varphi(b) = c, \varphi(d) = d$.*

Proof. Lemma 3 gives homeomorphisms $\zeta_0 : A_{ab}^{D_J} \to A_{ac}^{D_J}, \zeta_1 : A_{bd}^{D_J} \to A_{cd}^{D_J}$ such that $\zeta_0(a) = a, \zeta_0(b) = \zeta_1(b) = c, \zeta_1(d) = d$ and $ord(x, D_J) = ord(\zeta_i(x), D_J)$ for $i \in \{0, 1\}, x \in dom\zeta_i$. Let $\theta = \zeta_0 \cup \zeta_1 : A_{ad}^{D_J} \to A_{ad}^{D_J}$.

For each $u \in R(D_J) \cap A_{ad}^{D_J}$ there are either $ord(u, D_J) - 2$, if $ord(u, D_J)$ is finite, or $\aleph_0$ connected components $\{F_{un}\}_n$ of $D_J \setminus \{u\}$ disjoint from $A_{ad}^{D_J}$; moreover, each $D_{un} = F_{un} \cup \{u\}$ is homeomorphic to D_J by [CD94, Theorem 6.2] and has u as an end point. Fix a homeomorphism $\varphi_{un} : D_{un} \to D_{\theta(u)n}$ such that $\varphi_{un}(u) = \theta(u)$ (its existence can again be justified by [CD94, Theorem 6.2]). Then define

$$\varphi(x) = \begin{cases} \theta(x) & \text{if} \quad x \in A_{ad}^{D_J}, \\ \varphi_{un}(x) & \text{if} \quad x \in D_{un}. \end{cases}$$

Since for every $\varepsilon \in \mathbb{R}^+$ all but finitely many D_{un} have diameter less than ε, function φ is continuous. □

Lemma 5. *Let a be an end point of D_J and $b, c \in D_J \setminus \{a\}$ be such that neither of $A_{ab}^{D_J}, A_{ac}^{D_J}$ is a subarc of the other and $ord(b, D_J) = ord(c, D_J)$. Then there is a homeomorphism $\varphi : D_J \to D_J$ such that $\varphi(a) = a, \varphi(b) = c$.*

Proof. As a is an end point, $A_{ab}^{D_J} \cap A_{ac}^{D_J}$ is an arc. So, let $e \in D_J$ such that $A_{ab}^{D_J} \cap A_{ac}^{D_J} = A_{ae}^{D_J}$. Let f, g be end points of D_J such that $A_{ab}^{D_J} \subseteq A_{af}^{D_J}, A_{ac}^{D_J} \subseteq A_{ag}^{D_J}$. Using Lemma 3, construct a homeomorphism $\theta : A_{ef}^{D_J} \to A_{eg}^{D_J}$ such that

$$\theta(e) = e, \quad \theta(b) = c, \quad \forall x \in A_{ef}^{D_J} \; ord(x, D_J) = ord(\theta(x), D_J).$$

For each $u \in ((A_{ef}^{D_J} \cup A_{eg}^{D_J}) \cap R(D_J)) \setminus \{e\}$, let $\{F_{un}\}_n$ be an enumeration of the connected components of $D_J \setminus \{u\}$ disjoint from $A_{af}^{D_J} \cup A_{ag}^{D_J}$ and let $D_{un} = F_{un} \cup \{u\}$, which is homeomorphic to D_J. For $u \in (A_{ef}^{D_J} \cap R(D_J)) \setminus \{e\}$ fix homeomorphisms $\varphi_{un} : D_{un} \to D_{\theta(u)n}$ with $\varphi_{un}(u) = \theta(u)$. Finally, define:

$$\varphi(x) = \begin{cases} \theta(x) & \text{if } x \in A_{ef}^{D_J} \setminus \{e\}, \\ \theta^{-1}(x) & \text{if } x \in A_{eg}^{D_J} \setminus \{e\}, \\ \varphi_{un}(x) & \text{if } x \in D_{un}, u \in (A_{ef}^{D_J} \cap R(D_J)) \setminus \{e\}, \\ \varphi_{un}^{-1}(x) & \text{if } x \in D_{\theta(u)n}, u \in (A_{eg}^{D_J} \cap R(D_J)) \setminus \{e\}, \\ x & \text{otherwise.} \end{cases}$$

Function φ is a homeomorphism, similarly to the proof of Lemma 4. □

Corollary 6. *Let X, Y both be homeomorphic to D_J. Let $a, b \in X, c, d \in Y$ such that $a \neq b, c \neq d, ord(a, X) = ord(c, Y), ord(b, X) = ord(d, Y)$. Then there is a homeomorphism $\varphi : X \to Y$ such that $\varphi(a) = c, \varphi(b) = d$.*

Proof. Let $\{X_n\}_{n<ord(a,X)}, \{Y_n\}_{n<ord(c,Y)}$ be enumerations of the connected components of $X \setminus \{a\}, Y \setminus \{c\}$, respectively, with $b \in X_0, d \in Y_0$. Set $H_n = X_n \cup \{a\}, K_n = Y_n \cup \{c\}$: these are all homeomorphic to D_J. Let $\varphi_n : H_n \to K_n$ be a homeomorphism such that $\varphi_n(a) = c$. By applying Lemma 4 or Lemma 5, let $\psi : H_0 \to H_0$ be a homeomorphism such that $\psi(a) = a, \psi(b) = \varphi_0^{-1}(d)$. Set $\varphi = \varphi_0\psi \cup \bigcup_{n>0} \varphi_n$. Then φ is a homeomorphism, since for any $\varepsilon \in \mathbb{R}^+$ the diameters of H_n and K_n are eventually less than ε. □

Theorem 7. *Let $A = \{a_1, \ldots, a_n\} \subseteq D_J$ and let $H = G_A$ be the group of homeomorphisms of D_J fixing $a_1, \ldots, a_n$. Then the action of H on D_J has countably many orbits.*

Proof. It can be assumed that $n \geq 2$. Let T be the smallest subcontinuum of D_J containing $a_1, \ldots, a_n$. So T is a subtree of D_J; notice that $E(T) \subseteq A$. By enlarging A, if necessary, it can also be assumed that $R(T) \subseteq A$. Let $E_1, \ldots, E_m$ be subarcs of D_J such that, letting u_l, v_l be the end points of E_l:

- $u_l, v_l \in A$ for all l;

- each element of A is an end point of some E_l;
- if $l \neq l'$, then $E_l, E_{l'}$ intersect at most at one of their end points;
- $T = \bigcup_{l=1}^{m} E_l$.

For $l \in \{1, \ldots, m\}$, let $F_l = E_l \setminus \{u_l, v_l\}$. The statement will be proved by establishing the following claim.

Claim. The orbits of D_J under the action of H are:

(0) each singleton in A;
(1) each set $\{x \in F_l \mid ord(x, D_J) = k\}$ for $l \in \{1, \ldots, m\}, k \in \{2\} \cup J$;
(2) each set $\{x \in D_J \setminus \{a\} \mid r_T(x) = a, ord(x, D_J) = k\}$, for $a \in A, k \in \{1, 2\} \cup J$;
(3) each set $\{x \in D_J \setminus F_l \mid r_T(x) \in F_l, ord(r_T(x), D_J) = h, ord(x, D_J) = k\}$, for $l \in \{1, \ldots, m\}, h \in J, k \in \{1, 2\} \cup J$. □

Proof of claim. First notice that these sets are invariant under the action of H and their union is D_J. It remains to show that for any pair x, y of points in each of these, there is $\varphi \in H$ with $\varphi(x) = y$.

(1) If $x, y \in F_l$ are such that $ord(x, D_J) = ord(y, D_J)$, let $X = r_T^{-1}(F_l) \cup \{u_l, v_l\}$. Notice that X is homeomorphic to D_J, since each subarc of X contains points of all orders in J. So the claim follows by applying Lemma 4 to find a homeomorphism ψ of X fixing u_l, v_l and sending x to y; then define $\varphi : D_J \to D_J$ as being equal to ψ on X and to the identity on $D_J \setminus r_T^{-1}(F_l)$: this φ is continuous by the glueing lemma.
(2) If $a \in A$, let $X = r_T^{-1}(\{a\})$, which (if not a singleton) is homeomorphic to D_J. Let $x, y \in X \setminus \{a\}$ be such that $ord(x, D_J) = ord(y, D_J)$. Use Corollary 6 to establish a homeomorphism $\psi : X \to X$ such that $\psi(a) = a, \psi(x) = y$. Let $\varphi : D_J \to D_J$ agree with ψ on X and be the identity elsewhere.
(3) Let $x, y \in D_J \setminus F_l$ be such that
   - $r_T(x), r_T(y) \in F_l$,
   - $ord(r_T(x), D_J) = ord(r_T(y), D_J)$,
   - $ord(x, D_J) = ord(y, D_J)$.

   Applying Lemma 3, let $\theta : E_l \to E_l$ be a homeomorphism fixing the end points, such that $\forall z \in E_l$, $ord(z, D_J) = ord(\theta(z), D_J)$ and such that $\theta r_T(x) = r_T(y)$. For each $z \in (F_l \cap R(D_J)) \setminus \{r_T(x)\}$, fix a homeomorphism $\varphi_z : r_T^{-1}(\{z\}) \to r_T^{-1}(\{\theta(z)\})$ such that $\varphi_z(z) = \theta(z)$. Using Corollary 6, also let $\varphi_{r_T(x)} : r_T^{-1}(\{r_T(x)\}) \to r_T^{-1}(\{r_T(y)\})$ be a homeomorphism with $\varphi_{r_T(x)}(r_T(x)) = r_T(y), \varphi_{r_T(x)}(x) = y$. Now define the bijection $\varphi : D_J \to D_J$ as follows:

$$\varphi(u) = \begin{cases} u & \text{if} \quad u \notin r_T^{-1}(F_l), \\ \theta(u) & \text{if} \quad u \in F_l, \\ \varphi_z(u) & \text{if} \quad u \in r_T^{-1}(F_l) \setminus F_l, r_T(u) = z. \end{cases}$$

   Again, by the vanishing of the diameters of the $r_T^{-1}(\{z\})$ the continuity of φ follows. □

4. Ranks of dendrites

Fix a dendrite D and denote by G its group of homeomorphisms. The goal of this section is to show that if D has at least one uncountable orbit (in particular, if D is small), then the $\mathcal{NM}$-rank of (D,G) is 1.

Recall from [K10, Theorem 2.5(3)] that points $a \in Acl(A)$, that is, points whose orbits are countable under the action of G_A for some finite A, are nm-independent from B over A for any finite B with $A \subseteq B$. Consequently, if the orbit of a under G is countable, then $\mathcal{NM}(a,\emptyset)=0$. In particular, this holds for branch points of D. So it will be enough to compute $\mathcal{NM}(a,\emptyset)$ when $a \in D$ is such that $ord(a,D) \leq 2$.

Lemma 8. *Let (X,H) be any Polish structure, $\vec{a} \in X^n$ and let A,B be finite subsets of X with $A \subseteq B$. Suppose there is i such that $H_A a_i$ is uncountable and $H_B a_i$ is countable. Then $\vec{a}$ is nm-dependent on B over A. In particular $\mathcal{NM}(\vec{a},A) \geq 1$.*

Proof. Let $H_B a_i = \{h_0 a_i, h_1 a_i, \dots\}$ where each h_j is in H_B. In order to show that $\{g \in H_A \mid g\vec{a} \in H_B\vec{a}\}$ is meagre in H_A, observe that

$$\begin{aligned}
\{g \in H_A \mid g\vec{a} \in H_B\vec{a}\} &= \{g \in H_A \mid \vec{a} \in \{g^{-1}h_0\vec{a}, g^{-1}h_1\vec{a},\dots\}\}\\
&= \bigcup_j \{g \in H_A \mid \vec{a} = g^{-1}h_j\vec{a}\}\\
&= \bigcup_j \{g \in H_A \mid g^{-1}h_j \in H_{\{a_1,\dots,a_n\}}\}\\
&= \bigcup_j (h_j H_{\{a_1,\dots,a_n\}} \cap H_A)\\
&\subseteq \bigcup_j (h_j H_{\{a_i\}} \cap H_A).
\end{aligned}$$

Each term appearing in this last countable union, a coset of the stabiliser of a_i in H_A, is closed and is nowhere dense in H_A, since the index of $H_{\{a_i\}} \cap H_A$ in H_A is uncountable. □

Lemma 9. *Let (X,H) be any Polish structure, $\vec{a} \in X^n$ and let A,B be finite subsets of X with $A \subseteq B$. If for all i the orbit $H_A a_i$ is countable, then $\vec{a}$ is nm-independent from B over A.*

Proof. Notice that the hypothesis implies that $H_A\vec{a}$ is countable. So one can use the remark after [K10, Proposition 3.4] stating that [K10, Theorem 2.5] holds for imaginary extensions as well.

For convenience, however, the direct proof similar to [K10, Theorem 2.5(3)] is as follows. The index of $H_{A\cup\{a_1,\dots,a_n\}}$ in H_A is countable, so $H_{A\cup\{a_1,\dots,a_n\}}$ is non-meagre in H_A. Consequently, $H_B H_{A\cup\{a_1,\dots,a_n\}}$ is also non-meagre in H_A. Now apply [K10, Proposition 2.3]. □

Lemma 10. *Let $a \in E(D)$ and let B be a finite subset of D with $a \notin B$. Then $\{g \in G \mid g(a) \in G_B a\}$ contains a neighbourhood of the identity; in particular, a is nm-independent from B over $\emptyset$.*

Proof. If a is isolated in Ga, let $\varepsilon \in \mathbb{R}^+$ be such that there is no other point of Ga within ε of a. Then $\{g \in G \mid g(a) \in G_B a\}$ contains the open sphere in G centered in the identity and radius ε. So assume a is not isolated in Ga.

Let T be the smallest subtree of D containing B. Denote $p = r_T(a)$. Let C be a subdendrite of D such that C is a neighbourhood of a with diameter less than $d(a,p)$ and the boundary of C in D has exactly one element, say q. Then $q \in A^D_{ap}$. Pick $b \in E(C)\setminus\{a,q\}$; the existence of b is granted by the fact that a is not isolated in Ga. Let $c = r_{A^D_{aq}}(b)$, call L the connected component of b in $D\setminus\{c\}$ and let $K = L\cup\{c\}$. Similarly, let L' be the connected component of a in $D\setminus\{c\}$ and set

$K' = L' \cup \{c\}$. Since K is a neighbourhood of b and K' is a neighbourhood of a, let $\varepsilon \in \mathbb{R}^+$ be such that

- the open ball centered in b and radius ε is contained in K,
- the open ball centered in a and radius ε is contained in K',
- the open ball centered in p and radius ε is disjoint from C.

Fix any homeomorphism f of D less than ε apart from the identity, in order to show $f(a) \in G_B a$. Notice that $f(b) \in K, f(a) \in K', f(p) \notin C$. Moreover, any arc having an end point in K' and the other in $D \setminus (K \cup K')$ has c as a unique common point with K. So $A^D_{ap} \cap A^D_{bc} = \{c\}$; then $A^D_{f(a)f(p)} \cap A^D_{f(b)f(c)} = \{f(c)\}$, and $c \in A^D_{f(a)f(p)}$. Since $A^D_{f(a)f(p)}$ has an end point in K' and meets $A^D_{f(b)f(c)}$ in $f(c)$, this implies that $f(c) = c$. Consequently $f(K') = K'$. So if $g : D \to D$ is defined as f on K' and as the identity on $D \setminus K'$, one has $g(a) = f(a), g \in G_B$, whence $f(a) \in G_B a$. □

Corollary 11. *If $a \in E(D)$ and the orbit of a is uncountable, then $\mathcal{NM}(a, \emptyset) = 1$.*

Proof. By Lemmas 8 and 10, for B a finite subset of D, point a is nm-dependent on B over $\emptyset$ if and only if $a \in B$. Taken any finite $B \subseteq D$ with $a \in B$, by Lemma 9, a is nm-independent from C over B for any finite C with $B \subseteq C$. □

Lemma 12. *Let $a \in D$ with $ord(a, D) = 2$ and let B be a finite subset of D such that $a \notin B$. Then $\{g \in G \mid g(a) \in G_B a\}$ contains a neighbourhood of the identity. In particular, a is nm-independent from B over $\emptyset$.*

Proof. By possibly enlarging B it can be assumed that B intersects both connected components of $D \setminus \{a\}$; say B_1, B_2 are such intersections. For $j \in \{1, 2\}$ let T_j be the smallest subtree of D containing B_j and set $p_j = r_{T_j}(a)$.

Case 1. There is a neighbourhood of a, of the form $A^D_{bc} \subseteq A^D_{p_1p_2}$, all of whose points have order 2 in D.

If $\varepsilon \in \mathbb{R}^+$ is such that the ε-neighbourhood of a is included in A^D_{bc} and f is a homeomorphism of D less than ε apart from the identity, let $a^* = f(a)$. Let g be equal to the identity on $D \setminus A^D_{bc}$ and define $g|_{A^D_{bc}}$ as a homeomorphism of A^D_{bc} such that $g(b) = b, g(c) = c, g(a) = a^*$. Then $g \in G_B$ and thus $f(a) \in G_B a$.

Case 2. Point a is the limit of a sequence of branch points of D lying on $A^D_{p_1a}$, but there is $q \in A^D_{ap_2} \setminus \{a\}$ such that A^D_{aq} does not contain any branch point of D (or symmetrically, switching p_1, p_2). Pick $s, s', s'', r \in A^D_{p_1q} \setminus \{p_1, q\}$ such that

$$A^D_{p_1s} \subset A^D_{p_1s'} \subset A^D_{p_1a} \subset A^D_{p_1s''} \subset A^D_{p_1r}.$$

Fix $\varepsilon \in \mathbb{R}^+$ such that:

- the ε-neighbourhood of p_1 is included in $r^{-1}_{A^D_{p_1p_2}}(A^D_{p_1s})$;
- the ε-neighbourhood of a is included in $r^{-1}_{A^D_{p_1p_2}}(A^D_{s's''})$;
- the ε-neighbourhood of r is included in $A^D_{s''q}$.

Let f be a homeomorphism of D less than ε apart from the identity. By the choice of ε, $f(r) \in A^D_{s''q}$. Since A^D_{ar} does not contain branch points, so $A^D_{f(a)f(r)}$ does not contain such points as well; once again using the choice of ε, $f(a) \in A^D_{as''}$. Since points in $A^D_{as''} \setminus \{a\}$ are not limits of a sequence of branch points, whereas a is such a limit, the equality $f(a) = a$ is obtained. So $f(a) \in G_B a$.

Case 3. Point a is the limit of a sequence in $R(D) \cap A^D_{p_1 a}$ and of a sequence in $R(D) \cap A^D_{ap_2}$.

Pick points $r_1, s, s', r_2 \in A^D_{p_1 p_2} \setminus \{p_1, p_2\}$ such that

$$A^D_{p_1 r_1} \subset A^D_{p_1 s} \subset A^D_{p_1 a} \subset A^D_{p_1 s'} \subset A^D_{p_1 r_2}.$$

Let $\varepsilon_1 \in \mathbb{R}^+$ be such that:

- the ε_1-neighbourhood of p_j is included in $r^{-1}_{A^D_{p_1 p_2}}(A^D_{p_j r_j})$, for $j \in \{1, 2\}$;
- the ε_1-neighbourhood of a is included in $r^{-1}_{A^D_{p_1 p_2}}(A^D_{ss'})$.

For $j \in \{1, 2\}$, pick $b_j \in A^D_{p_j a} \cap R(D)$ with $d(a, b_j) < \varepsilon_1$ and take $c_j \in r^{-1}_{A^D_{p_1 p_2}}(\{b_j\}) \setminus \{b_j\}$. Let $\varepsilon_2 < \varepsilon_1$ be such that the ε_2-neighbourhood of c_j is contained in $r^{-1}_{A^D_{p_1 p_2}}(\{b_j\})$ and let f be any homeomorphism of D less than ε_2 apart from the identity. Then $f(b_j) = b_j$, since $A^D_{c_j b_j} = A^D_{p_1 c_j} \cap A^D_{p_2 c_j}$ and $A^D_{f(c_j) b_j} = A^D_{f(p_1) f(c_j)} \cap A^D_{f(p_2) f(c_j)}$. Consequently, $f(r^{-1}_{A^D_{b_1 b_2}}(A^D_{b_1 b_2} \setminus \{b_1, b_2\})) = r^{-1}_{A^D_{b_1 b_2}}(A^D_{b_1 b_2} \setminus \{b_1, b_2\})$. Let $g : D \to D$ be equal to the identity on $r^{-1}_{A^D_{b_1 b_2}}(\{b_1, b_2\})$ and to f on $r^{-1}_{A^D_{b_1 b_2}}(A^D_{b_1 b_2} \setminus \{b_1, b_2\})$. Then $g \in G_B, g(a) = f(a)$, granting $f(a) \in G_B a$. □

Corollary 13. *If $a \in D, ord(a, D) = 2$ and the orbit of a is uncountable, then $\mathcal{NM}(a, \emptyset) = 1$.*

Proof. As for Corollary 11, but using Lemmas 8, 12 and 9. □

Corollary 14. *Let D be a dendrite. Then:*

- *if all orbits of D are countable, then $\mathcal{NM}(D) = 0$;*
- *if there is an uncountable orbit in D, then $\mathcal{NM}(D) = 1$.*

Proof. By Corollaries 11, 13 and the initial remark about points whose orbits are countable. □

5. Existence of independent extensions

One of the reasons for the importance of small Polish structures is that they satisfy the existence of nm-independent extensions: if the Polish structure (X, H) is small, $\vec{a} \in X^n$, and A, B are finite subsets of X with $A \subseteq B$, then there exists $\vec{b} \in H_A \vec{a}$ such that $\vec{b}$ is nm-independent from B over A. The proof of this is in [K10], together with the discussion of its significance and examples of non-small Polish structures that admit (or do not admit) nm-independent extensions.

The situation for dendrites is that they do satisfy this property, even non-small ones. So this section is concerned with proving the following theorem, which exploits again arguments such as those in Lemmas 10 and 12.

Theorem 15. *Let D be a dendrite and G its group of homeomorphisms. Then for all $\vec{a} \in D^n$, for all finite subsets $A, B \subseteq D$ with $A \subseteq B$, there is $\vec{b} \in G_A \vec{a}$ such that $\vec{b}$ is nm-independent from B over A.*

Proof. Given $\vec{a}, A, B$ as in the statement of the theorem, pick $\vec{b} \in G_A \vec{a}$ such that for each i, if $G_A a_i$ is uncountable, then $b_i \notin B$. The existence of $\vec{b}$ can be justified as follows: let i_0 be least such that $G_A a_{i_0}$ is uncountable but $a_{i_0} \in B$; then arbitrarily close to the identity there are elements of G_A that move a_{i_0}. By finiteness of B, it is possible to pick $g \in G_A$ so that $g(a_{i_0}) \notin B$ and, if $a_j \notin B$, then $g(a_j) \notin B$. Now $g\vec{a}$ has at least one component less than $\vec{a}$ having uncountable G_A-orbit and belonging to B. Continuing this way, the tuple $\vec{b}$ is recovered. Now the aim is to show that $\vec{b}$ is nm-independent from B over A.

Let $\varepsilon > 0$ be less than all distances between pairwise distinct elements of $B \cup \{b_1, \dots, b_n\}$. By (5) of section 2, for each $i \in \{1, \dots, n\}$ let D_i be a dendrite such that

- $diam(D_i) < \frac{\varepsilon}{2}$,
- D_i is a neighbourhood of b_i,
- the boundary of D_i in D is finite.

Let B' be the union of B and the boundaries of all D_i for $i \in \{1, \dots, n\}$. Now notice that for all i there is $\delta_i > 0$ such that $\{g \in G_A \mid g(b_i) \in G_{B'} b_i\}$ contains the δ_i-neighbourhood in G_A of the identity. Indeed, if $G_A b_i$ is countable (this includes the case $b_i \in B$), apply the proof of [K10, Theorem 2.5(3)]. If instead $G_A b_i$ is uncountable, then $ord(b_i, D) \leq 2$; now apply either Lemma 10 or Lemma 12 to get a δ_i-neighbourhood in G of the identity contained in $\{g \in G \mid g(b_i) \in G_{B'} b_i\}$ and thus the claim.

Let $\delta < \min(\delta_1, \dots, \delta_n)$ be such that for each i the δ-ball centered in b_i is contained in D_i. The proof of the theorem will be concluded by showing that for all $g \in G_A$, if g is less than δ apart from the identity, then there is $h \in G_{B'}$ with $h\vec{b} = g\vec{b}$. So fix such a g. Let $h_i \in G_{B'}$ be such that $g(b_i) = h_i(b_i)$. Notice that $g(b_i) \in D_i$; thus h_i is a homeomorphism of D fixing all points of B' and sending b_i to $g(b_i)$, both of these points being interior to D_i. This entails $h_i(D_i) = D_i$: for one inclusion, if $p \in D_i$ was such that $h_i(p) \notin D_i$, then $A^D_{pb_i} \cap B' = \emptyset$, while $h_i(A^D_{pb_i}) \cap h_i(B') = A^D_{h_i(p)h_i(b_i)} \cap B' \neq \emptyset$; the other inclusion uses a similar argument on h_i^{-1}. It is now enough to define $h : D \to D$ by letting

$$h(x) = \begin{cases} h_i(x) & \text{if } \quad x \in D_i, \\ x & \text{if } \quad x \notin \bigcup_{i=1}^{n} D_i. \end{cases}$$

□

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