

**A ONE-PARAMETER FAMILY OF PICK FUNCTIONS DEFINED
BY THE GAMMA FUNCTION AND RELATED TO THE
VOLUME OF THE UNIT BALL IN n -SPACE**

CHRISTIAN BERG AND HENRIK L. PEDERSEN

(Communicated by Walter Van Assche)

ABSTRACT. We show that

$$F_a(x) = \frac{\ln \Gamma(x+1)}{x \ln(ax)}$$

can be considered as a Pick function when $a \geq 1$, i.e. extends to a holomorphic function mapping the upper half-plane into itself. We also consider the function

$$f(x) = \left(\frac{\pi^{x/2}}{\Gamma(1+x/2)} \right)^{1/(x \ln x)}$$

and show that $\ln f(x+1)$ is a Stieltjes function and that $f(x+1)$ is completely monotonic on $]0, \infty[$. In particular, $f(n) = \Omega_n^{1/(n \ln n)}$, $n \geq 2$, is a Hausdorff moment sequence. Here Ω_n is the volume of the unit ball in Euclidean n -space.

1. INTRODUCTION AND RESULTS

Since the appearance of the paper [3], monotonicity properties of the functions

$$(1) \quad F_a(x) = \frac{\ln \Gamma(x+1)}{x \ln(ax)}, \quad x > 0, a > 0,$$

have attracted the attention of several authors in connection with monotonicity properties of the sequence $\{\Omega_n\}$ of volumes of the unit ball in Euclidean n -space. Recent papers about inequalities involving Ω_n are [2], [15], [18]. See also the survey paper [4].

Let us first consider the case $a = 1$. In [10] the authors proved that F_1 is a *Bernstein function*, which means that it is positive and has a completely monotonic derivative, i.e.,

$$(2) \quad (-1)^n F_1^{(n+1)}(x) \geq 0, \quad x > 0, n \geq 0.$$

This extended monotonicity and concavity are proved in [5] and [13] respectively.

We actually proved a stronger statement than (2), namely that the reciprocal function $x \ln x / \ln \Gamma(x+1)$ is a Stieltjes transform, i.e. belongs to the Stieltjes cone

Received by the editors December 10, 2009 and, in revised form, June 9, 2010.

2010 *Mathematics Subject Classification*. Primary 33B15; Secondary 30E20, 30E15.

Key words and phrases. Gamma function, completely monotonic function, Pick function.

Both authors acknowledge support by grant 272-07-0321 from the Danish Research Council for Nature and Universe.

\mathcal{S} of functions of the form

$$(3) \quad g(x) = c + \int_0^\infty \frac{d\mu(t)}{x+t}, \quad x > 0,$$

where $c \geq 0$ and μ is a non-negative measure on $[0, \infty[$ satisfying

$$\int_0^\infty \frac{d\mu(t)}{1+t} < \infty.$$

The result was obtained using the holomorphic extension of the function F_1 to the cut plane $\mathcal{A} = \mathbb{C} \setminus]-\infty, 0]$, leading to an explicit formula for the measure μ in (3). Our derivation used the fact that the holomorphic function $\log \Gamma(z)$ vanishes in \mathcal{A} only at the points $z = 1$ and $z = 2$, a result interesting in itself and included as an appendix in [10]. A simpler proof of the non-vanishing of $\log \Gamma(z)$ appeared in [11], where we also proved that F_1 is a Pick function and obtained the following representation formula:

$$(4) \quad F_1(z) = 1 - \int_0^\infty \frac{d_1(t)}{z+t} dt, \quad z \in \mathcal{A},$$

with

$$(5) \quad d_1(t) = \frac{\ln |\Gamma(1-t)| + (k-1) \ln t}{t((\ln t)^2 + \pi^2)} \quad \text{for } t \in]k-1, k[, \quad k = 1, 2, \dots,$$

and $d_1(t)$ tends to infinity when t approaches $1, 2, \dots$. Since $d_1(t) > 0$ for $t > 0$, (2) is an immediate consequence of (4).

We recall that a Pick function is a holomorphic function φ in the upper half-plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$ satisfying $\Im \varphi(z) \geq 0$ for $z \in \mathbb{H}$; cf. [12].

For $a = 2$ Anderson and Qiu proved in [5] that F_2 is strictly increasing on $[1, \infty[$, thereby proving a conjecture from [3]. Alzer proved in [2] that F_2 is concave on $[46, \infty[$. In [16] the concavity was extended to the optimal interval $] \frac{1}{2}, \infty[$.

We will now describe the main results of the present paper.

First a few words about notation. We use \ln for the natural logarithm but only applied to positive numbers. The holomorphic extension of \ln from the open half-line $]0, \infty[$ to the cut plane $\mathcal{A} = \mathbb{C} \setminus]-\infty, 0]$ is denoted $\text{Log } z = \ln |z| + i \text{Arg } z$, where $-\pi < \text{Arg } z < \pi$ is the principal argument. The holomorphic branch of the logarithm of $\Gamma(z)$ for z in the simply connected domain \mathcal{A} , equal to $\ln \Gamma(x)$ for $x > 0$, is denoted $\log \Gamma(z)$. This branch can also be defined as the integral

$$\log \Gamma(z) = \int_1^z \frac{\Gamma'(w)}{\Gamma(w)} dw, \quad z \in \mathcal{A},$$

where the integration is along the segment from 1 to z . The imaginary part of $\log \Gamma(z)$ is a continuous branch of argument of $\Gamma(z)$ which we denote $\arg \Gamma(z)$, i.e.,

$$\log \Gamma(z) = \ln |\Gamma(z)| + i \arg \Gamma(z), \quad z \in \mathcal{A}.$$

The expression

$$(6) \quad F_a(z) = \frac{\log \Gamma(z+1)}{z \text{Log}(az)}$$

clearly defines a holomorphic extension of (1) to $\mathcal{A} \setminus \{1/a\}$, and $z = 1/a$ is a simple pole unless $a = 1$, where the residue $\ln \Gamma(1 + 1/a)$ vanishes. Thus $z = 1$ is a removable singularity for F_1 .

Using the residue theorem we shall obtain:

Theorem 1.1. For $a > 0$ the function F_a has the integral representation

$$(7) \quad F_a(z) = 1 + \frac{\ln \Gamma(1 + 1/a)}{z - 1/a} - \int_0^\infty \frac{d_a(t)}{z + t} dt, \quad z \in \mathcal{A} \setminus \{1/a\},$$

where

$$(8) \quad d_a(t) = \frac{\ln |\Gamma(1 - t)| + (k - 1) \ln(at)}{t((\ln(at))^2 + \pi^2)} \quad \text{for } t \in]k - 1, k[, \quad k = 1, 2, \dots,$$

and $d_a(0) = 0, d_a(k) = \infty, k = 1, 2, \dots$. We have $d_a(t) \geq 0$ for $t \geq 0, a \geq 1/2$. (This is slightly improved in Remark 2.6 below.) Furthermore, F_a is a Pick function for $a \geq 1$ but not for $0 < a < 1$.

From this follows the monotonicity property conjectured in [16]:

Corollary 1.2. Assume $a \geq 1$. Then

$$(9) \quad (-1)^n F_a^{(n+1)}(x) > 0, \quad x > 1/a, n = 0, 1, \dots$$

In particular, F_a is strictly increasing and strictly concave on the interval $]1/a, \infty[$.

The function

$$(10) \quad f(x) = \left(\frac{\pi^{x/2}}{\Gamma(1 + x/2)} \right)^{1/(x \ln x)}$$

has been studied because the volume Ω_n of the unit ball in \mathbb{R}^n is

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(1 + n/2)}, n = 1, 2, \dots$$

We prove the following integral representation of the extension of $\ln f(x + 1)$ to the cut plane \mathcal{A} .

Theorem 1.3. For $z \in \mathcal{A}$ we have

$$(11) \quad \log f(z + 1) = -\frac{1}{2} + \frac{\ln(2/\sqrt{\pi})}{z} + \frac{\ln(\sqrt{\pi})}{\text{Log}(z + 1)} + \frac{1}{2} \int_1^\infty \frac{d_2((t - 1)/2)}{z + t} dt.$$

In particular, $1/2 + \ln f(x + 1)$ is a Stieltjes function and $f(x + 1)$ is completely monotonic.

We recall that completely monotonic functions $\varphi :]0, \infty[\rightarrow \mathbb{R}$ are characterized by Bernstein's theorem as

$$(12) \quad \varphi(x) = \int_0^\infty e^{-xt} d\mu(t),$$

where μ is a positive measure on $[0, \infty[$ such that the integrals above make sense for all $x > 0$.

We also recall that a sequence $\{a_n\}_{n \geq 0}$ of positive numbers is called a Hausdorff moment sequence if it has the form

$$(13) \quad a_n = \int_0^1 x^n d\sigma(x), \quad n \geq 0,$$

where σ is a positive measure on the unit interval. Note that $\lim_{n \rightarrow \infty} a_n = \sigma(\{1\})$. Hausdorff proved that these sequences are exactly the same as completely monotonic sequences; see [20, p. 108] or [8, p. 134]. It is easy to see that if φ is completely

monotonic with the integral representation (12), then $a_n = \varphi(n + 1), n \geq 0$, is a Hausdorff moment sequence, because

$$a_n = \int_0^\infty e^{-(n+1)t} d\mu(t) = \int_0^1 x^n d\sigma(x),$$

where σ is the image measure of $e^{-t} d\mu(t)$ under e^{-t} . Since $\lim_{x \rightarrow \infty} f(x+1) = e^{-1/2}$ we get

Corollary 1.4. *The sequence*

$$(14) \quad f(n + 2) = \Omega_{n+2}^{1/((n+2)\ln(n+2))}, n = 0, 1, \dots,$$

is a Hausdorff moment sequence tending to $e^{-1/2}$.

A Hausdorff moment sequence $\{a_n\}_{n \geq 0}$ is easily seen to be decreasing and convex, and it is even logarithmically convex, meaning that $a_n^2 \leq a_{n-1}a_{n+1}, n \geq 1$. In fact, by the Cauchy-Schwarz inequality

$$\begin{aligned} a_n^2 &= \left(\int_0^1 x^n d\sigma(x) \right)^2 \\ &= \left(\int_0^1 x^{(n-1)/2} x^{(n+1)/2} d\sigma(x) \right)^2 \leq \int_0^1 x^{n-1} d\sigma(x) \int_0^1 x^{n+1} d\sigma(x) = a_{n-1}a_{n+1}. \end{aligned}$$

The logarithmic convexity of $\{a_n\}_{n \geq 0}$ was obtained in [16] in a different way.

2. PROPERTIES OF THE FUNCTION F_a

In this section we will study the holomorphic extension (6) of the function F_a defined in (1). We shall use the following property of $\log \Gamma(z)$; cf. [10, Lemma 2.1].

Lemma 2.1. *We have, for any $k \geq 1$,*

$$\lim_{z \rightarrow t, \Im z > 0} \log \Gamma(z) = \ln |\Gamma(t)| - i\pi k$$

for $t \in]-k, -k + 1[$ and

$$\lim_{z \rightarrow t, \Im z > 0} |\log \Gamma(z)| = \infty$$

for $t = 0, -1, -2, \dots$

Lemma 2.2. *For $a > 0$ and $t \leq 0$ we have*

$$(15) \quad \lim_{y \rightarrow 0^+} \Im F_a(t + iy) = \pi d_a(-t),$$

where d_a is given by (8).

Proof. For $-1 < t < 0$ we get

$$\lim_{y \rightarrow 0^+} F_a(t + iy) = \frac{\ln \Gamma(1 + t)}{t(\ln(a|t|) + i\pi)};$$

hence $\lim_{y \rightarrow 0^+} \Im F_a(t + iy) = \pi d_a(-t)$. For $-k < t < -k + 1, k = 2, 3, \dots$, we find, using Lemma 2.1, that

$$\lim_{y \rightarrow 0^+} F_a(t + iy) = \frac{\ln |\Gamma(1 + t)| - i(k - 1)\pi}{t(\ln(a|t|) + i\pi)}.$$

Hence $\lim_{y \rightarrow 0^+} \Im F_a(t + iy) = \pi d_a(-t)$ also in this case.

For $t = -k, k = 1, 2, \dots$, we have

$$|F_a(-k + iy)| \geq \frac{|\ln |\Gamma(-k + 1 + iy)||}{| -k + iy| |\text{Log}(a(-k + iy))|} \rightarrow \infty$$

for $y \rightarrow 0^+$ because $\Gamma(z)$ has poles at $z = 0, -1, \dots$. Finally, for $t = 0$ we get (15) from the next lemma. □

Lemma 2.3. *For $a > 0$ we have*

$$\lim_{z \rightarrow 0, z \in \mathcal{A}} |F_a(z)| = 0.$$

Proof. Since $\log \Gamma(z + 1)/z$ has a removable singularity for $z = 0$, the result follows because $|\text{Log}(az)| \geq |\ln(a|z|)| \rightarrow \infty$ for $|z| \rightarrow 0, z \in \mathcal{A}$. □

Lemma 2.4. *For $a > 0$ we have the radial behaviour*

$$(16) \quad \lim_{r \rightarrow \infty} F_a(re^{i\theta}) = 1 \text{ for } -\pi < \theta < \pi,$$

and there exists a constant $C_a > 0$ such that for $k = 1, 2, \dots$ and $-\pi < \theta < \pi$,

$$(17) \quad |F_a((k + \frac{1}{2})e^{i\theta})| \leq C_a.$$

Proof. We first note that

$$(18) \quad F_a(z) = F_1(z) \frac{\text{Log}(z)}{\text{Log}(az)},$$

and since

$$\lim_{|z| \rightarrow \infty, z \in \mathcal{A}} \frac{\text{Log}(z)}{\text{Log}(az)} = 1,$$

it is enough to prove the results for $a = 1$. We do this by using a method introduced in [10, Prop. 2.4].

Define

$$R_k = \{z = x + iy \in \mathbb{C} \mid -k \leq x < -k + 1, 0 < y \leq 1\} \text{ for } k \in \mathbb{Z}$$

and

$$R = \bigcup_{k=0}^{\infty} R_k, \quad S = \{z = x + iy \in \mathbb{C} \mid x \leq 1, |y| \leq 1\}.$$

The function F_1 is continuous on the punctured circle $|z| = (k + \frac{1}{2})e^{i\theta}, \theta \in]-\pi, \pi[$, and by Lemma 2.1 it has limits for $\theta \rightarrow \pm\pi$. These limits are complex conjugate of each other, and therefore $|F_1|$ has a continuous extension to the circle $|z| = k + \frac{1}{2}$. Hence

$$(19) \quad M_k = \sup_{|\theta| < \pi} |F_1((k + \frac{1}{2})e^{i\theta})| < \infty$$

for each $k = 1, 2, \dots$. It is then enough to prove that M_k is bounded for $k \rightarrow \infty$.

Stieltjes ([19, formula 20]) found the following formula for $\log \Gamma(z)$ for z in the cut plane \mathcal{A} :

$$(20) \quad \log \Gamma(z + 1) = \ln \sqrt{2\pi} + (z + 1/2) \text{Log } z - z + \mu(z).$$

Here

$$\mu(z) = \sum_{n=0}^{\infty} h(z + n) = \int_0^{\infty} \frac{P(t)}{z + t} dt,$$

where $h(z) = (z + 1/2) \operatorname{Log}(1 + 1/z) - 1$ and P is periodic with period 1 and $P(t) = 1/2 - t$ for $t \in [0, 1[$. A derivation of these formulas can also be found in [6]. The integral above is improper, and integration by parts yields

$$(21) \quad \mu(z) = \frac{1}{2} \int_0^\infty \frac{Q(t)}{(z+t)^2} dt,$$

where Q is periodic with period 1 and $Q(t) = t - t^2$ for $t \in [0, 1[$. Note that by (21) μ is a completely monotonic function. For further properties of Binet’s function μ ; see [14].

We claim that

$$|\mu(z)| \leq \frac{\pi}{8} \text{ for } z \in \mathcal{A} \setminus S.$$

In fact, since $0 \leq Q(t) \leq 1/4$, we get for $z = x + iy \in \mathcal{A}$,

$$|\mu(z)| \leq \frac{1}{8} \int_0^\infty \frac{dt}{(t+x)^2 + y^2}.$$

For $x > 1$ we have

$$\int_0^\infty \frac{dt}{(t+x)^2 + y^2} \leq \int_0^\infty \frac{dt}{(t+1)^2} = 1,$$

and for $x \leq 1, |y| \geq 1$ we have

$$\int_0^\infty \frac{dt}{(t+x)^2 + y^2} = \int_x^\infty \frac{dt}{t^2 + y^2} < \int_{-\infty}^\infty \frac{dt}{t^2 + 1} = \pi.$$

Since

$$F_1(z) = 1 + \frac{\ln \sqrt{2\pi} + 1/2 \operatorname{Log} z - z + \mu(z)}{z \operatorname{Log} z},$$

for $z \in \mathcal{A}$, we immediately get (16) and

$$(22) \quad |F_1(z)| \leq 2$$

for all $z \in \mathcal{A} \setminus S$ for which $|z|$ is sufficiently large. In particular, there exists $N_0 \in \mathbb{N}$ such that

$$(23) \quad |F_1((k + \frac{1}{2})e^{i\theta})| \leq 2 \text{ for } k \geq N_0, (k + \frac{1}{2})e^{i\theta} \in \mathcal{A} \setminus S.$$

By continuity the quantity

$$(24) \quad c = \sup \{ |\log \Gamma(z)| \mid z = x + iy, \frac{1}{2} \leq x \leq 1, 0 \leq y \leq 1 \}$$

is finite.

We will now estimate the quantity $|F_1((k + \frac{1}{2})e^{i\theta})|$ when $(k + \frac{1}{2})e^{i\theta} \in S$, and since $F_1(\bar{z}) = \overline{F_1(z)}$, it is enough to consider the case when $(k + \frac{1}{2})e^{i\theta} \in R_{k+1}$. To do this we use the relation

$$(25) \quad \log \Gamma(z + 1) = \log \Gamma(z + k + 1) - \sum_{l=1}^k \operatorname{Log}(z + l)$$

for $z \in \mathcal{A}$ and $k \in \mathbb{N}$. Equation (25) follows from the fact that the functions on both sides of the equality sign are holomorphic functions in \mathcal{A} , and they agree on the positive half-line by repeated applications of the functional equation for the Gamma function.

For $z = (k + \frac{1}{2})e^{i\theta} \in R_{k+1}$ we get $|\log \Gamma(z + k + 1)| \leq c$ by (24), and hence by (25)

$$|\log \Gamma(z + 1)| \leq c + \sum_{l=1}^k |\text{Log}(z + l)| \leq c + k\pi + \sum_{l=1}^k |\ln |z + l||.$$

For $l = 1, \dots, k-1$ we have $k-l < |z+l| < k+2-l$; hence $0 < \ln |z+l| < \ln(k+2-l)$. Furthermore, $1/2 \leq |z+k| \leq \sqrt{2}$; hence $-\ln 2 < \ln |z+k| \leq (\ln 2)/2$. Inserting these inequalities, we get

$$|\log \Gamma(z + 1)| \leq c + k\pi + \sum_{j=2}^{k+1} \ln j < c + k\pi + k \ln(k + 1).$$

From this we get for $z = (k + \frac{1}{2})e^{i\theta} \in R_{k+1}$

$$(26) \quad |F_1(z)| \leq \frac{c + k\pi + k \ln(k + 1)}{(k + \frac{1}{2}) \ln(k + \frac{1}{2})},$$

which tends to 1 for $k \rightarrow \infty$. Combined with (23) we see that there exists $N_1 \in \mathbb{N}$ such that

$$|F_1((k + \frac{1}{2})e^{i\theta})| \leq 2 \text{ for } k \geq N_1, -\pi < \theta < \pi,$$

which shows that M_k from (19) is a bounded sequence. □

Lemma 2.5. *Let $a > 0$. For $k = 1, 2, \dots$ there exists an integrable function $f_{k,a} :]-k, -k + 1[\rightarrow [0, \infty[$ such that*

$$(27) \quad |F_a(x + iy)| \leq f_{k,a}(x) \text{ for } -k < x < -k + 1, 0 < y \leq 1.$$

Proof. For $z = x + iy$ as above we get using (25)

$$|\log \Gamma(z + 1)| \leq |\log \Gamma(z + k + 1)| + \sum_{l=1}^k |\text{Log}(z + l)| \leq L + k\pi + \sum_{l=1}^k |\ln |z + l||,$$

where L is the maximum of $|\log \Gamma(z)|$ for $z \in \overline{R_{-1}}$. We only treat the case $k \geq 2$ because the case $k = 1$ is a simple modification combined with Lemma 2.3.

For $l = 1, \dots, k - 2$ we have $1 < |z + l| < 1 + k - l$, and for $l = k - 1, k$ we have $\ln |x + l| \leq \ln |z + l| \leq (1/2) \ln 2$, so we find

$$(28) \quad |\log \Gamma(z + 1)| \leq L + k\pi + \sum_{j=2}^k \ln j + |\ln |x + k - 1|| + |\ln |x + k||,$$

so as $f_{k,1}$ we can use the right-hand side of (28) divided by $(k - 1) \ln(k - 1)$. Using (18) we next define

$$f_{k,a}(x) = f_{k,1}(x) \max_{z \in \overline{R_k}} \frac{|\text{Log } z|}{|\text{Log}(az)|}. \quad \square$$

Proof of Theorem 1.1. For fixed $w \in \mathcal{A} \setminus \{1/a\}$ we choose $\varepsilon > 0, k \in \mathbb{N}$ such that $\varepsilon < |w|, 1/a < k + \frac{1}{2}$ and consider the positively oriented contour $\gamma(k, \varepsilon)$ in \mathcal{A} consisting of the half-circle $z = \varepsilon e^{i\theta}, \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and the half-lines $z = x \pm i\varepsilon, x \leq 0$ until they cut the circle $|z| = k + \frac{1}{2}$, which closes the contour. By the residue theorem we find that

$$\frac{1}{2\pi i} \int_{\gamma(k, \varepsilon)} \frac{F_a(z)}{z - w} dz = F_a(w) + \frac{\ln \Gamma(1 + 1/a)}{1/a - w}.$$

We now let $\varepsilon \rightarrow 0$ in the contour integration. By Lemma 2.3 the contribution from the half-circle with radius ε will tend to zero, and by Lemma 2.2 and Lemma 2.5 we get

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{F_a((k + \frac{1}{2})e^{i\theta})}{(k + \frac{1}{2})e^{i\theta} - w} (k + \frac{1}{2})e^{i\theta} d\theta + \int_{-k-\frac{1}{2}}^0 \frac{d_a(-t)}{t - w} dt = F_a(w) + \frac{\ln \Gamma(1 + 1/a)}{1/a - w}.$$

For $k \rightarrow \infty$ the integrand in the first integral converges to 1 for each $\theta \in]-\pi, \pi[$ and by Lemma 2.4 Lebesgue’s theorem on dominated convergence can be applied, so we finally get

$$F_a(w) = 1 + \frac{\ln \Gamma(1 + 1/a)}{w - 1/a} - \int_0^\infty \frac{d_a(t)}{t + w} dt.$$

The last integral above appears as an improper integral, but we shall see that the integrand is Lebesgue integrable. We show below that $d_a(t) \geq 0$ when $a \geq 1/2$, and for these values of a the integrability is obvious. The function d_a tends to 0 for $t \rightarrow 0$ and has a logarithmic singularity at $t = 1$, so d_a is integrable over $]0, 1[$. For $k - 1 < t < k$, $k \geq 2$ we have

$$(29) \quad d_a(t) = \frac{(\ln(t))^2 + \pi^2}{(\ln(at))^2 + \pi^2} d_1(t) + \frac{(k - 1) \ln a}{t((\ln(at))^2 + \pi^2)},$$

and the factor in front of $d_1(t)$ is a bounded continuous function with limit 1 at 0 and at infinity. Therefore

$$\int_1^\infty \frac{|d_a(t)|}{t} dt < \infty$$

follows from the finiteness of the corresponding integral for $a = 1$ provided that we establish

$$K := \sum_{k=2}^\infty (k - 1) \int_{k-1}^k \frac{dt}{t^2((\ln(at))^2 + \pi^2)} < \infty.$$

Choosing $N \in \mathbb{N}$ such that $aN > 1$, we can estimate

$$\begin{aligned} K &< \sum_{k=1}^\infty \int_{ka}^{(k+1)a} \frac{dt}{t(\ln^2(t) + \pi^2)} < \int_a^{Na} \frac{dt}{t(\ln^2(t) + \pi^2)} + \sum_{k=N}^\infty \int_{ka}^{(k+1)a} \frac{dt}{t \ln^2(t)} \\ &= \int_a^{Na} \frac{dt}{t(\ln^2(t) + \pi^2)} + \frac{1}{\ln(aN)} < \infty. \end{aligned}$$

We next examine positivity of d_a .

For $0 < t < 1$ we have

$$d_a(t) = \frac{\ln |\Gamma(1 - t)|}{t((\ln(at))^2 + \pi^2)} > 0$$

because $\Gamma(s) > 1$ for $0 < s < 1$.

For $k \geq 2$ and $t \in]k - 1, k[$ the numerator N_a in d_a can be written as

$$N_a(t) = \ln \Gamma(k - t) + \sum_{l=1}^{k-1} \ln \frac{ta}{t - l},$$

where we have used the functional equation for Γ . Hence

$$N_a(t) \geq \sum_{l=1}^{k-1} \ln \frac{k}{k - l} + (k - 1) \ln a = (k - 1) \ln k - \ln \Gamma(k) + (k - 1) \ln a,$$

because $\Gamma(k - t) > 1$ and $t/(t - l)$ is decreasing for $k - 1 < t < k$. From (20) we get

$$(30) \quad \ln \Gamma(k) = \ln \sqrt{2\pi} + (k - 1/2) \ln k - k + \mu(k),$$

and in particular for $k = 2$

$$\mu(2) = 2 - \frac{3}{2} \ln 2 - \ln \sqrt{2\pi}.$$

Using (30) we find that

$$N_a(t) \geq k - \frac{1}{2} \ln k - \ln \sqrt{2\pi} - \mu(k) + (k - 1) \ln a \geq k - \frac{1}{2} \ln k - 2 + \frac{3}{2} \ln 2 + (k - 1) \ln a,$$

because μ is decreasing on $]0, \infty[$ as shown by (21).

For $a \geq 1/2$ and $k - 1 < t < k$ with $k \geq 2$ we then get

$$N_a(t) \geq k(1 - \ln 2) - \frac{1}{2} \ln k + \frac{5}{2} \ln 2 - 2 \geq 0,$$

because the sequence $c_k, k \geq 2$, on the right-hand side is increasing with $c_2 = 0$.

We also see that $d_a(t)$ tends to infinity for t approaching the end points of the interval $]k - 1, k[$. For $z = 1/a + iy, y > 0$, we get from (7)

$$\Im F_a(1/a + iy) = -\frac{\ln \Gamma(1 + 1/a)}{y} + \int_0^\infty \frac{y d_a(t)}{(1/a + t)^2 + y^2} dt.$$

The last term tends to 0 for $y \rightarrow 0$, while the first term tends to $-\infty$ when $0 < a < 1$. This shows that F_a is not a Pick function for these values of a . □

Remark 2.6. We proved in Theorem 1.1 that $d_a(t)$ is non-negative on $[0, \infty[$ for $a \geq 1/2$. This is not best possible, and we shall explain that the smallest value of a for which $d_a(t)$ is non-negative is $a_0 = 0.3681154742\dots$

Replacing k by $k + 1$ in the numerator N_a for d_a given by (8), we see that

$$N_a(t) = \ln |\Gamma(1 - t)| + k \ln(at) \text{ for } t \in]k, k + 1[, \quad k = 1, 2, \dots,$$

is non-negative if and only if

$$\ln(1/a) \leq \ln(k + s) + \frac{1}{k} \ln |\Gamma(1 - k - s)| \text{ for } s \in]0, 1[, \quad k = 1, 2, \dots,$$

and using the reflection formula for Γ this is equivalent to $\ln(1/a) \leq \rho(k, s)$ for all $0 < s < 1$ and all $k = 1, 2, \dots$, where

$$(31) \quad \rho(k, s) = \ln(k + s) - \frac{1}{k} \ln \left(\Gamma(k + s) \frac{\sin(\pi s)}{\pi} \right).$$

Using Stieltjes' formula (20), we find that

$$(32) \quad \rho(k, s) = 1 + \frac{\ln(\pi/2)}{2k} - (1/k) [(s - 1/2) \ln(s + k) + \ln \sin(\pi s) - s + \mu(s + k)]$$

for all $s \in]0, 1[$ and $k = 1, 2, \dots$. For fixed $s \in]0, 1[$ we see that $\rho(k, s) \rightarrow 1$ as $k \rightarrow \infty$, so $\ln(1/a) \leq 1$ is a necessary condition for non-negativity of $d_a(t)$. This condition is not sufficient, because for $\ln(1/a) = 1$ the inequality $1 \leq \rho(k, s)$ is equivalent to

$$0 \geq (1/2) \ln(2/\pi) + (s - 1/2) \ln(s + k) + \ln \sin(\pi s) - s + \mu(s + k),$$

which does not hold when k is sufficiently large and $1/2 < s < 1$.

For each $k = 1, 2, \dots$ it is easy to verify that the function $\rho_k(s) = \rho(k, s)$ has a unique minimum m_k over $]0, 1[$, and clearly

$$(33) \quad \ln(1/a_0) = \inf\{m_k, k \geq 1\}$$

determines the smallest value of a for which $d_a(t)$ is non-negative. Using Maple one obtains that m_k is decreasing for $k = 1, \dots, 510$ and increasing for $k \geq 510$ with limit 1. Therefore $m_{510} = \inf m_k = 0.9993586013\dots$ corresponding to $a_0 = 0.3681154742\dots$. We add that $m_1 = 1.6477352344\dots, m_{178} = 1.0000028637\dots, m_{179} = 0.9999936630\dots$

3. PROPERTIES OF THE FUNCTION f

Proof of Theorem 1.3. The function

$$\ln f(x) = \frac{(x/2) \ln \pi - \ln \Gamma(1 + x/2)}{x \ln x}$$

clearly has a meromorphic extension to $\mathcal{A} \setminus 1$ with a simple pole at $z = 1$ with residue $\ln 2$. We denote this meromorphic extension as $\log f(z)$ and have

$$\log f(z+1) = \frac{\ln \sqrt{\pi}}{\text{Log}(z+1)} - \frac{1}{2} F_2 \left(\frac{z+1}{2} \right).$$

Using the representation (7), we immediately get (11). It is well-known that $1/\text{Log}(z+1)$ is a Stieltjes function (cf. [9, p. 130]), and the integral representation is

$$(34) \quad \frac{1}{\text{Log}(z+1)} = \int_1^\infty \frac{dt}{(z+t)((\ln(t-1))^2 + \pi^2)}.$$

It follows that $\ln(\sqrt{e}f(x+1))$ is a Stieltjes function, in particular completely monotonic, showing that $\sqrt{e}f(x+1)$ belongs to the class \mathcal{L} of logarithmically completely monotonic functions studied in [17] and in [7]. Therefore $f(x+1)$ is also completely monotonic. \square

4. REPRESENTATION OF $1/F_a$

For $a > 0$ we consider the function

$$(35) \quad G_a(z) = 1/F_a(z) = \frac{z \text{Log}(az)}{\log \Gamma(z+1)},$$

which is holomorphic in \mathcal{A} with an isolated singularity at $z = 1$, which is a simple pole with residue $\ln a/\Psi(2) = \ln a/(1-\gamma)$ if $a \neq 1$, while it is a removable singularity when $a = 1$. Here $\Psi(z) = \Gamma'(z)/\Gamma(z)$ and γ is Euler's constant.

Theorem 4.1. *For $a > 0$ the function G_a has the integral representation*

$$(36) \quad G_a(z) = 1 + \frac{\ln a}{(1-\gamma)(z-1)} + \int_0^\infty \frac{\rho_a(t)}{z+t} dt, \quad z \in \mathcal{A} \setminus \{1\},$$

where

$$(37) \quad \rho_a(t) = t \frac{\ln |\Gamma(1-t)| + (k-1) \ln(at)}{(\ln |\Gamma(1-t)|)^2 + ((k-1)\pi)^2} \quad \text{for } t \in]k-1, k[, \quad k = 1, 2, \dots,$$

and $\rho_a(0) = 1/\gamma, \rho_a(k) = 0, k = 1, 2, \dots$, which makes ρ_a continuous on $[0, \infty[$. We have $\rho_a(t) \geq 0$ for $t \geq 0, a \geq a_0 = 0.3681154742\dots$ (cf. Remark 2.6), and $G_a(x+1)$ is a Stieltjes function for $a \geq 1$ but not for $0 < a < 1$.

Proof. We notice that for $-k < t < -k + 1$, $k = 1, 2, \dots$, we get using Lemma 2.1

$$\lim_{y \rightarrow 0^+} G_a(t + iy) = \frac{t(\ln(a|t|) + i\pi)}{\ln|\Gamma(1+t)| - i(k-1)\pi},$$

and for $t = -k$, $k = 1, 2, \dots$, we get

$$\lim_{y \rightarrow 0^+} |G_a(-k + iy)| = 0$$

because of the poles of Γ ; hence $\lim_{y \rightarrow 0^+} \Im G_a(t + iy) = -\pi\rho_a(-t)$ for $t < 0$.

For fixed $w \in \mathcal{A} \setminus \{1\}$ we choose $\varepsilon > 0$, $k \in \mathbb{N}$ such that $\varepsilon < |w|$, $1 < k + \frac{1}{2}$, and we consider the positively oriented contour $\gamma(k, \varepsilon)$ in \mathcal{A} , which was used in the proof of Theorem 1.1.

By the residue theorem we find that

$$\frac{1}{2\pi i} \int_{\gamma(k, \varepsilon)} \frac{G_a(z)}{z - w} dz = G_a(w) + \frac{\ln a}{(1 - \gamma)(1 - w)}.$$

We now let $\varepsilon \rightarrow 0$ in the contour integration. The contribution from the ε -half circle tends to 0, and we get

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{G_a((k + \frac{1}{2})e^{i\theta})}{(k + \frac{1}{2})e^{i\theta} - w} (k + \frac{1}{2})e^{i\theta} d\theta - \int_{-k-\frac{1}{2}}^0 \frac{\rho_a(-t)}{t - w} dt = G_a(w) + \frac{\ln a}{(1 - \gamma)(1 - w)}.$$

Finally, letting $k \rightarrow \infty$ we get (36), leaving the details to the reader. Clearly, $\rho_a \geq 0$ if and only if d_a defined in (8) is non-negative. It follows that $G_a(x + 1)$ is a Stieltjes function for $a \geq 1$ but not for $0 < a < 1$, since in the latter case $\Im G_a(1 + iy) > 0$ for $y > 0$ sufficiently small. \square

Remark 4.2. The integral representation in Theorem 4.1 was established in [10, (6)] in the case of $a = 1$. Since

$$G_a(z) = G_1(z) + \ln(a) \frac{z}{\log \Gamma(z + 1)},$$

the formula for G_a can be deduced from the formula for G_1 and the formula

$$(38) \quad \frac{z}{\log \Gamma(z + 1)} = \frac{1}{(1 - \gamma)(z - 1)} + \int_0^\infty \frac{\tau(t) dt}{z + t}, \quad z \in \mathcal{A} \setminus \{1\},$$

where

$$(39) \quad \tau(t) = \frac{(k - 1)t}{(\ln|\Gamma(1 - t)|)^2 + ((k - 1)\pi)^2} \quad \text{for } t \in]k - 1, k[, \quad k = 1, 2, \dots$$

REFERENCES

[1] N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*. English translation, Oliver and Boyd, Edinburgh, 1965. MR0184042 (32:1518)
 [2] H. Alzer, *Inequalities for the Volume of the Unit Ball in \mathbb{R}^n . II*, *Mediterr. J. Math.* **5** (2008), 395–413. MR2465568 (2009j:26018)
 [3] G. D. Anderson, M. K. Vamanamurthy, M. Vuorinen, *Special functions of quasiconformal theory*, *Expo. Math.* **7** (1989), 97–136. MR1001253 (90k:30032)
 [4] G. D. Anderson, M. K. Vamanamurthy, M. Vuorinen, *Topics in special functions*, *Papers on analysis*, 5–26, Rep. Univ. Jyväskylä Dep. Math. Stat., 83, 2001, Univ. Jyväskylä. MR1886609 (2003e:33001)
 [5] G. D. Anderson, S.-L. Qiu, *A monotonicity property of the gamma function*, *Proc. Amer. Math. Soc.* **125** (1997), 3355–3362. MR1425110 (98h:33001)
 [6] E. Artin, *The Gamma Function*. Holt, Rinehart and Winston, New York, 1964. MR0165148 (29:2437)

- [7] C. Berg, *Integral representation of some functions related to the Gamma function*, *Mediterr. J. Math.* **1** (2004), 433–439. MR2112748 (2005h:33002)
- [8] C. Berg, J. P. R. Christensen, P. Ressel, *Harmonic analysis on semigroups. Theory of positive definite and related functions*. Graduate Texts in Mathematics, vol. 100, Springer-Verlag, Berlin-Heidelberg-New York, 1984. MR747302 (86b:43001)
- [9] C. Berg, G. Forst, *Potential Theory on Locally Compact Abelian Groups*, Springer, Berlin-Heidelberg-New York, 1975. MR0481057 (58:1204)
- [10] C. Berg, H. L. Pedersen, *A completely monotone function related to the Gamma function*, *J. Comput. Appl. Math.* **133** (2001), 219–230. MR1858281 (2003k:33001)
- [11] C. Berg, H. L. Pedersen, *Pick functions related to the gamma function*, *Rocky Mount. J. Math.* **32** (2002), 507–525. MR1934903 (2004h:33007)
- [12] W. F. Donoghue Jr., *Monotone Matrix Functions and Analytic Continuation*. Springer-Verlag, Berlin-Heidelberg-New York, 1974. MR0486556 (58:6279)
- [13] A. Elbert, A. Laforgia, *On some properties of the Gamma function*, *Proc. Amer. Math. Soc.* **128** (2000), 2667–2673. MR1694859 (2000m:33002)
- [14] S. Koumandos, H. L. Pedersen, *Completely monotonic functions of positive order and asymptotic expansions of the logarithm of Barnes double gamma function and Euler’s gamma function*, *J. Math. Anal. Appl.* **355** (2009), 33–40. MR2514449 (2010e:33004)
- [15] C. Mortici, *Monotonicity properties of the volume of the unit ball in \mathbb{R}^n* , *Optim. Lett.* **4** (2010), 457–464.
- [16] Feng Qi, Bai-Ni Guo, *Monotonicity and logarithmic convexity relating to the volume of the unit ball*, arXiv:0902.2509v1[math.CA].
- [17] Feng Qi, Bai-Ni Guo, Chao-Ping Chen, *Some completely monotonic functions involving the gamma and polygamma functions*, *J. Aust. Math. Soc.* **80** (2006), no. 1, 81–88. MR2212317 (2007b:33004)
- [18] S.-L. Qiu, M. Vuorinen, *Some properties of the gamma and psi functions, with applications*, *Math. Comp.* **74** (2005), 723–742. MR2114645 (2005i:33002)
- [19] T. J. Stieltjes, *Sur le développement de $\log \Gamma(a)$* , *J. Math. Pures Appl., Sér. 4*, **5** (1889), 425–444. In *Collected Papers, Volume II*, pp. 215–234, edited by G. van Dijk, Springer, Berlin-Heidelberg-New York, 1993. MR1272017 (95g:01033)
- [20] D. V. Widder, *The Laplace Transform*. Princeton University Press, Princeton, NJ, 1941. MR0005923 (3:232d)

INSTITUTE OF MATHEMATICAL SCIENCES, UNIVERSITY OF COPENHAGEN, UNIVERSITETSPARKEN
5, DK-2100 KØBENHAVN Ø, DENMARK

E-mail address: `berg@math.ku.dk`

DEPARTMENT OF BASIC SCIENCES AND ENVIRONMENT, FACULTY OF LIFE SCIENCES, UNIVERSITY
OF COPENHAGEN, THORVALDSSENSVEJ 40, DK-1871 FREDERIKSBERG C, DENMARK

E-mail address: `henrikp@dina.kvl.dk`