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THE TITS BUILDING AND AN APPLICATION TO ABSTRACT CENTRAL EXTENSIONS OF p-ADIC ALGEBRAIC GROUPS BY FINITE p-GROUPS

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ABSTRACT. For a connected, semisimple, simply connected algebraic group G defined and isotropic over a field k, the corresponding Tits building is used to study central extensions of the abstract group $G(k)$. When k is a non-Archimedean local field and A is a finite, abelian p-group where p is the characteristic of the residue field of k, then with G of k-rank at least 2, we show that the group $H^2(G(k), A)$ of abstract central extensions injects into a finite direct sum of $H^2(H(k), A)$ for certain semisimple k-subgroups H of smaller k-ranks. On the way, we prove some results which are valid over a general field k; for instance, we prove that the analogue of the Steinberg module for $G(k)$ has no nonzero $G(k)$-invariants.

INTRODUCTION AND STATEMENT OF MAIN THEOREM

Certain problems on algebraic groups over global fields like the congruence subgroup problem involve the determination of topological central extensions of the adelic group which, in turn, leads naturally to the study of topological central extensions of p-adic Lie groups by finite groups like the group of roots of unity in the p-adic field. Moreover, central extensions of semisimple p-adic Lie groups often come from a subgroup of small rank like SL_2, which has the interesting property that abstract central extensions of the locally compact group $SL_2(k)$ for a p-adic field k by a finite group turn out to be automatically topological. Thus, it may be of some interest to look at abstract central extensions of p-adic Lie groups by finite groups. Let k be a non-Archimedean local field and let A be a finite, abelian group. Consider a connected, semisimple, simply connected algebraic group G defined over k. For the trivial action of $G(k)$ on A, one has the group $H^2(G(k), A)$ of abstract central extensions of the locally compact group $G(k)$ by the finite group and its subgroup $H^2_{top}(G(k), A)$ of topological central extensions. If G is quasi-split, these two groups coincide; this was noticed in [Su93]. The equality, in general, is a question posed by Gopal Prasad. In [PR84], Gopal Prasad and M.S. Raghunathan have proved, among several other things, that for a semisimple, simply connected k-isotropic algebraic group G, the group $H^2_{top}(G(k), A)$ of topological central extensions of $G(k)$ by any group A maps injectively under restriction maps into a

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direct sum of $H^2_{top}(H(k), A)$ over k-rank 1 subgroups H; they used the Bruhat-Tits building of G over k.

Here, we use the corresponding Tits buiding to prove an analogous injectivity result for the group $H^2(G(k), A)$ of abstract central extensions when the group A is a finite, abelian p-group, where p is the characteristic of the residue field of k. These ideas have also been employed [P82] by Gopal Prasad earlier in the context of groups over finite fields. Here, we imitate this in the context of local fields. Moreover, this implies that if the abstract central extensions are automatically topological for all groups of a certain k-rank $r \geq 1$, then the same holds for groups of k-rank $r+1$. Actually, many of the results proved on the way hold over arbitrary fields under some conditions (see remark (b) after Lemma 2 for a precise statement). For instance, Lemma 1 below shows for general k and A that the analogue of the Steinberg module for $G(k)$ with values in A has no nonzero $G(k)$-invariants. We prove the following theorem.

Theorem. *Let k be a non-Archimedean local field and A a finite, abelian p-group, where p is the characteristic of the residue field of k. Let G be an absolutely almost simple, simply connected algebraic group defined over k with k-rank$(G) = r \geq 2$. Then there exist semisimple k-subgroups $G_1, \dots, G_r$ without k-anisotropic factors and each of k-rank equal to k-rank$(G)-1$ and semisimple k-subgroups G_{ij} of $G_i \cap G_j$ such that the 'restriction' map*

$$H^2(G(k), A) \to \bigoplus_{i=1}^{r} H^2(G_i(k), A)$$

of abstract central extensions is injective and injects into

$$\mathrm{Ker}\Big(\bigoplus_{i \leq r} H^2(G_i(k), A) \to \bigoplus_{i<j} H^2(G_{ij}(k), A)\Big).$$

In particular, if the abstract central extensions are automatically topological for all k-subgroups of k-rank $r-1$, then the same holds for G.

Remarks. (i) If $G = SL_{n,D}$ for a k-central division algebra with $n \geq 3$, the theorem produces $G_1, \dots, G_{n-1}$, each of which is k-isomorphic to $SL_{n-1,D}$.

(ii) The technique cannot address k-anisotropic groups. For instance, it is an interesting hard, open problem to determine the central extensions of $SL(1, D)$ by $\mathbf{F}_p$, where D is a division algebra over a p-adic field.

1. Modus operandi: using the Tits building

In proving the theorem, we adopt the technique from ([P82]) and ([PR84], chapter 4). Basically, to compute $H^2(G(k), A)$ we use a natural injective resolution of the $G(k)$-module A provided by the Tits building associated to G over k. In [PR84], the Bruhat-Tits building is used to compute topological central extensions; note that the Bruhat-Tits building is contractible. We often work with general fields and crucially use a result due to Solomon and Tits [G73] asserting that the Tits building of G over k has the homotopy type of a bouquet of (possibly infinitely many) spheres, each of dimension $r-1$. In order to define the Tits building of G over k and to recall basic properties that we need from [BS76], let us recall the information provided by the Borel-Tits structure theory. Let k, G, r, A be as above and consider the trivial action of $G(k)$ on A. Let S be a maximal k-split torus of G and $T \supset S$, a maximal k-torus of G. Let Δ denote the corresponding set of simple k-roots and Φ^+ be the positive k-roots. For a subset Θ of Δ, write $S^\Theta = (\bigcap_{\theta\in\Theta} \operatorname{Ker}\theta)^\circ$ and $M^\Theta = C_G(S^\Theta)$. The latter is a connected reductive k-group in which S^Θ is the maximal k-split central torus. Note that $[M^\Theta, M^\Theta]$ is a semisimple, simply connected k-group. Further, if $\mathfrak{U}_\Theta = \sum_{\alpha\in\Phi^+-\langle\Theta\rangle} \mathfrak{g}_\alpha$ and if U_Θ is the corresponding connected unipotent group normalized by T, then observe that M^Θ normalizes U_Θ. Also, $P^\Theta := M^\Theta U_\Theta$ is a parabolic k-subgroup of G with U_Θ as its unipotent radical. Note that $\Theta = \emptyset$ corresponds to a minimal parabolic k-subgroup and the above correspondence between the set of subsets of Δ and the set of parabolic k-subgroups containing $P^\emptyset$ is bijective and inclusion-preserving.

In what follows, $H^n(G(k), A)$ stands for the abstract group cohomology $\operatorname{Ext}^n(G(k), A)$. To compute this, we shall use a resolution of the $G(k)$-module provided by the simplicial Tits building. We recall the definition and properties of the Tits building of G over k following Borel-Serre [BS73]. This is a simplicial complex of dimension $r-1$ (where $r = k\text{-rank}(G)$) whose vertices are maximal parabolic k-subgroups. A set $\{P_1, \ldots, P_d\}$ of vertices forms a simplex if and only if the intersection $\bigcap_{i=1}^d P_i$ is a parabolic k-subgroup; this parabolic k-subgroup is precisely the stabilizer of the simplex. $G(k)$ acts on parabolic k-subgroups by conjugation. As recalled above, once a simple system Δ of k-roots has been fixed (equivalently, an r-simplex of the Tits complex is fixed) the parabolic k-subgroups can be defined in terms of the subsets of this set Δ. In order to parametrize the simplices by parabolic k-subgroups, it is more convenient to work with $P_\Theta = P^{\Delta-\Theta}$ and $M_\Theta := M^{\Delta-\Theta}$, $U^\Theta := U_{\Delta-\Theta}$. Then the set of s-dimensional simplices in the Tits building is $G(k)$-equivariantly parametrized by $\bigcup_{|\Theta|=s+1} G(k)/P_\Theta(k)$. Since $G(k)$ acts simplicially on the Tits building, we have a complex of $G(k)$-modules

$$0 \to A \to C^0(A) \to C^1(A) \to \cdots \to C^{r-1}(A) \to 0,$$

where $C^i(A)$ is the group of simplicial i-cochains of the Tits building, with coefficients in A. Therefore, $C^i(A) = \bigoplus_{|\Theta|=i+1} \operatorname{Ind}_{P_\Theta(k)}^{G(k)}(A)$ as a $G(k)$-module. Here, $\operatorname{Ind}_{P_\Theta(k)}^{G(k)}(A)$ stands for the $G(k)$-module induced by the trivial action of $P_\Theta(k)$ on A. By a theorem of Solomon and Tits (see [G73], Appendix II), the Tits building of G over k is of the homotopy type of a bouquet of (possibly infinitely many) spheres, each of dimension $r-1$. Indeed, this was proved in [G73] for any Tits system with finite Weyl group; when the Weyl group of a Tits system is infinite, the corresponding Tits building is even contractible. Thus, the Bruhat-Tits building of G over k (which we have not defined as we do not need it here) is contractible, whereas the Tits building of G over k is not. So, the simplicial cohomology groups of the Tits building of G over k with coefficients in A are all zero except for the 0-th and the $(r-1)$-th ones. This top cohomology, denoted by $St(A)$, is called the Steinberg module of G over k with coefficients in A as in [P82]. Therefore, the $G(k)$-complex

$$0 \to A \to C^0(A) \to \cdots \to C^{r-1}(A) \to St(A) \to 0$$

is exact. The associated spectral sequence which computes $H^*(G(k), A)$ has its $E_2^{i,j}$-term to be the i-th cohomology of the complex

$$0 \to H^j\left(G(k), C^0(A)\right) \to H^j\left(G(k), C^1(A)\right) \to \cdots \to H^j(G(k), St(A)) \to 0.$$

Using Shapiro's lemma, this is just the complex

$$0 \to \bigoplus_{|\Theta|=1} H^j(P_\Theta(k), A) \to \cdots \to H^j(P_\Delta(k), A) \to H^j(G(k), St(A)) \to 0.$$

Throughout what follows, k is local and A is a p-group unless specified otherwise.

2. Key step: Reduction to parabolic k-subgroups

The proof of the theorem proceeds in steps, each step reducing the computation of the relevant cohomology groups to a computation for subgroups of a particular kind such as parabolic subgroups, then their Levi subgroups, and finally groups of smaller k-ranks. The crucial step is really the following proposition.

Proposition. $E_\infty^{1,1} = 0 = E_\infty^{2,0}$. *Further,*

$$H^2(G(k), A) \hookrightarrow \operatorname{Ker}(\bigoplus_{|\Theta|=1} H^2(P_\Theta(k), A) \to \bigoplus_{|\Theta|=2} H^2(P_\Theta(k), A)).$$

For the proof, we will require a result of independent interest which we prove for a general field k.

Lemma. *For an arbitrary field k, let $St(A)$ be defined as the top-dimensional cohomology with coefficients in A of the Tits building of G over k. Assume that if A has even order, then k is infinite. Then, we have $St(A)^{G(k)} = 0$.*

In the proof of this lemma, we shall use another result of independent interest. This is:

Sublemma. Let k be any infinite field and G be a semisimple, algebraic k-group which is k-isotropic; and let S be a maximal k-split torus and P be a minimal parabolic k-subgroup of G containing S. Let W denote the k-Weyl group and U^- denote the unipotent radical of the parabolic k-subgroup which is opposite to P. Then, $\bigcap_{w\in W} U^-(k)P(k)w \neq \emptyset$.

Proof. Now, $W \leq G(k)$ and for each $w \in W, U^-Pw$ is a nonempty open subvariety of G. Since G is irreducible, the finite intersection $\bigcap_{w\in W} U^-Pw$ is a nonempty open subvariety of G. Now if k is infinite, $G(k)$ is Zariski-dense in G, and hence $G(k)\cap\bigcap_{w\in W} U^-Pw$ is nonempty. But as $(U^-Pw)(k) = U^-(k)P(k)w$, we conclude that $\bigcap_{w\in W} U^-(k)P(k)w$ is nonempty. This proves our results. $\square$

Remarks. When k is finite, the above result may or may not hold. For example, if $G = SL_n$ over a field k which has at least n elements $a_1, \ldots, a_n$, and if P is the upper triangular Borel subgroup, then the intersection $\bigcap_{w\in W} U^-(k)P(k)w$ contains a matrix of the form

$$\begin{pmatrix} \lambda & \lambda & \cdots & \lambda \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \ddots & & & \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{pmatrix},$$

where $\lambda^{-1} = \prod_{i>j}(a_i - a_j)$. The author thanks K.N. Raghavan for the above example. On the other hand, if $|k| = 2$, then $SL_3(k)$ does not satisfy the property mentioned in the sublemma, as can be checked easily.

Proof of Lemma 1. Note that $St(A) \cong C^{r-1}(A)/\operatorname{Im}(C^{r-2}(A) \to C^{r-1}(A))$. Also, we observed that $C^{r-1}(A)$ is the set F of all functions $\phi : G(k) \to A$ which are right invariant under $P(k)$ (here, we denote the minimal parabolic k-subgroup P_Δ for simplicity as P). Also, for each simple reflection s_i (corresponding to each simple

k-root), there is a parabolic k-subgroup $P_i := P\langle s_i\rangle P$. From the set of functions from $G(k)$ to A which are right invariant under $P_i(k)$, there is evidently a natural map to F. Call the image F_i. Then, $St(A)$ can be identified with the quotient $F/\sum_i F_i$, and the action of $G(k)$ is induced by left multiplication. We will show that the only $G(k)$-invariant element in $F/\sum_i F_i$ is zero.

Define for any $\phi \in F$, the function $\tilde{\phi}$ from $G(k)/C_G(S)(k)$ to A as

$$\tilde{\phi}(x) := \sum_{w\in W} (-1)^{l(w)}\phi(xw).$$

Clearly this is well-defined and the map $\phi \mapsto \tilde{\phi}$ respects the left $G(k)$-action. Further, the image lies in the subgroup E of functions $\psi : G(k)/C_G(S)(k) \to A$ which satisfy

$$\psi(xw) = (-1)^{l(w)}\psi(x).$$

We claim that $\sum_i F_i$ maps to zero.

For this, note that for any simple reflection s_i, W is partitioned as $W = W_{s_i} \sqcup W_{s_i}s_i$, where $W_{s_i} = \{w \in W : l(ws_i) > l(w)\}$, and that $(-1)^{l(ws_i)} = -(-1)^{l(w)}$. Thus, elements from the image of F_i map to the zero function under $\phi \mapsto \tilde{\phi}$, and there is a map $\tilde{\pi}$ from $F/\sum_i F_i$ to E.

Consider the restriction map π_{res} from the group $F(G(k)/C_G(S)(k))$ of A-valued functions on $G(k)/C_G(S)(k)$ to the group $F(P(k)/C_G(S)(k))$ of A-valued functions on $P(k)/C_G(S)(k)$. This respects the left $P(k)$-actions. On the other hand, the group $F(P(k)/C_G(S)(k))$ is evidently isomorphic to the subgroup F_0 of F consisting of all elements which have support in $R_u(P)(k)w_0P(k)/P(k)$; the isomorphism is given by the mapping

$$\theta \mapsto \Big(uw_0P(k) \mapsto \theta(uC_G(S)(k))\Big).$$

Here, w_0 is the longest element. Now, the composite

$$(\spadesuit) \qquad F_0 \hookrightarrow F \to F/\sum_i F_i \to E \to F(P(k)/C_G(S)(k)) \to F_0 \cdots$$

is clearly the multiplication map by $(-1)^{l(w_0)}$. The above map from F_0 to $F/\sum_i F_i$ is surjective by Proposition 3.5 and Remark 3.6 of [BS76] and it also respects the $R_u(P)(k)$-action. For completeness, we give the argument here. We have the Bruhat decomposition $G(k) = \bigsqcup_w R_u(P)(k)wP(k)$. If $w \neq w_0$, then there exists a simple reflection s_i such that $l(ws_i) > l(w)$. As $R_u(P)(k)wP(k)s_iP(k) = R_u(P)(k)ws_iP(k)$, we can easily see that the map $R_u(P)(k)wP(k) \to R_u(P)(k)wP_i(k)$, obtained from the natural projection $G(k)/P(k) \to G(k)/P_i(k)$, is bijective. Hence, any right $P(k)$-invariant function with support in $R_u(P)(k)wP(k)$ extends uniquely to a $P_i(k)$-invariant function with support in $R_u(P)(k)wP_i(k)$. This latter function lies in F_i. In other words, we have $F = F_0 + \sum_i F_i$, which shows the surjectivity of the map from F_0 to $F/\sum_i F_i$. In particular, since the composite of the 5 maps in ($\spadesuit$) is an automorphism of F_0, the $G(k)$-equivariant map from $F/\sum_i F_i$ to E is injective. Hence, the $G(k)$-invariants in $F/\sum_i F_i$ map into the $G(k)$-invariants in E. But, the latter contains only constant functions on $G(k)/C_G(S)(k)$ as $G(k)$ acts transitively on $G(k)/C_G(S)(k)$. However, since the functions ψ in E satisfy $\psi(xs) = -\psi(x)$ for all $s \in W$ of order 2, it follows that if A has odd order, the only constant function ψ in E is the zero function. This completes the proof of the lemma when A has odd order. If A has even order, the same reasoning implies that the only possible nonzero constant functions in E are those which identically take the value 'a' for an element a of order 2. To rule out this final case when k is infinite, we use the above sublemma in the following manner. Indeed, suppose that a constant function a in E is the image of a function ϕ in F under the map from F to E above. We may assume that ϕ is in F_0; that is, it is supported on $R_u(P)(k)w_0P(k)/P(k)$. Now, if $g \in R_u(P)(k)w_0C_G(S)(k)$, then $gw \in R_u(P)(k)w_0wC_G(S)(k)$, and so $gw \cap R_u(P)(k)w_0P(k) = \emptyset$ unless w is the identity. Thus, $a = \widetilde{\phi}(g) = \phi(g)$, which means that ϕ is the constant function equal to a. Now, we use the sublemma to pick an element $g_0 \in R_u(P)(k)w_0P(k)$ such that $g_0w \in R_u(P)(k)w_0P(k)$ for all $w \in W$. Then

$$a = \tilde{\phi}(g_0) = a \sum_{w \in W} (-1)^{l(w)} = 0.$$

The lemma is proved. □

Proof of Proposition 1. The complex whose i-th cohomology (for $i > 0$) computes the $E_2^{i,0}$-th term of the spectral sequence is

$$0 \to \bigoplus_{|\Theta|=1} A \to \cdots \to \bigoplus_{|\Theta|=r} A \to St(A)^{G(k)} \to 0.$$

As

$$0 \to A \to \bigoplus_{|\Theta|=1} A \to \cdots \to \bigoplus_{|\Theta|=r} A \to 0$$

is simply the augmented simplicial cochain complex of an $(r-1)$-simplex, it is exact. Now, the lemma shows that the i-th cohomology $E_2^{i,0} = 0$ for all $i > 0$. Thus, we get

$$E_\infty^{2,0} = E_3^{2,0} = E_2^{2,0} / \operatorname{Im}(E_2^{0,1} \to E_2^{2,0}) = 0.$$

Towards proving $E_\infty^{1,1} = 0$, we need some information about $E_2^{1,1}$. As this involves $H^1(P(k), A)$ for parabolic k-subgroups P, we start by observing a special property of parabolic k-subgroups in our case, where k is local and A is a p-group. □

Lemma. *For any parabolic k-subgroup $P = P_\Theta$, consider the Levi subgroup $M = M_\Theta$ as above and let $D = [M, M]$. Then, $D(k)/[D(k), D(k)]$ is a finite abelian group of order prime to p and* $\operatorname{Hom}(P(k), A) \cong \operatorname{Hom}((P/[P, P])(k), A)$.

Proof. We will use here the assumption that G is simply connected. Look at the decomposition $P = M \propto U$ where U is the unipotent radical. We know that M, U are defined over k. Moreover, $D = [M, M]$ is a semisimple, simply connected k-group. We digress briefly to indicate how simple connectivity follows. Indeed, if $\alpha : S \to GL_1$ is a simple k-root and $\widehat{\alpha} : GL_1 \to S$ is the corresponding coroot, then consider the image $\widehat{S_\alpha}$ of the latter. The simple connectivity of G is equivalent to the product map $\prod_{\alpha \in \Delta} \widehat{S_\alpha} \to S$ being an isomorphism. Therefore, for each Θ as above, the product $\prod_{\alpha \in \Theta} \widehat{S_\alpha}$ maps isomorphically onto the corresponding maximal k-split torus of D, which shows that D is simply connected.

To continue with the proof, we shall first show that $P(k) \to (P/[P, P])(k)$ is a surjection. In other words, looking at the Galois cohomology sequence

$$1 \to [P, P](k) \to P(k) \to (P/[P, P])(k) \to H^1(k, [P, P])$$

corresponding to

$$1 \to [P, P] \to P \to P/[P, P] \to 1,$$

we shall show that $H^1(k, [P, P]) = 0$.

Since $P = M \cdot U$, we have $[P, P] = [M, M] \cdot U$ as $U = [M, U]$.

Now $P \to P/[P, P]$ has the kernel $[P, P] = [M, M]U$. We crucially use the fact that $D = [M, M]$ is a simply connected group. First, a result due to Kneser (and due to Bruhat and Tits when char $k > 0$) on Galois cohomology implies that $H^1(k, D) = 0$. Moreover, $H^1(k, U) = 0$, as U is a k-split unipotent group. So, we obtain $H^1(k, D.U) = 0$; that is, $H^1(k, [P, P]) = 0$.

Thus, we have shown that $P(k) \to (P/[P, P])(k)$ is a surjection.

On the other hand, the kernel of the above map $P(k) \to (P/[P, P])(k)$ is

$$[P, P](k) = [M, M](k) \cdot U(k) \supseteq [P(k), P(k)].$$

Now, we use the validity of the Kneser-Tits conjecture for the simply connected, semisimple k-isotropic group $D = [M, M]$.

Note that, being simply connected, this group is a direct product of its k-simple factors and each k-isotropic k-simple factor H satisfies $H(k) = [H(k), H(k)]$ by the validity of the Kneser-Tits conjecture ([M91], 2.3.2(b)). For a k-anisotropic factor J, we have $J(k)/[J(k), J(k)]$ being a subquotient of F^*, where F is the residue field of k. Therefore, in particular, $D(k)/[D(k), D(k)]$ is a finite, abelian group of order prime to p.

Finally, since $[P(k), P(k)]$ contains $U(k)$ as well as $[D(k), D(k)]$, we have that $[P, P](k)/[P(k), P(k)]$ is a quotient of $D(k)/[D(k), D(k)]$.

Hence $[P, P](k)/[P(k), P(k)]$ is also a finite, abelian group of order prime to p. The latter group being the kernel of $P(k)/[P(k), P(k)] \to (P/[P, P])(k)$ and, with A having only p-power torsion, we get

$$\mathrm{Hom}((P/[P, P])(k), A) \cong \mathrm{Hom}(P(k)/[P(k), P(k)], A) = \mathrm{Hom}(P(k), A).$$

This proves the lemma. □

Remarks. (a) Note that Lemma 2 above is analogous (but dual!) to the situation in chapter 4 of [PR84]. There, the groups are over the residue field of k and have p-power order while the coefficients are considered with prime-to-p torsion.

(b) Analyzing the proof of this lemma, it follows that even when k is an arbitrary field, the lemma as well as the proposition and injectivity assertion of the main theorem go through provided the quotient groups $D(k)/[D(k), D(k)]$ have orders coprime to that of the coefficient group A where $D = [M, M]$ and M runs through the Levi parts of the standard parabolic k-subgroups.

(c) In view of this lemma, it will be convenient to use the following notation in the proof of the proposition. For any set Θ of simple k-roots, let $\overline{P_\Theta}$ denote the abelian group $(P_\Theta)_{ab}(k)$. For any $\alpha \in \Delta$, if we let P^*_α denote the abelian group, then $\mathrm{Hom}((P_\alpha)_{ab}(k), A) = \mathrm{Hom}(P_\alpha(k), A)$. Then, for any set Θ of simple k-roots, we may identify $\overline{P_\Theta}$ with $\prod_{\alpha\in\Theta} \overline{P_\alpha}$ and $\mathrm{Hom}(P_\Theta(k), A)$ with the direct product $\prod_{\alpha\in\Theta} P^*_\alpha$ in view of the following result, which is similar to §4.6 of [PR84]:

Lemma. *For each set Θ of simple k-roots, the map $\overline{P_\Theta} \to \prod_{\alpha\in\Theta} \overline{P_\alpha}$ is an isomorphism. More generally, for two disjoint subsets Θ, Θ' of Δ, the map*

$$P_{\Theta\cup\Theta'}/[P_{\Theta\cup\Theta'}, P_{\Theta\cup\Theta'}] \to P_\Theta/[P_\Theta, P_\Theta] \times P_{\Theta'}/[P_{\Theta'}, P_{\Theta'}]$$

is a k-isomorphism of k-algebraic groups.

Proof. Since the latter map is defined over k, it suffices to prove that it is an isomorphism over $\bar{k}$. The idea of the proof is to produce tori R_Θ (defined over $\bar{k}$) inside the parabolic k-subgroups P_Θ which are evidently seen to have the asserted isomorphism property and which map isomorphically onto the abelianization under the natural map $P_\Theta \to P_\Theta/[P_\Theta, P_\Theta]$. We shall consider the various corresponding subgroups over $\bar{k}$ as we did over k earlier. Let T be a maximal k-torus of G containing S and let Δ_T denote the set of simple roots with respect to T. For a subset $\Theta \subseteq \Delta_T$, let us write $T^\Theta = (\bigcap_{\alpha\in\Theta} \mathrm{Ker}(\alpha))^0$ and $N_\Theta = C_G(T^\Theta)$. Now, the torus $T_\Theta := (T \cap [N_\Theta, N_\Theta])^0$ has dimension equal to $|\Theta|$ and, moreover (as G is simply connected) is a direct product of all T_α's for $\alpha \in \Theta$. In particular, T itself is isomorphic to the direct product of all T_α as α runs over Δ_T. Now, if $\alpha \in \Delta$ (that is, it is a simple k-root), look at the set of all $\tilde{\alpha} \in \Delta_T$ such that $\tilde{\alpha}|_S = \alpha$. Let R_α be the subtorus of T generated by all such $T_{\tilde{\alpha}}$'s. More generally, for any set Θ of simple k-roots, we have a subtorus R_Θ of T and evidently $R_{\Theta\cup\Theta'} \cong R_\Theta \times R_{\Theta'}$. Returning to our parabolic k-subgroups P_Θ, we note that R_Θ is a maximal torus of M_Θ which intersects $[M_\Theta, M_\Theta]$ only trivially (and hence also $[P_\Theta, P_\Theta]$ only trivially). So, the quotient homomorphism from P_Θ to its abelianization is injective on R_Θ and maps surjectively onto $P_\Theta/[P_\Theta, P_\Theta]$. The lemma follows. □

In view of the last lemma, it is meaningful to write P_Θ^* for

$$\prod_{\alpha\in\Theta} P_\alpha^* = \prod_{\alpha\in\Theta} \mathrm{Hom}((P_\alpha)_{ab}(k), A).$$

Completion of proof of Proposition 1. We are trying to prove here that $E_\infty^{1,1} = 0 = E_\infty^{2,0}$ and, further, that

$$H^2(G(k), A) \hookrightarrow \mathrm{Ker}(\bigoplus_{|\Theta|=1} H^2(P_\Theta(k), A) \to \bigoplus_{|\Theta|=2} H^2(P_\Theta(k), A))$$

holds. We have already shown that $E_\infty^{2,0} = 0$. Now, $E_2^{i,1} = i$-th homology of the complex

$$0 \to \bigoplus_{|\Theta|=1} H^1(P_\Theta(k), A) \to \cdots \to \bigoplus_{|\Theta|=r} H^1(P_\Theta(k), A) \to H^1(G(k), St(A)) \to 0,$$

which is

$$0 \to \bigoplus_{|\Theta|=1} P_\Theta^* \to \cdots \to \bigoplus_{|\Theta|=r} P_\Theta^* \to H^1(G(k), St(A)) \to 0.$$

By lemmata 2 and 3, we have $P_\Theta^* = \prod_{\alpha\in\Theta} P_\alpha^*$.

We write this complex B as the direct sum of complexes $B_\alpha, \alpha \in \Delta$ as follows. Consider the $(r-1)$-simplex whose vertices are the elements of Δ and the coefficients are considered in the abelian group P_α^*. Then, excepting the last term, we see that the above complex B is just the direct sum of the relative cochain complexes B_α of this $(r-1)$-simplex relative to the $(r-2)$-dimensional face obtained by throwing out the vertex α. Hence, each B_α is exact, and so is B except at the last term. So, we have $E_2^{i,1} = 0$ for all $i \le r-1$. In particular,

$$E_\infty^{1,1} = E_3^{1,1} = \mathrm{Ker}(E_2^{1,1} \to E_2^{3,0}) \subseteq E_2^{1,1} = 0.$$

Also using the fact that $E_\infty^{2,0} = 0$, we will have

$$E_\infty^{0,2} = E_4^{0,2} \subseteq E_3^{0,2} \subseteq E_2^{0,2}.$$

That is,

$$H^2(G(k),A) \hookrightarrow \mathrm{Ker}(\bigoplus_{|\Theta|=1} H^2(P_\Theta(k),A) \to \bigoplus_{|\Theta|=2} H^2(P_\Theta(k),A)).$$

Thus, the proposition is proved.

3. Reduction to Levi parts of parabolic subgroups

In this section, k is arbitrary.

Proposition.

$$H^2(G(k),A) \hookrightarrow \mathrm{Ker}(\bigoplus_{|\Theta|=1} H^2(M_\Theta(k),A) \to \bigoplus_{|\Theta|=2} H^2(M_\Theta(k),A)).$$

Proof. We first claim that for any parabolic k-subgroup $P = M \cdot U$, the restriction map gives an isomorphism:

$$H^2(P(k),A) \cong H^2(M(k),A).$$

We use the Hochschild-Serre spectral sequence for $U \leq P$. We claim that

$$H^i(M, H^{2-i}(U,A)) = 0, \qquad i = 0,1.$$

Now the central torus of $M (= M_\Theta$ say) acts nontrivially on $U (= U_\Theta)$ as S^Θ has no fixed points on U^Θ; that is, there exists t in the central torus of M such that $t-1$ acts as an automorphism on $\mathrm{Hom}\,(U,A)$. By [R72], p. 121, for any group Γ and a Γ-module V, if there exists $t \in Z(\Gamma)$ so that $t-1$ acts as automorphisms on V, then $H^i(\Gamma, V) = 0$ for all i. Hence, we conclude in our case that $H^1(M, H^1(U,A)) = 0$. Thus, to complete the proof of the proposition, we are left to show that $H^2\,(U,A)^M = 0$.

If $U = U_\Theta$, then we shall apply induction using Hochschild-Serre successively for the connected unipotent groups corresponding to the eigenspaces $\mathbf{g}_\alpha$ for $\alpha \in \Phi^+ - \langle\Theta\rangle$. It suffices to show that $H^2\,(U_1,A)^M = 0$ if U_1 is 1-dimensional. But, if U_1 corresponds to the root α, the central torus of M acts on $H^2(U_1,A) = \mathrm{Ext}^2\,(U_1,A)$ through the character 2α. Since we can choose t in the central torus of M such that $(2\alpha)(t) \neq 1$, it follows that $H^2(U_1,A)^M = \{0\}$. To recapitulate, we have shown that

$$H^i(M, H^{2-i}(U,A)) = 0, \qquad i = 0,1.$$

Therefore, we have shown that

$$H^2(G(k),A) \hookrightarrow \mathrm{Ker}(\bigoplus_{|\Theta|=1} H^2(M_\Theta(k),A) \to \bigoplus_{|\Theta|=2} H^2(M_\Theta(k),A)). \qquad \square$$

4. Reduction to semisimple subgroups of smaller rank

Proposition. *For $\emptyset \neq \Theta \subseteq \Delta$, let $D_\Theta = [M_\Theta, M_\Theta]$. Then, the D_Θ are semisimple, simply connected k-groups having k-ranks lower than that of G and satisfying*

$$H^2(G(k),A) \hookrightarrow \mathrm{Ker}(\bigoplus_{|\Theta|=1} H^2(D_\Theta(k),A) \to \bigoplus_{|\Theta|=2} H^2(D_\Theta(k),A)).$$

Proof. We shall again use the Hochschild-Serre spectral sequence, this time corresponding to $D_\Theta \leq M_\Theta$. Recall that we have a canonical isomorphism $\overline{P_\Theta} \cong \bigoplus_{\alpha\in\Theta}(P_\alpha/[P_\alpha, P_\alpha])(k)$. Further, they are isomorphic with $M_\Theta(k)/D_\Theta(k)$. We have again used the vanishing of the Galois cohomology $H^1(k, D_\Theta)$. Since $H^1(D_\Theta(k), A) = 0$, we have an exact sequence coming from the inflation and restriction maps:

$$0 \to H^2(\overline{P_\Theta}, A) \to H^2(M_\Theta(k), A) \to H^2(D_\Theta(k), A).$$

Consider the following commutative diagram:

$$\begin{array}{ccccccc} 0 \to & \bigoplus_{|\Theta|=1} H^2(\overline{P_\Theta}, A) & \to & \bigoplus_{|\Theta|=1} H^2(M_\Theta(k), A) & \to & \bigoplus_{|\Theta|=1} H^2(D_\Theta(k), A) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \to & \bigoplus_{|\Theta|=2} H^2(\overline{P_\Theta}, A) & \to & \bigoplus_{|\Theta|=2} H^2(M_\Theta(k), A) & \to & \bigoplus_{|\Theta|=2} H^2(D_\Theta(k), A) \end{array}$$

When $\Theta = \{\alpha, \beta\}$, the natural maps $(M_\Theta/D_\Theta)(k) \to (M_\alpha/D_\alpha)(k)$ and $(M_\Theta/D_\Theta)(k) \to (M_\beta/D_\beta)(k)$ are simply the projections of $\overline{P_\Theta}$ to $\overline{P_\alpha}$ and $\overline{P_\beta}$ respectively. So, the left-most vertical arrow in the diagram is injective. Therefore, it follows from the inclusion

$$H^2(G(k), A) \hookrightarrow \mathrm{Ker}(\bigoplus_{|\Theta|=1} H^2(M_\Theta(k), A) \to \bigoplus_{|\Theta|=2} H^2(M_\Theta(k), A))$$

of Proposition 2 that we have

$$H^2(G(k), A) \hookrightarrow \mathrm{Ker}(\bigoplus_{|\Theta|=1} H^2(D_\Theta(k), A) \to \bigoplus_{|\Theta|=2} H^2(D_\Theta(k), A)).$$

The proposition is proved. □

5. Reduction to subgroups without k-anisotropic factors

Finally, we want to reduce the above injectivity result to one where the D_Θ are replaced by connected normal k-subgroups which have no k-anisotropic factors. Now, since D_Θ is semisimple and simply connected, it is a direct product of its connected k-simple normal subgroups; in particular, D_Θ is a direct product of its maximal connected normal k-anisotropic subgroup H_Θ and a connected normal k-subgroup G_Θ which has no k-anisotropic factors. If $\Theta \subseteq \Theta' \subseteq \Delta$, then H_Θ is a direct factor of $H_{\Theta'}$. Indeed, recall that the Tits index of H_Θ over k is obtained from that of G by removing all orbits corresponding to Θ and the edges containing them. Hence $G_{\Theta'} \leq G_\Theta$ for $\Theta \subseteq \Theta'$. Now, if Θ_1, Θ_2 are disjoint subsets of Δ, then

$$H_{\Theta_1} \leq G_{\Theta_2}, \qquad H_{\Theta_2} \leq G_{\Theta_1}.$$

So, the subgroups $H_{\Theta_1}, H_{\Theta_2}$ commute element-wise and intersect trivially if $\Theta_1 \cap \Theta_2 = \emptyset$. Thus, $H_\alpha H_\beta$ is a direct factor of $H_{\{\alpha,\beta\}}$ for all $\alpha, \beta \in \Delta$. Now, $D_\Theta(k)$ is the direct product $H_\Theta(k) \cdot G_\Theta(k)$ for each $\Theta \subseteq \Delta$.

If $\Theta \subseteq \Theta' \subseteq \Delta$, then consider the homomorphism

$$D_{\Theta'}(k)/G_{\Theta'}(k) \to D_\Theta(k)/G_\Theta(k)$$

induced by the inclusions $D_{\Theta'} \subseteq D_\Theta, G_{\Theta'} \subseteq G_\Theta$. It is just the natural projection of $H_{\Theta'}(k)$ onto the factor $H_\Theta(k)$ when we identify $D_\Theta(k)/G_\Theta(k)$ with $H_\Theta(k)$. Moreover, $H^1(G_\Theta(k), A) = 0 = H^1(D_\Theta(k)/G_\Theta(k), A)$ since $D_\Theta(k)/G_\Theta(k) \cong H_\Theta(k)$ and since (as we observed in the course of the proof of Lemma 2) the abelianization is finite or of order prime to p.

Look at the commutative diagram

$$\begin{array}{ccccccc} 0 \to & \bigoplus_{|\Theta|=1} H^2(H_\Theta(k), A) & \to & \bigoplus_{|\Theta|=1} H^2(D_\Theta(k), A) & \to & \bigoplus_{|\Theta|=1} H^2(G_\Theta(k), A) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \to & \bigoplus_{|\Theta|=2} H^2(H_\Theta(k), A) & \to & \bigoplus_{|\Theta|=2} H^2(D_\Theta(k), A) & \to & \bigoplus_{|\Theta|=2} H^2(G_\Theta(k), A) \end{array}$$

Since $H_\alpha H_\beta$ is a direct factor of $H_{\alpha,\beta}$, we have the injectivity of the left-most vertical arrow, so the kernel of the middle vertical arrow injects into the kernel of the right-most vertical arrow; that is,

$$H^2(G(k), A) \hookrightarrow \mathrm{Ker}(\bigoplus_{\alpha\in\Delta} H^2(G_\alpha(k), A) \to \bigoplus_{|\Theta|=2} H^2(G_\Theta(k), A)).$$

Thus, we have proved the injectivity assertion of the theorem. Finally, the last implication can be deduced from Moore's results, which assert that the group $H^2_{top}(G(k), A)$ of topological central extensions forms the subgroup of $H^2(G(k), A)$ defined via measurable cochains and that the restriction maps $H^2(G(k), A) \to H^2(G_i(k), A)$ restrict to the corresponding restriction maps $H^2_{top}(G(k), A) \to H^2_{top}(G_i(k), A)$.

Remarks on rank 1 *groups.* The main (perhaps the only) thrust of this paper has been to use the theory of Tits buildings to reduce the study of abstract central extensions for higher rank groups over p-adic fields to that for k-rank 1 groups. As mentioned in the introduction, the determination of abstract central extensions for k-rank 1 groups is largely unknown apart from the case of split and quasi-split groups where the classical work of Moore applies. This seems to be a difficult problem requiring some new techniques that are different from that of this paper. Perhaps the first new example one could try to look at is the group $SL_{2,D}$, where D is a division algebra over k. For this group, the abstract central extensions have been determined by U. Rehmann [Re86], and his theorem (Theorem 2.1) in this context asserts:

Let D be a skew field such that $SL_2(D)$ is perfect. Let U be the group presented by generators $c(u, v), u, v \in D^*$ and relations

$$c(u, v)c(vu, w) = c(u, vw)c(v, w),$$
$$c(u, v) = c(uvu, u^{-1}), c(x, y)c(u, v)c(x, y)^{-1} = c([x, y]u, v)c(v, [x, y]),$$
$$c(u, v) = c(u, v(1 - u)) \ (1 - u \in D^*).$$

Then $c(u, v) \mapsto [u, v]$ defines a central extension of $[D^*, D^*]$ with kernel isomorphic to $H_2(SL_2(D), \mathbf{Z})$.

Rehmann also shows in that paper that when k is a p-adic field, the topological fundamental group of $SL_2(D)$ is $\mu(k)$, the group of roots of unity.

However, it still seems difficult to compare abstract and topological central extensions by a finite group, and we are unable to prove a result in this direction.

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In the context of computing central extensions and the congruence subgroup problem, the technique of using the combinatorial or topological buildings was initiated and fruitfully exploited by Gopal Prasad and M.S. Raghunathan in the 1980's. Recently, there has been a revival of sorts in comparing abstract and continuous cohomology in [FKRS04] and [N07]. I am indebted to these authors for rekindling my interest in this topic. In particular, I thank Nikolay Nikolov for inviting me to give a talk on this in the London Algebra seminar at Imperial College, London in June 2008. I owe a lot to Gopal Prasad and M.S. Raghunathan, who patiently shared their insights on this topic. I am particularly indebted to Gopal Prasad for answering several queries and pointing out inaccuracies in earlier versions. In particular, he showed how the results of Borel and Serre can be used in Lemma 1. Thanks are also due to Dipendra Prasad for providing me the proof of the sublemma used in Lemma 1. I am told that a result similar to the main result of this paper has been obtained by Gopal Prasad and M.S. Raghunathan but has not been put down in writing. This work began in the late 1980's but was in cold storage until it was revived in 2008 and essentially completed while visiting the Max-Planck-Institut für Mathematik in Bonn during February–March 2009. Finally, it is a pleasure to thank the referee for useful comments.

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