

ON MATHER'S α -FUNCTION OF MECHANICAL SYSTEMS

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ABSTRACT. We study Mather's α -function for mechanical systems. We show that for mechanical systems, the α -function is differentiable at $c = 0$ in at least one direction. We also give a topological condition on the potential function to guarantee the existence of a flat part near $c = 0$ for general mechanical systems. Some examples are also given.

1. INTRODUCTION

Let $L : T\mathbb{T}^n \rightarrow \mathbb{R}$ be the Tonelli Lagrangian on the n -torus satisfying the following properties:

(L1) SMOOTHNESS: $L : T\mathbb{T}^n \rightarrow \mathbb{R}$ is of class at least C^2 .

(L2) CONVEXITY: The Hessian $\frac{\partial^2 L}{\partial \dot{x}^2}(x, \dot{x})$ is positively definite on each fibre $T_x \mathbb{T}^n$.

(L3) SUPERLINEARITY:

$$\lim_{|\dot{x}| \rightarrow \infty} \frac{L(x, \dot{x})}{|\dot{x}|} = \infty, \quad \text{uniformly on } x \in \mathbb{T}^n.$$

Let $\mathcal{M}(L)$ be the set of Φ_t -invariant Borel probability measures on $T\mathbb{T}^n$, where Φ_t is the Euler-Lagrange flow of L . For every $\mu \in \mathcal{M}(L)$, we can define its *average action*

$$(1) \quad A(\mu) = \int_{\mathbb{T}^n} L \, d\mu.$$

The integral is defined since L is bounded below. If $A(\mu) < +\infty$, we may associate to μ its *rotation vector* $\rho(\mu) \in H_1(\mathbb{T}^n, \mathbb{R}) = \mathbb{R}^n$. The rotation vector $\rho(\mu)$ is uniquely characterized by

$$\langle c, \rho(\mu) \rangle = \int \eta_c \, d\mu, \quad \text{for all } c \in H^1(\mathbb{T}^n, \mathbb{R}),$$

where η_c is a representative of the de Rham cohomological class $c \in H^1(\mathbb{T}^n, \mathbb{R}) \cong \mathbb{R}^n$ and the bracket on the left side of the equality above is the canonical pairing of $H^1(\mathbb{T}^n, \mathbb{R})$ and $H_1(\mathbb{T}^n, \mathbb{R})$. The integral on the right is well defined, since an addition of an exact form to η_c does not change the integral (see [Mat1, Mat2]).

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For every $h \in H_1(\mathbb{T}^n, \mathbb{R})$, we define Mather's β -function, $\beta : H_1(\mathbb{T}^n, \mathbb{R}) \rightarrow \mathbb{R}$, as

$$(2) \quad \beta(h) = \inf\{A(\mu) : \mu \in \mathcal{M}(L), \rho(\mu) = h\}.$$

It is easy to see that $\beta(h)$ is a convex function on $H_1(\mathbb{T}^n, \mathbb{R})$ with superlinear growth. We define Mather's α -function, $\alpha : H^1(\mathbb{T}^n, \mathbb{R}) \rightarrow \mathbb{R}$, as the Fenchel transformation of the β -function, i.e.,

$$(3) \quad \alpha(c) = \max\{\langle c, h \rangle - \beta(h) : h \in H_1(\mathbb{T}^n, \mathbb{R})\}, \quad \text{for all } c \in H^1(\mathbb{T}^n, \mathbb{R}).$$

From the basic facts in convex analysis, $\alpha(c)$ is also a convex function on $H^1(\mathbb{T}^n, \mathbb{R})$ with superlinear growth. It is well known that

$$(4) \quad \alpha(c) = - \inf_{\mu \in \mathcal{M}(L)} \int_{T\mathbb{T}^n} L - c \, d\mu.$$

The following inf-max formula for the α -function is also useful (see [CIPP]):

$$(5) \quad \alpha(c) = \inf_{u \in C^1(\mathbb{T}^n)} \max_{x \in \mathbb{T}^n} H(x, du(x) + c).$$

Many authors contributed to the structure of the α -functions or β -functions; see e.g. [BIK, C, LPV, Mas1, Mas2, O]. This paper is motivated by the two following problems:

Problem 1. Is it true that for any autonomous Tonelli Lagrangian the α -function is differentiable in at least one direction everywhere?

Problem 2. For mechanical systems with the form $L(x, v) = \frac{1}{2}|v|^2 - U(x)$, is there some relation between the topological structure of the level set $\{x \in \mathbb{T}^n : U(x) = \max_{x \in \mathbb{T}^n} U(x)\}$ and the regularity properties of the α -function?

In this paper, we give partial answers for the problems above in sections 2 and 3 respectively. We have:

- (1) (Corollary 1) For the Tonelli Lagrangian in the form $\ell(v) - U(x)$, where ℓ is strictly convex, $\ell(0) = 0 = \min_{v \in \mathbb{R}^n} \ell(v)$ and $\max_{x \in \mathbb{T}^n} U(x) = 0$, the α -function is differentiable in some directions at $c = 0$.
- (2) (Theorem 3) If there exist k vector fields X_i on \mathbb{T}^n independently, $1 \leq k \leq n$ such that the X_i 's satisfy (9) and (10) with $\int_{\mathbb{T}^n} X_i(x) dx \neq 0$, $i = 1, \dots, k$, then the α -function has k -dimensional flat part near $c = 0$.
- (3) (Theorem 4) If the critical set E of U of the system $L(x, v) = \frac{1}{2}|v|^2 - U(x)$ does not contain a simple closed homotopically nontrivial smooth curve, then the α -function has fully dimensional flat part near $c = 0$.

In section 3, we also give some examples of mechanical systems with arguments to expose the link between the topological structure of the projected Aubry set \mathcal{A}_0 and the regularity properties of the α -function.

2. DIFFERENTIABILITY OF THE α -FUNCTION

This section is motivated by the famous Hedlund example on the geodesic flow on \mathbb{T}^3 ; see [Ba] for details or [Y] for a similar example of mechanical systems.

Given any $c \in H^1(\mathbb{T}^n, \mathbb{R}) \cong \mathbb{R}^n$, c is specified to a real vector in \mathbb{R}^n as its representative element in the cohomology class c . From now on, we identify the cohomology class c with $c \in \mathbb{R}^n$ for convenience, since the exact 1-forms do not contribute to the action.

Now let $L(x, v) = \frac{1}{2}|v|^2 - U(x)$ be a Tonelli Lagrangian of the mechanical system on \mathbb{T}^n , where $U : \mathbb{T}^n \rightarrow \mathbb{R}$ is a C^2 function and $\max_{x \in \mathbb{T}^n} U(x) = 0$. Define

$$E = \{x \in \mathbb{T}^n : U(x) = 0 = \max_{x \in \mathbb{T}^n} U(x)\}.$$

It is clear that $E \subset \mathbb{T}^n$ is a compact set.

For convenience, let us lift the torus to its universal covering space. Let $Q = [0, 1)^n$ be the fundamental domain of the covering space \mathbb{R}^n , $\tilde{E} \subset Q$ be the natural quotient of the lift of E in Q .

Since the fundamental group $\pi_1(\mathbb{T}^n) \cong \mathbb{Z}^n$, for any $\mathbf{n} \in \mathbb{Z}^n$, let $\Gamma_{\mathbf{n}}$ be the C^1 closed curve in \mathbb{T}^n whose lifts to the universal covering space \mathbb{R}^n is a straight line segment having the endpoints \tilde{x}_1 and $\tilde{x}_2 = \tilde{x}_1 + \mathbf{n}$.

Theorem 1. *Let E admit a homotopically nontrivial C^1 simple closed curve $\Gamma_{\mathbf{n}}$ for some nonzero $\mathbf{n} \in \mathbb{Z}^n$; i.e., the natural quotient of $\tilde{\Gamma}_{\mathbf{n}}$ in Q is \tilde{E} . Then the corresponding α -function in the direction \mathbf{n} can be represented by*

$$\alpha(r\mathbf{n}) = \frac{1}{2}|r\mathbf{n}|^2, \quad r \in \mathbb{R}.$$

Proof. For convenience, we lift the Hamiltonian $H(x, p) = \frac{1}{2}|p|^2 + U(x)$ to the universal covering space $\mathbb{R}^n \times \mathbb{R}^n$ of $\mathbb{T}^* \mathbb{T}^n$; that is, U can be regarded as a \mathbb{Z}^n -periodic function with respect to x . Given any smooth \mathbb{Z}^n -periodic function u on \mathbb{R}^n , there exists $t \in [0, 1]$ such that the directional derivative with respect to \mathbf{n} of u on $\tilde{\Gamma}_{\mathbf{n}}(t)$ is 0. Then for $c = c(r) = r\mathbf{n}$, $r > 0$, the inf-max formulae (5) of the α -function implies that

$$\max_{x \in Q} \frac{1}{2}|du + c|^2 + U(x) \geq \max_{x \in Q} \frac{1}{2}|c|^2 + U(x) = \frac{1}{2}|c|^2.$$

Then we have $\alpha(c) \geq \frac{1}{2}|c|^2$. On the other hand, we have $\alpha(c) \leq \frac{1}{2}|c|^2$ by choosing u to be any constant function. □

Lemma 1. *Let $L_{U,\ell}$ be the Tonelli Lagrangian in the form $L_{U,\ell}(x, v) = \ell(v) - U(x)$ with strictly convex kinetic energy ℓ and potential U . Suppose $U(x) \leq \tilde{U}(x)$ for any $x \in \mathbb{T}^n$ and $\ell(v) \geq \tilde{\ell}(v)$ for any $v \in \mathbb{R}^n$. Then the relation between the α -function of the systems $L_{U,\ell}$ and $L_{\tilde{U},\tilde{\ell}}$ satisfies*

$$\alpha_{U,\ell} \leq \alpha_{\tilde{U},\tilde{\ell}}.$$

Proof. This is deduced directly from the definition and formula (4). □

Now suppose the potential U is not trivially constant. Then there exist both a minimum and a maximum of U since \mathbb{T}^n is compact. Let y and $z \in \mathbb{T}^n$ be such that

$$(6) \quad U(y) = \min_{x \in \mathbb{T}^n} U(x) < U(z) = \max_{x \in \mathbb{T}^n} U(x).$$

Then we can find an alternative potential function U_δ on \mathbb{T}^n for small $\delta > 0$ such that

$$(7) \quad \max_{x \in \mathbb{T}^n} U_\delta(x) = U_\delta(z) = U(z), \quad U_\delta(x) \geq U(x) \text{ for any } x \in \mathbb{T}^n$$

and

$$(8) \quad U_\delta(x) \equiv U_\delta(z) \text{ for } x \notin B(y, \delta).$$

The potential U_δ can be obtained by the smooth Urysohn lemma since E is a compact subset in \mathbb{T}^n and there exists a ball $B(y, \delta) \subset \mathbb{T}^n \setminus E$.

Theorem 2. *For any mechanical Tonelli Lagrangian $\frac{1}{2}|v|^2 - U(x)$, the corresponding α -function $\alpha(c)$ is differentiable in some directions at $c = 0$.*

Proof. If $U \equiv C$ is a constant, then the system is completely integrable and $\alpha(c) = \frac{1}{2}|c|^2$, which is trivial. Now suppose $U \not\equiv C$.

We apply Theorem 1 to the system $\frac{1}{2}|v|^2 - U_\delta(x)$ at first for some small $\delta > 0$. In that case, $\tilde{E} = Q \setminus B(y, \delta)$. Then there exists $\tilde{\Gamma}_{\mathbf{n}} \subset \tilde{E}$ as in Theorem 1 for some $\mathbf{n} \in \mathbb{Z}^n$. Each $\tilde{\Gamma}_{\mathbf{n}}$ determines a direction \mathbf{n} such that $\alpha_\delta(r\mathbf{n}) = \frac{1}{2}|r\mathbf{n}|^2$, where α_δ is the α -function of the system with potential U_δ . The number of such \mathbf{n} 's is decided by δ . Here we omit the argument of that number. Then we have that α_δ is quadratic in the direction \mathbf{n} and is differentiable at $c = 0$.

From the construction of the potential U_δ , together with Lemma 1, we easily have $\alpha_\delta(c) \geq \alpha(c)$ for all c . Then the differentiability of α at $c = 0$ along the direction \mathbf{n} can be obtained since $\alpha_\delta(0) = \alpha(0)$ and both of them are convex functions. \square

Corollary 1. *For the Tonelli Lagrangian in the form $\ell(v) - U(x)$, where ℓ is strictly convex, $\ell(0) = 0 = \min_{v \in \mathbb{R}^n} \ell(v)$ and $\max_{x \in \mathbb{T}^n} U(x) = 0$, the α -function is differentiable in some directions at $c = 0$.*

Proof. We only discuss the nontrivial case. Let $L_1(x, v) = \ell(v) - U(x)$, $L_2(x, v) = \frac{\lambda^2}{2}|v|^2 - U_\delta(x)$ for some $\lambda > 0$ and small $\delta > 0$ as before. Then $L_1 \geq L_2$ by the strict convexity assumption of ℓ and the definition of U_δ . Denoting by α_i , $i = 1, 2$, the α -function of L_i respectively, we have $\alpha_1 \leq \alpha_2$, and the differentiability of α_1 in some directions follows from that of α_2 , which has been proved in Theorem 2. \square

Remark 1. If Problem 1 in section 1 can be solved so as to be true, then the main result of [BC] can be improved. We will try it in other papers in the future.

3. FLAT PART OF THE α -FUNCTION

Let $L^0(x, v) = \frac{1}{2}|v|^2 - U(x)$ with the potential $U(x) \leq 0$ and $\max_{x \in \mathbb{T}^n} U(x) = 0$, $\kappa(x) = \sqrt{2(-U(x))}$. We want to find a C^1 vector field $X(x)$ as a function $X : \mathbb{T}^n \rightarrow \mathbb{R}^n$ such that

$$(9) \quad |X(x)| = \kappa(x)$$

and

$$(10) \quad dX(x) = dX^*(x).$$

Condition (10) means that X is a *gradient-like vector field* corresponding to a closed 1-form on \mathbb{T}^n in the following sense: the closed 1-form ω is defined by

$$\omega(x)(v) = \langle X(x), v \rangle, \quad v \in T_x \mathbb{T}^n \cong \mathbb{R}^n.$$

Now let us recall some basic facts on the construction of the closed 1-form on a closed smooth manifold M . Letting $f : M \rightarrow \mathbb{T}^1$ be a smooth map, the circle \mathbb{T}^1 is equipped with the canonical angular form $d\theta$, where $d\theta$ is a closed 1-form, which cannot be represented as a differential of a smooth function on \mathbb{T}^1 . The pullback $f^*(d\theta)$ is a closed 1-form on M . It is not hard to show that a closed 1-form ω on M can be represented by this form if and only if the de Rham cohomology class $[\omega] \in H^1(M, \mathbb{Z})$ (see [Fab]).

In particular in the case of $M = \mathbb{T}^n$, we can construct the required closed 1-form as follows: Letting $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ be the local coordinates of $T\mathbb{T}^n$, any closed 1-form is locally exact by Poincaré's lemma. Then there exists an open cover $\{\Omega_i\}$ of \mathbb{T}^n such that there exist smooth functions f_i defined on Ω_i such that $\omega|_{\Omega_i} = df_i|_{\Omega_i}$. More precisely, for the purpose of conditions (9) and (10), define the local vector field $X_i(x) = (0, \dots, \frac{\partial f_i}{\partial x_i}, \dots, 0)$ on Ω_i , where f_i is a smooth function on \mathbb{T}^n such that $|\frac{\partial f_i}{\partial x_i}| = \kappa(x)$. The smoothness of the f_i 's can be guaranteed if $\max_{x \in \mathbb{T}^n} U(x) < 0$. The global vector fields can be given by partition of unity. It is clear that the X_i 's are independent.

Then define a new Lagrangian

$$(11) \quad L^1(x, v) = \frac{1}{2}|v - X(x)|^2 = L^0(x, v) - \langle X(x), v \rangle.$$

Denote by α_0 and α_1 the α -functions of L^0 and L^1 respectively.

Lemma 2. *If for the potential U of L^0 there exists a C^1 vector field X satisfying (9) and (10) and $c = \int_{\mathbb{T}^n} X(x)dx$, then*

$$(12) \quad \alpha_0(c) = \alpha_1(0) = 0$$

with $|c| \leq \int_{\mathbb{T}^n} \kappa(x)dx$. Consequently, with the conditions above, the α -function has a flat part near 0 in the direction c if $c \neq 0$.

Proof. It is well known that for a positive definite Lagrangian L , $L - \lambda$ has the same Euler-Lagrange equation if the 1-form λ is closed. It is clear from (11) that if $c = \int_{\mathbb{T}^n} X(x)dx$, then $\alpha_1(0) = \alpha_0(c) = 0$. $|c| \leq \int_{\mathbb{T}^n} \kappa(x)dx$ follows easily from the definition of the vector field X . If $c \neq 0$, then (12) implies that $\alpha_0(rc) = 0$ for $0 \leq r \leq 1$ since $\alpha_0(0) = \min_{c \in \mathbb{R}^n} \alpha_0(c)$ and the α -function is convex. \square

Theorem 3. *If there exist k vector fields X_i on \mathbb{T}^n independently, $1 \leq k \leq n$, such that the X_i 's satisfy (9) and (10) with $\int_{\mathbb{T}^n} X_i(x)dx \neq 0$, $i = 1, \dots, k$, then the α -function has k -dimensional flat part near $c = 0$.*

Proof. This is a direct consequence of Lemma 2. \square

Now we apply Lemma 2 to some examples.

Example 1. When $n = 1$ and $\max_{x \in \mathbb{T}^1} U(x) = 0$, the flat part $|c| \leq \int_{\mathbb{T}^1} \kappa(x)dx$ of the α -function is well known; see e.g. [LPV]. Let $V_\varepsilon(x) = U(x) - \varepsilon$ for $\varepsilon > 0$. Then

$$L_\varepsilon^0(x, v) - \langle X_\varepsilon(x), v \rangle = \frac{1}{2}|v|^2 - V_\varepsilon(x) - \langle X_\varepsilon(x), v \rangle = \frac{1}{2}|v - X_\varepsilon(x)|^2 = L_\varepsilon^1(x, v),$$

where X_ε satisfies (9) and (10) for the potential V_ε . The existence of such an X_ε can easily be obtained by $X_\varepsilon = \sqrt{-V_\varepsilon}$. Denote by α_ε^0 and α_ε^1 the α -function of L_ε^0 and L_ε^1 respectively. Then we have $\alpha_\varepsilon^1(0) = \alpha_\varepsilon^0(c_\varepsilon) = 0$ by Lemma 2, where $c_\varepsilon = \int_{\mathbb{T}^1} X_\varepsilon(x)dx$. This implies that $\alpha^0(c_\varepsilon) - \varepsilon = 0$ for any $\varepsilon > 0$. So $\alpha(c_0) = 0$ by the continuity of c_ε with respect to ε , and $c_0 \neq 0$ if $U \not\equiv 0$. Thus the α -function has flat part on $[0, c_0]$, and the case of $[-c_0, 0]$ is similar by choosing $X_\varepsilon = -\sqrt{-V_\varepsilon}$.

Example 2. Let U be a smooth function on \mathbb{T}^n , and suppose that the critical set E defined in section 2 does not contain a simple closed homotopically nontrivial C^1 curve; e.g., U is a function of Morse type. Then the ε -trick in Example 1 and the construction of independent vector fields X_i as before imply the following:

Theorem 4. *If the critical set E of U of the system $L(x, v) = \frac{1}{2}|v|^2 - U(x)$ does not contain a simple closed homotopically nontrivial smooth curve, then the α -function has fully dimensional flat part near $c = 0$.*

Proof. If the critical set E of U does not contain a simple closed homotopically nontrivial smooth curve, then there exist n independent gradient-like vector fields $\{X_{i,\varepsilon}\}_{i=1}^n$ for $V_\varepsilon(x) = U(x) - \varepsilon$ as in Example 1. Applying the argument in Example 1 to each $X_{i,\varepsilon}$, we can see that there exists $c_{0,i} \neq 0$ such that the α -function has flat part in the direction of $c_{0,i}$. The independency of $X_{i,\varepsilon}$ means the independency of such $c_{0,i}$'s; thus we get the conclusion. \square

Remark 1. For the mechanical systems with the form $L(x, v) = \frac{1}{2}|v|^2 - U(x)$, the critical set $E = \{x : U(x) = \max_{x \in \mathbb{T}^n} U(x)\}$ is exactly the projected Aubry set \mathcal{A}_0 . For the definition of *Aubry set* and *projected Aubry set*, see e.g. [Mat3, Be, Fat, FS]. So the existence of the flat part near $c = 0$ is closely related to the topological structure of \mathcal{A}_0 . Actually a very complicated structure of \mathcal{A}_0 exists, e.g., Mather's striking example ([Mat3]).

Example 3. Let U be a smooth function on \mathbb{T}^n , and let the critical set E defined in section 2 contain a simple closed homotopically nontrivial C^1 curve $\Gamma_{\mathbf{n}}$, i.e., the example described in Theorem 1. It is clear from the construction of gradient-like vector fields before that there exist n gradient-like vector fields X_1, \dots, X_n on \mathbb{T}^n for the ε -perturbed system as in Example 1 such that the closed 1-form related to X_n is exact and X_1, \dots, X_{n-1} may be independent. Then a similar argument shows that there is no flat part along the direction of $\mathbf{n} \in \mathbb{Z}^n$ since X_n is a gradient field. Rewriting

$$\frac{1}{2}|v|^2 - U(x) - \langle c, v \rangle = \frac{1}{2}|v - c|^2 - \frac{1}{2}|c|^2 - U(x),$$

we also have that the α -function is quadratic in the direction of \mathbf{n} , which is the same statement as in Theorem 1.

Remark 2. When we consider the case that the projected Aubry set $\mathcal{A}_0 = E$ contains a general closed homotopically nontrivial C^1 curve, we need to find the obstacle to the existence of such a gradient-like vector field. The author hopes to try it in the future.

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REFERENCES

- [Ba] Bangert, V., *Minimal geodesics*, Ergodic Theory Dynam. Systems, **10**(1990), no. 2, 263–286. MR1062758 (91j:58126)
- [Be] Bernard, P., *Connecting orbits of time dependent Lagrangian systems*, Ann. Inst. Fourier (Grenoble), **52**(2002), no. 5, 1533–1568. MR1935556 (2003m:37088)
- [BC] Bernard, P. and Contreras, G., *A generic property of families of Lagrangian systems*, Ann. of Math. (2), **167**(2008), no. 3, 1099–1108. MR2415395 (2009d:37113)
- [BIK] Burago, D., Ivanov, S. and Kleiner, B., *On the structure of the stable norm of periodic metrics*, Math. Res. Lett. **4**(1997), no. 6, 791–808. MR1492121 (98k:53051)
- [C] Carneiro, M. J. Dias, *On minimizing measures of the action of autonomous Lagrangians*, Nonlinearity, **8**(1995), no. 6, 1077–1085. MR1363400 (96j:58062)

- [CIPP] Contreras, G., Iturriaga, R., Paternain, G. P. and Paternain, M., *Lagrangian graphs, minimizing measures and Mañé's critical values*, *Geom. Funct. Anal.*, **8**(1998), no. 5, 788–809. MR1650090 (99f:58075)
- [Fab] Farber, M., *Topology of closed one-forms*. Mathematical Surveys and Monographs, 108. American Mathematical Society, Providence, RI, 2004. MR2034601 (2005c:58023)
- [Fat] Fathi, A., *Weak KAM theorem in Lagrangian dynamics*, to be published by Cambridge University Press.
- [FS] Fathi, A. and Siconolfi, A., *Existence of C^1 critical subsolutions of the Hamilton-Jacobi equation*, *Invent. Math.*, **155**(2004), no. 2, 363–388. MR2031431 (2004m:37114)
- [LPV] Lions, P. L., Papanicolaou, G. and Varadhan, S. R. S., *Homogenization of Hamilton-Jacobi equations*, unpublished manuscript, 1988.
- [Man1] Mañé, R., *Generic properties and problems of minimizing measures of Lagrangian systems*, *Nonlinearity*, **9**(1996), no. 2, 273–310. MR1384478 (97d:58118)
- [Mas1] Massart, D., *On Aubry sets and Mather's action functional*, *Israel J. Math.*, **134**(2003), 157–171. MR1972178 (2004g:37088)
- [Mas2] Massart, D., *Vertices of Mather's beta function. II*, *Ergodic Theory Dynam. Systems*, **29**(2009), no. 4, 1289–1307. MR2529650
- [Mat1] Mather, John N., *Action minimizing invariant measures for positive definite Lagrangian systems*, *Math. Z.*, **207**(1991), no. 2, 169–207. MR1109661 (92m:58048)
- [Mat2] Mather, John N., *Variational construction of connecting orbits*, *Ann. Inst. Fourier (Grenoble)*, **43**(1993), no. 5, 1349–1386. MR1275203 (95c:58075)
- [Mat3] Mather, John N., *Examples of Aubry sets*, *Ergodic Theory Dynam. Systems*, **24**(2004), no. 5, 1667–1723. MR2104599 (2005h:37143)
- [O] Osuna, O., *Vertices of Mather's beta function*, *Ergodic Theory Dynam. Systems*, **25**(2005), no. 3, 949–955. MR2142954 (2005m:37150)
- [Y] Yu, Y., *Properties of effective Hamiltonian and connections with Aubry-Mather theory in two dimension*, preprint, 2009.

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