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ON MATHER'S α-FUNCTION OF MECHANICAL SYSTEMS

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ABSTRACT. We study Mather's α-function for mechanical systems. We show that for mechanical systems, the α-function is differentiable at $c = 0$ in at least one direction. We also give a topological condition on the potential function to guarantee the existence of a flat part near $c = 0$ for general mechanical systems. Some examples are also given.

1. INTRODUCTION

Let $L : T\mathbb{T}^n \to \mathbb{R}$ be the Tonelli Lagrangian on the n-torus satisfying the following properties:

(L1) SMOOTHNESS: $L : T\mathbb{T}^n \to \mathbb{R}$ is of class at least C^2.

(L2) CONVEXITY: The Hessian $\frac{\partial^2 L}{\partial \dot{x}^2}(x, \dot{x})$ is positively definite on each fibre $T_x\mathbb{T}^n$.

(L3) SUPERLINEARITY:

$$\lim_{|\dot{x}|\to\infty} \frac{L(x, \dot{x})}{|\dot{x}|} = \infty, \quad \text{uniformly on } x \in \mathbb{T}^n.$$

Let $\mathscr{M}(L)$ be the set of Φ_t-invariant Borel probability measures on $T\mathbb{T}^n$, where Φ_t is the Euler-Lagrange flow of L. For every $\mu \in \mathscr{M}(L)$, we can define its *average action*

$$A(\mu) = \int_{\mathbb{T}^n} L \, d\mu. \tag{1}$$

The integral is defined since L is bounded below. If $A(\mu) < +\infty$, we may associate to μ its *rotation vector* $\rho(\mu) \in H_1(\mathbb{T}^n, \mathbb{R}) = \mathbb{R}^n$. The rotation vector $\rho(\mu)$ is uniquely characterized by

$$\langle c, \rho(\mu)\rangle = \int \eta_c \, d\mu, \quad \text{for all} \quad c \in H^1(\mathbb{T}^n, \mathbb{R}),$$

where η_c is a representative of the de Rham cohomological class $c \in H^1(\mathbb{T}^n, \mathbb{R}) \cong \mathbb{R}^n$ and the bracket on the left side of the equality above is the canonical pairing of $H^1(\mathbb{T}^n, \mathbb{R})$ and $H_1(\mathbb{T}^n, \mathbb{R})$. The integral on the right is well defined, since an addition of an exact form to η_c does not change the integral (see [Mat1, Mat2]).

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For every $h \in H_1(\mathbb{T}^n, \mathbb{R})$, we define Mather's β-function, $\beta : H_1(\mathbb{T}^n, \mathbb{R}) \to \mathbb{R}$, as

$$\beta(h) = \inf\{A(\mu) : \mu \in \mathscr{M}(L),\ \rho(\mu) = h\}. \tag{2}$$

It is easy to see that $\beta(h)$ is a convex function on $H_1(\mathbb{T}^n, \mathbb{R})$ with superlinear growth. We define Mather's α-function, $\alpha : H^1(\mathbb{T}^n, \mathbb{R}) \to \mathbb{R}$, as the Fenchel transformation of the β-function, i.e.,

$$\alpha(c) = \max\{\langle c, h\rangle - \beta(h) : h \in H_1(\mathbb{T}^n, \mathbb{R})\}, \quad \text{for all} \quad c \in H^1(\mathbb{T}^n, \mathbb{R}). \tag{3}$$

From the basic facts in convex analysis, $\alpha(c)$ is also a convex function on $H^1(\mathbb{T}^n, \mathbb{R})$ with superlinear growth. It is well known that

$$\alpha(c) = -\inf_{\mu \in \mathscr{M}(L)} \int_{T\mathbb{T}^n} L - c\, d\mu. \tag{4}$$

The following inf-max formula for the α-function is also useful (see [CIPP]):

$$\alpha(c) = \inf_{u \in C^1(\mathbb{T}^n)} \max_{x \in \mathbb{T}^n} H(x, du(x) + c). \tag{5}$$

Many authors contributed to the structure of the α-functions or β-functions; see e.g. [BIK, C, LPV, Mas1, Mas2, O]. This paper is motivated by the two following problems:

Problem 1. Is it true that for any autonomous Tonelli Lagragian the α-function is differentiable in at least one direction everywhere?

Problem 2. For mechanical systems with the form $L(x, v) = \frac{1}{2}|v|^2 - U(x)$, is there some relation between the topological structure of the level set $\{x \in \mathbb{T}^n : U(x) = \max_{x \in \mathbb{T}^n} U(x)\}$ and the regularity properties of the α-function?

In this paper, we give partial answers for the problems above in sections 2 and 3 respectively. We have:

(1) (Corollary 1) For the Tonelli Lagrangian in the form $\ell(v) - U(x)$, where ℓ is strictly convex, $\ell(0) = 0 = \min_{v \in \mathbb{R}^n} \ell(v)$ and $\max_{x \in \mathbb{T}^n} U(x) = 0$, the α-function is differentiable in some directions at $c = 0$.

(2) (Theorem 3) If there exist k vector fields X_i on $\mathbb{T}^n$ independently, $1 \leqslant k \leqslant n$ such that the X_i's satisfy (9) and (10) with $\int_{\mathbb{T}^n} X_i(x)dx \neq 0$, $i = 1, \dots, k$, then the α-function has k-dimensional flat part near $c = 0$.

(3) (Theorem 4) If the critical set E of U of the system $L(x, v) = \frac{1}{2}|v|^2 - U(x)$ does not contain a simple closed homotopically nontrivial smooth curve, then the α-function has fully dimensional flat part near $c = 0$.

In section 3, we also give some examples of mechanical systems with arguments to expose the link between the topological structure of the projected Aubry set $\mathscr{A}_0$ and the regularity properties of the α-function.

2. Differentiability of the α-function

This section is motivated by the famous Hedlund example on the geodesic flow on $\mathbb{T}^3$; see [Ba] for details or [Y] for a similar example of mechanical systems.

Given any $c \in H^1(\mathbb{T}^n, \mathbb{R}) \cong \mathbb{R}^n$, c is specified to a real vector in $\mathbb{R}^n$ as its representative element in the cohomology class c. From now on, we identify the cohomology class c with $c \in \mathbb{R}^n$ for convenience, since the exact 1-forms do not contribute to the action.

Now let $L(x,v)=\frac{1}{2}|v|^2-U(x)$ be a Tonelli Lagrangian of the mechanical system on $\mathbb{T}^n$, where $U:\mathbb{T}^n\to\mathbb{R}$ is a C^2 function and $\max_{x\in\mathbb{T}^n}U(x)=0$. Define

$$E=\{x\in\mathbb{T}^n:\ U(x)=0=\max_{x\in\mathbb{T}^n}U(x)\}.$$

It is clear that $E\subset\mathbb{T}^n$ is a compact set.

For convenience, let us lift the torus to its universal covering space. Let $Q=[0,1)^n$ be the fundamental domain of the covering space $\mathbb{R}^n$, $\tilde{E}\subset Q$ be the natural quotient of the lift of E in Q.

Since the fundamental group $\pi_1(\mathbb{T}^n)\cong\mathbb{Z}^n$, for any $\mathbf{n}\in\mathbb{Z}^n$, let $\Gamma_{\mathbf{n}}$ be the C^1 closed curve in $\mathbb{T}^n$ whose lifts to the universal covering space $\mathbb{R}^n$ is a straight line segment having the endpoints $\tilde{x}_1$ and $\tilde{x}_2=\tilde{x}_1+\mathbf{n}$.

Theorem 1. *Let E admit a homotopically nontrivial C^1 simple closed curve $\Gamma_{\mathbf{n}}$ for some nonzero $\mathbf{n}\in\mathbb{Z}^n$; i.e., the natural quotient of $\tilde{\Gamma}_{\mathbf{n}}$ in Q is $\tilde{E}$. Then the corresponding α-function in the direction $\mathbf{n}$ can be represented by*

$$\alpha(r\mathbf{n})=\frac{1}{2}|r\mathbf{n}|^2,\quad r\in\mathbb{R}.$$

Proof. For convenience, we lift the Hamiltonian $H(x,p)=\frac{1}{2}|p|^2+U(x)$ to the universal covering space $\mathbb{R}^n\times\mathbb{R}^n$ of $\mathbb{T}^*\mathbb{T}^n$; that is, U can be regarded as a $\mathbb{Z}^n$-periodic function with respect to x. Given any smooth $\mathbb{Z}^n$-periodic function u on $\mathbb{R}^n$, there exists $t\in[0,1]$ such that the directional derivative with respect to $\mathbf{n}$ of u on $\tilde{\Gamma}_{\mathbf{n}}(t)$ is 0. Then for $c=c(r)=r\mathbf{n}$, $r>0$, the inf-max formulae (5) of the α-function implies that

$$\max_{x\in Q}\frac{1}{2}|du+c|^2+U(x)\geqslant\max_{x\in Q}\frac{1}{2}|c|^2+U(x)=\frac{1}{2}|c|^2.$$

Then we have $\alpha(c)\geqslant\frac{1}{2}|c|^2$. On the other hand, we have $\alpha(c)\leqslant\frac{1}{2}|c|^2$ by choosing u to be any constant function. □

Lemma 1. *Let $L_{U,\ell}$ be the Tonelli Lagrangian in the form $L_{U,\ell}(x,v)=\ell(v)-U(x)$ with strictly convex kinetic energy ℓ and potential U. Suppose $U(x)\leqslant\tilde{U}(x)$ for any $x\in\mathbb{T}^n$ and $\ell(v)\geqslant\tilde{\ell}(v)$ for any $v\in\mathbb{R}^n$. Then the relation between the α-function of the systems $L_{U,\ell}$ and $L_{\tilde{U},\tilde{\ell}}$ satisfies*

$$\alpha_{U,\ell}\leqslant\alpha_{\tilde{U},\tilde{\ell}}.$$

Proof. This is deduced directly from the definition and formula (4). □

Now suppose the potential U is not trivially constant. Then there exist both a minimum and a maximum of U since $\mathbb{T}^n$ is compact. Let y and $z\in\mathbb{T}^n$ be such that

$$U(y)=\min_{x\in\mathbb{T}^n}U(x)<U(z)=\max_{x\in\mathbb{T}^n}U(x).\tag{6}$$

Then we can find an alternative potential function U_δ on $\mathbb{T}^n$ for small $\delta>0$ such that

$$\max_{x\in\mathbb{T}^n}U_\delta(x)=U_\delta(z)=U(z),\quad U_\delta(x)\geqslant U(x)\text{ for any }x\in\mathbb{T}^n\tag{7}$$

and

$$U_\delta(x)\equiv U_\delta(z)\text{ for }x\notin B(y,\delta).\tag{8}$$

The potential U_δ can be obtained by the smooth Urysohn lemma since E is a compact subset in $\mathbb{T}^n$ and there exists a ball $B(y,\delta)\subset\mathbb{T}^n\setminus E$.

Theorem 2. *For any mechanical Tonelli Lagrangian* $\frac{1}{2}|v|^2-U(x)$*, the corresponding* α*-function* $\alpha(c)$ *is differentiable in some directions at* $c=0$*.*

Proof. If $U\equiv C$ is a contant, then the system is completely integrable and $\alpha(c)=\frac{1}{2}|c|^2$, which is trivial. Now suppose $U\not\equiv C$.

We apply Theorem 1 to the system $\frac{1}{2}|v|^2-U_\delta(x)$ at first for some small $\delta>0$. In that case, $\tilde{E}=Q\setminus B(y,\delta)$. Then there exists $\tilde{\Gamma}_{\mathbf{n}}\subset\tilde{E}$ as in Theorem 1 for some $\mathbf{n}\in\mathbb{Z}^n$. Each $\tilde{\Gamma}_{\mathbf{n}}$ determines a direction $\mathbf{n}$ such that $\alpha_\delta(r\mathbf{n})=\frac{1}{2}|r\mathbf{n}|^2$, where α_δ is the α-function of the system with potential U_δ. The number of such $\mathbf{n}$'s is decided by δ. Here we omit the argument of that number. Then we have that α_δ is quadratic in the direction $\mathbf{n}$ and is differentiable at $c=0$.

From the construction of the potential U_δ, together with Lemma 1, we easily have $\alpha_\delta(c)\geqslant\alpha(c)$ for all c. Then the differentiability of α at $c=0$ along the direction $\mathbf{n}$ can be obtained since $\alpha_\delta(0)=\alpha(0)$ and both of them are convex functions. $\square$

Corollary 1. *For the Tonelli Lagrangian in the form* $\ell(v)-U(x)$*, where* ℓ *is strictly convex,* $\ell(0)=0=\min_{v\in\mathbb{R}^n}\ell(v)$ *and* $\max_{x\in\mathbb{T}^n}U(x)=0$*, the* α*-function is differentiable in some directions at* $c=0$*.*

Proof. We only discuss the nontrivial case. Let $L_1(x,v)=\ell(v)-U(x)$, $L_2(x,v)=\frac{\lambda^2}{2}|v|^2-U_\delta(x)$ for some $\lambda>0$ and small $\delta>0$ as before. Then $L_1\geqslant L_2$ by the strict convexity assumption of ℓ and the definition of U_δ. Denoting by α_i, $i=1,2$, the α-function of L_i respectively, we have $\alpha_1\leqslant\alpha_2$, and the differentiability of α_1 in some directions follows from that of α_2, which has been proved in Theorem 2. $\square$

Remark 1. If Problem 1 in section 1 can be solved so as to be true, then the main result of [BC] can be improved. We will try it in other papers in the future.

3. Flat part of the α-function

Let $L^0(x,v)=\frac{1}{2}|v|^2-U(x)$ with the potential $U(x)\leqslant 0$ and $\max_{x\in\mathbb{T}^n}U(x)=0$, $\kappa(x)=\sqrt{2(-U(x))}$. We want to find a C^1 vector field $X(x)$ as a function $X:\mathbb{T}^n\to\mathbb{R}^n$ such that

$$|X(x)|=\kappa(x) \tag{9}$$

and

$$dX(x)=dX^*(x). \tag{10}$$

Condition (10) means that X is a *gradient-like vector field* corresponding to a closed 1-form on $\mathbb{T}^n$ in the following sense: the closed 1-form ω is defined by

$$\omega(x)(v)=\langle X(x),v\rangle,\quad v\in T_x\mathbb{T}^n\cong\mathbb{R}^n.$$

Now let us recall some basic facts on the construction of the closed 1-form on a closed smooth manifold M. Letting $f:M\to\mathbb{T}^1$ be a smooth map, the circle $\mathbb{T}^1$ is equipped with the canonical angular form $d\theta$, where $d\theta$ is a closed 1-form, which cannot be represented as a differential of a smooth function on $\mathbb{T}^1$. The pullback $f^*(d\theta)$ is a closed 1-form on M. It is not hard to show that a closed 1-form ω on M can be represented by this form if and only if the de Rham cohomology class $[\omega]\in H^1(M,\mathbb{Z})$ (see [Fab]).

In particular in the case of $M = \mathbb{T}^n$, we can construct the required closed 1-form as follows: Letting $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ be the local coordinates of $T\mathbb{T}^n$, any closed 1-form is locally exact by Poincaré's lemma. Then there exists an open cover $\{\Omega_i\}$ of $\mathbb{T}^n$ such that there exist smooth functions f_i defined on Ω_i such that $\omega|_{\Omega_i} = df_i|_{\Omega_i}$. More precisely, for the purpose of conditions (9) and (10), define the local vector field $X_i(x) = (0, \dots, \frac{\partial f_i}{\partial x_i}, \dots, 0)$ on Ω_i, where f_i is a smooth function on $\mathbb{T}^n$ such that $|\frac{\partial f_i}{\partial x_i}| = \kappa(x)$. The smoothness of the f_i's can be guaranteed if $\max_{x\in\mathbb{T}^n} U(x) < 0$. The global vector fields can be given by partition of unity. It is clear that the X_i's are independent.

Then define a new Lagrangian

$$L^1(x,v) = \frac{1}{2}|v - X(x)|^2 = L^0(x,v) - \langle X(x), v\rangle. \tag{11}$$

Denote by α_0 and α_1 the α-functions of L^0 and L^1 respectively.

Lemma 2. *If for the potential U of L^0 there exists a C^1 vector field X satisfying* (9) *and* (10) *and $c = \int_{\mathbb{T}^n} X(x)dx$, then*

$$\alpha_0(c) = \alpha_1(0) = 0 \tag{12}$$

with $|c| \leqslant \int_{\mathbb{T}^n} \kappa(x)dx$. Consequently, with the conditions above, the α-function has a flat part near 0 *in the direction c if $c \neq 0$.*

Proof. It is well known that for a positive definite Lagrangian L, $L - \lambda$ has the same Euler-Lagrange equation if the 1-form λ is closed. It is clear from (11) that if $c = \int_{\mathbb{T}^n} X(x)dx$, then $\alpha_1(0) = \alpha_0(c) = 0$. $|c| \leqslant \int_{\mathbb{T}^n} \kappa(x)dx$ follows easily from the definition of the vector field X. If $c \neq 0$, then (12) implies that $\alpha_0(rc) = 0$ for $0 \leqslant r \leqslant 1$ since $\alpha_0(0) = \min_{c\in\mathbb{R}^n} \alpha_0(c)$ and the α-function is convex. □

Theorem 3. *If there exist k vector fields X_i on $\mathbb{T}^n$ independently, $1 \leqslant k \leqslant n$, such that the X_i's satisfy* (9) *and* (10) *with $\int_{\mathbb{T}^n} X_i(x)dx \neq 0$, $i = 1, \dots, k$, then the α-function has k-dimensional flat part near $c = 0$.*

Proof. This is a direct consequence of Lemma 2. □

Now we apply Lemma 2 to some examples.

Example 1. When $n = 1$ and $\max_{x\in\mathbb{T}^n} U(x) = 0$, the flat part $|c| \leqslant \int_{\mathbb{T}^1} \kappa(x)dx$ of the α-function is well known; see e.g. [LPV]. Let $V_\varepsilon(x) = U(x) - \varepsilon$ for $\varepsilon > 0$. Then

$$L^0_\varepsilon(x,v) - \langle X_\varepsilon(x), v\rangle = \frac{1}{2}|v|^2 - V_\varepsilon(x) - \langle X_\varepsilon(x), v\rangle = \frac{1}{2}|v - X_\varepsilon(x)|^2 = L^1_\varepsilon(x,v),$$

where X_ε satisfies (9) and (10) for the potential V_ε. The existence of such an X_ε can easily be obtained by $X_\varepsilon = \sqrt{-V_\varepsilon}$. Denote by α^0_ε and α^1_ε the α-function of L^0_ε and L^1_ε respectively. Then we have $\alpha^1_\varepsilon(0) = \alpha^0_\varepsilon(c_\varepsilon) = 0$ by Lemma 2, where $c_\varepsilon = \int_{\mathbb{T}^1} X_\varepsilon(x)dx$. This implies that $\alpha^0(c_\varepsilon) - \varepsilon = 0$ for any $\varepsilon > 0$. So $\alpha(c_0) = 0$ by the continuity of c_ε with respect to ε, and $c_0 \neq 0$ if $U \not\equiv 0$. Thus the α-function has flat part on $[0, c_0]$, and the case of $[-c_0, 0]$ is similar by choosing $X_\varepsilon = -\sqrt{-V_\varepsilon}$.

Example 2. Let U be a smooth function on $\mathbb{T}^n$, and suppose that the critical set E defined in section 2 does not contain a simple closed homotopically nontrivial C^1 curve; e.g., U is a function of Morse type. Then the ε-trick in Example 1 and the construction of independent vector fields X_i as before imply the following:

Theorem 4. *If the critical set E of U of the system $L(x,v)=\frac{1}{2}|v|^2-U(x)$ does not contain a simple closed homotopically nontrivial smooth curve, then the α-function has fully dimensional flat part near $c=0$.*

Proof. If the critical set E of U does not contain a simple closed homotopically nontrivial smooth curve, then there exist n independent gradient-like vector fields $\{X_{i,\varepsilon}\}_{i=1}^n$ for $V_\varepsilon(x)=U(x)-\varepsilon$ as in Example 1. Applying the argument in Example 1 to each $X_{i,\varepsilon}$, we can see that there exists $c_{0,i}\neq 0$ such that the α-function has flat part in the direction of $c_{0,i}$. The independency of $X_{i,\varepsilon}$ means the independency of such $c_{0,i}$'s; thus we get the conclusion. □

Remark 1. For the mechanical systems with the form $L(x,v)=\frac{1}{2}|v|^2-U(x)$, the critical set $E=\{x: U(x)=\max_{x\in\mathbb{T}^n}U(x)\}$ is exactly the projected Aubry set $\mathscr{A}_0$. For the definition of *Aubry set* and *projected Aubry set*, see e.g. [Mat3, Be, Fat, FS]. So the existence of the flat part near $c=0$ is closely related to the topological structure of $\mathscr{A}_0$. Actually a very complicated structure of $\mathscr{A}_0$ exists, e.g., Mather's striking example ([Mat3]).

Example 3. Let U be a smooth function on $\mathbb{T}^n$, and let the critical set E defined in section 2 contain a simple closed homotopically nontrivial C^1 curve $\Gamma_{\mathbf{n}}$, i.e., the example described in Theorem 1. It is clear from the construction of gradient-like vector fields before that there exist n gradient-like vector fields $X_1,\dots,X_n$ on $\mathbb{T}^n$ for the ε-perturbed system as in Example 1 such that the closed 1-form related to X_n is exact and $X_1,\dots,X_{n-1}$ may be independent. Then a similar argument shows that there is no flat part along the direction of $\mathbf{n}\in\mathbb{Z}^n$ since X_n is a gradient field. Rewriting

$$\frac{1}{2}|v|^2-U(x)-\langle c,v\rangle=\frac{1}{2}|v-c|^2-\frac{1}{2}|c|^2-U(x),$$

we also have that the α-function is quadratic in the direction of $\mathbf{n}$, which is the same statement as in Theorem 1.

Remark 2. When we consider the case that the projected Aubry set $\mathscr{A}_0=E$ contains a general closed homotopically nontrivial C^1 curve, we need to find the obstacle to the existence of such a gradient-like vector field. The author hopes to try it in the future.

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