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WEIGHTED ESTIMATES FOR POWERS OF THE AHLFORS-BEURLING OPERATOR

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ABSTRACT. We prove that for any $n \in \mathbb{Z}\backslash\{0\}$, $p > 1$ and any weight w from the Muckenhoupt A_p class, the norm of the n-th power of the Ahlfors-Beurling operator T on the weighted Lebesgue space $L^p(w)$ is majorized by $C(p)\,|n|^3[w]_p^{\max\{1,1/(p-1)\}}$, where $[w]_p$ is the A_p characteristic of w. We apply this estimate for a result concerning the spectrum of T on $L^p(w)$.

1. Introduction

For any locally integrable function f on $\mathbb{C}$ and any bounded set $Q \subset \mathbb{C}$ with positive Lebesgue measure $|Q|$ denote by $\langle f\rangle_Q$ the average of f on Q,

$$\langle f\rangle_Q = \frac{1}{|Q|}\int_Q f(x)\,dx\,.$$

If w is a positive locally integrable function on $\mathbb{C}$, introduce

$$[w]_p := \sup_{Q\subset\mathbb{R}^2} \langle w\rangle_Q \big\langle w^{-\frac{1}{p-1}}\big\rangle_Q^{p-1}\,,$$

where the supremum is taken over all squares in $\mathbb{C}$ regardless of their orientation. Let $L^p(w)$ be the space of all functions p-integrable with respect to the weight w, i.e., $\int |f|^p w < \infty$. When $w \equiv 1$ we will simply write L^p.

From now on we assume that w belongs to the Muckenhoupt class A_p, defined as

$$w \in A_p \overset{\text{def}}{\iff} [w]_p < \infty\,.$$

The other fundamental object of this paper is the Ahlfors-Beurling operator T, which acts on test functions as

$$Tf(z) = -\frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{C}} \frac{f(\zeta)}{(\zeta - z)^2}\, dA(\zeta)\,.$$

Equivalently, it can be introduced in terms of the Fourier transforms:

$$\widehat{Tf}(\xi) = \frac{\bar{\xi}}{\xi}\,\hat{f}(\xi)\,. \tag{1}$$

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The importance that the estimates of T on classical L^p spaces bear in the theory of quasiconformal mappings has been known for many decades and is by now a classical fact; see e.g. [1, 3]. Iwaniec [10] conjectured that the exact norm of T on L^p is p^*-1, where $p^*=\max\{p,q\}$ and $q=p/(p-1)$, which would have far-reaching consequences as to the optimal exponent pertaining to the area distortion of quasiconformal mappings. The latter problem was posed by Gehring and Reich in the 1960s and solved by Astala in 1994 [2], though the Iwaniec conjecture remains open. The best current estimate is due to Bañuelos and Janakiraman [5].

It was the paper by Astala, Iwaniec and Saksman [4] that revealed deep connections between quasiconformal mappings and *weighted* estimates of T. The question was raised there whether T admits linear estimates on $L^p(w)$ when $p\geqslant 2$. This problem was first solved by Petermichl and Volberg [12] who confirmed that, for some $C>0$ and all $w\in A_2$,

$$\|T\|_{B(L^2(w))}\leqslant C[w]_2\,. \tag{2}$$

This estimate is sharp. By means of the extrapolation technique [7, Theorem 2] one then obtains, for $p>1$,

$$\|T\|_{B(L^p(w))}\leqslant C(p)[w]_p^{p^*/p}\,.$$

Their proof of (2) utilized the Bellman function method and the sharp weighted estimate for martingale transforms M_σ, due to Wittwer [13]:

$$\|M_\sigma\|_{B(L^2(w))}\leqslant C[w]_2\,. \tag{3}$$

Later on, Volberg and the present author [9] gave another proof of (2) by showing that T lies in the weakly closed linear span of M_σ and then applying (3).

The approach from [9] was then pursued in another direction, namely, for obtaining estimates for powers of the Ahlfors-Beurling operator on (unweighted) L^p; see [8]. Again the main result was that arbitrary powers T^n can be represented as averages of M_σ; cf. Theorem 2 there. The constants turned out to be linear in $n\in\mathbb{N}$, and since, owing to a well-known inequality of Burkholder, the norms of M_σ on L^p are not greater than p^*-1, we obtained $\|T^n\|_p\leqslant Cnp^*$; see [8, Theorem 1]. Finally, in [8] it was conjectured that for $p\geqslant 2$ and any $n\in\mathbb{N}$,

$$\|T^n\|_p=\frac{(1/q)_n}{(1/p)_n}\,,$$

where $(a)_n=a(a+1)\cdot\ldots\cdot(a+n-1)$ is the Pochhammer symbol; see also [6].

Based on the above results we find it natural to consider estimating powers of T on weighted spaces. It turns out that an adequate modification of the techniques from [9] and [8] yields the following generalization of the main result of [12]:

Theorem 1. *For every $p>1$ there is $C(p)>0$ such that for every $n\in\mathbb{Z}\backslash\{0\}$ and $w\in A_p$ we have*

$$\|T^n\|_{B(L^p(w))}\leqslant C(p)\,|n|^3\,[w]_p^{p^*/p}\,. \tag{4}$$

The estimates of T and T^n on L^p and T on $L^p(w)$ were all motivated by various aspects of the quasiconformal theory; see [10, 11, 4], respectively. As indicated in Theorem 1, the averaging method developed in [9] and [8] can be equally successfully applied for estimating T^n on $L^p(w)$ for arbitrary $n\in\mathbb{Z}$, $p>1$, $w\in A_p$. Thus to complete the picture one might want to pose a question in the "reverse" direction, i.e., to ask about possible complex-analytic implications of an estimate such as (4).

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Since powers of an operator feature prominently in the spectral radius formula, we are led to inquire whether Theorem 1 offers any meaningful information about the spectrum of T on $L^p(w)$.

Fix $p > 1$ and $w \in A_p$. Denote by $\sigma(T)$ the spectrum of T on $L^p(w)$, i.e., the set of all $\lambda \in \mathbb{C}$ for which $T - \lambda I$ is not invertible in $B(L^p(w))$, and by Δ the open unit disk in $\mathbb{C}$.

Corollary 1. *For any $p > 1$ and $w \in A_p$, we have $\sigma(T) = \partial\Delta$.*

In the unweighted case this result already appeared in [4, Proposition 2] and [3, Theorem 14.1.1]. In order to prove it the authors first showed (by a method different from the one in this paper) that (a) $\sigma(T) \subset \partial\Delta$, and then argued that (b) any $\lambda \in \partial\Delta$ is an eigenvalue for T. Yet the claim (b) cannot hold, which can for $p = 2$ be seen either by taking the Fourier transform or by recalling that any normal bounded operator on a separable Hilbert space has an at most countable point spectrum. A closer inspection of the proofs in [4] and [3] reveals that their candidate for an eigenvector of $T - \zeta I$ is actually a zero function if $|\zeta| = 1$. It is true, however, that $\sigma(T)$ consists of *approximate* eigenvalues, not only in L^p but in all $L^p(w)$.

It should be emphasized that the proofs from [4] and [3] have the advantage of bringing up an explicit formula for the inverse of $T - \zeta I$; see [3, identity (14.9)]. Our proof does not entail that.

These remarks motivate us to classify $\sigma(T)$ in the unweighted case, following the usual way of decomposing spectra of bounded operators on Banach spaces. Let $\sigma_c(T)$ be the continuous spectrum of T on L^p, i.e., the set of all $\lambda \in \sigma(T)$ for which $T - \lambda I$ has trivial kernel and dense range in L^p.

Theorem 2. *On any $L^p(\mathbb{R}^2)$, $p > 1$, we have $\sigma(T) = \sigma_c(T)$.*

2. Proofs

We first address Theorem 1. All the concepts we need for the proof are actually known already. We intend to average martingale transforms corresponding to translated, dilated and rotated dyadic lattices. In [9] this was done in order to achieve weighted estimates for T, while in [8] our goals were (unweighted) estimates of powers T^n. Therefore for our purpose it is most natural to try to combine these two paths.

Typically Haar functions are defined on intervals in $\mathbb{R}$. By working in $\mathbb{R}^2$ one is led to define them on squares [9]. It turns out, however, that the averaging method adjusted for powers T^n also calls for Haar systems defined on rectangles. When dealing with averaging processes it is desirable to make the objects of averaging have the same norm estimate. In the unweighted case [8] this was achieved by a construction which permitted a *uniform* application of the sharp Burkholder's result on the norms of general differentially subordinated martingales. Doing so, all the martingale transforms we considered were, regardless of their symbols, on L^p bounded by $p^* - 1$. This time we will apply Wittwer's result on weighted estimates of martingale transforms [13]. Adapting it to different rectangular Haar systems translates into dealing with different weights. This will at the end roughly explain why the linear estimate in terms of n from [8] turns into a cubic one.

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Thus the idea of the proof below is to combine and appropriately modify approaches from [9] and [8] while paying special attention to the way this affects the estimates of norms.

Proof of Theorem 1. We start by treating the case $p = 2$. First we need to introduce some auxiliary notions and objects. For $b > 0$ set

$$\mathcal{P}_b := \{\text{all rectangles in } \mathbb{R}^2 \text{ with the ratio of sides equal to } b\}\,.$$

We emphasize that this definition also includes rectangles whose sides are not parallel to the coordinate axes. Define

$$[w]_{A_2(b)} := \sup_{Q \subset \mathcal{P}_b} \langle w \rangle_Q \langle w^{-1} \rangle_Q\,.$$

When $b = 1$ we retain the notation $[w]_2$ or $[w]_{A_2}$. It is easy yet important to discern how different A_2 characteristics relate to each other:

$$\frac{1}{\varphi(b)^2} \leqslant \frac{[w]_{A_2(b)}}{[w]_{A_2}} \leqslant \varphi(b)^2\,, \tag{5}$$

where $\varphi(b) = e^{|\log b|} = \max\{b, 1/b\}$.

For each interval $I \subset \mathbb{R}$ let I_-, I_+ be its left and its right half, respectively. Denote by χ_I its characteristic function and by h_I its Haar function, which is to say $h_I = |I|^{-1/2}(\chi_{I_+} - \chi_{I_-})$. An interval in $\mathbb{R}$ is called dyadic if it is of the form $[k2^m, (k+1)2^m)$ for some integers k, m. Let $\mathcal{L}$ be the standard dyadic lattice in $\mathbb{R}^2$, i.e., the collection of squares $Q = I \times J$, where both I and J are dyadic intervals of the same length. The corresponding Haar functions are then

$$h_Q^0 = \frac{1}{\sqrt{|I|}}\, \chi_I \otimes h_J\,, \qquad h_Q^+ = \sqrt{\frac{2}{|J|}}\, h_I \otimes \chi_{J_+}\,, \qquad h_Q^- = \sqrt{\frac{2}{|J|}}\, h_I \otimes \chi_{J_-}\,.$$

The constants in the front are chosen so that the functions are normalized in (the usual) L^2.

In analogy with [8, Section 2.2] we define martingale transforms M_σ to be the operators

$$M_\sigma f(\zeta) = \sum_{\substack{Q \in \mathcal{L} \\ * \in \{0,+,-\}}} \sigma_Q^* \langle f, h_Q^* \rangle h_Q^*(\zeta)\,, \tag{6}$$

where $\sigma_Q^* \in \partial\Delta$ and h_Q^* are the Haar functions.

The result (3) addresses martingale transforms on the line and associated with the standard dyadic lattice in $\mathbb{R}$, but a careful reading of [13] shows that the same result holds for the "planar" operators M_σ defined above: there exists $C > 0$ such that for any family of coefficients σ and any $w \in A_2$,

$$\|M_\sigma\|_{B(L^2(w))} \leqslant C[w]_{A_2}\,. \tag{7}$$

This fact was already used implicitly in [9]. We omit its proof since it "only" requires following Wittwer's argument from [13] and checking that it holds in higher dimensions.

Again by drawing parallels with [8], this time to Section 3.1, we will work with one sole set of coefficients, namely $\sigma_Q^0 = 1$ and $\sigma_Q^+ = \sigma_Q^- = -1$ for all Q. In other words, our starting operator will be

$$M = \mathcal{P}_0 - \mathcal{P}_\pm\,,$$

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where $\mathcal{P}_0$ and $\mathcal{P}_\pm$ are orthogonal projections in L^2 onto the subspaces generated by all h_Q^0 and h_Q^+, h_Q^-, respectively.

As mentioned above, we need not only Haar systems built upon squares, but also those on (rotated and translated) rectangles. Precisely, fix $b > 0$, $\vartheta \in [0, 2\pi)$ and $\eta \in \mathbb{R}^2$. Let B be a composition of rotation by ϑ and dilation by b in the y-direction:

$$B = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix} \begin{bmatrix} 1 & \\ & b \end{bmatrix}.$$

Let $\Lambda(\zeta) = B\zeta + \eta$. Introduce $S = S_\Lambda$ by $Sg = g \circ \Lambda$ and

$$M_\Lambda = S^{-1} \circ M \circ S\,. \tag{8}$$

These operators can again be represented in the form of (6), save that now we are dealing with Haar functions over rectangles from $\Lambda\mathcal{L}$. We may still call them martingale transforms. Of course, $M_I = M$.

Let us denote by $[\cdot]_{A_2^r}$ the characteristic taken over *regular* squares only, i.e., those whose sides are parallel with the coordinate axes. One readily verifies that

$$[w]_{A_2^r} \leqslant [w]_{A_2} \leqslant 4[w]_{A_2^r} \quad \text{and} \quad [Sw]_{A_2^r} \leqslant [w]_{A_2(b)}\,.$$

Consequently,

$$[Sw]_{A_2} \leqslant 4[w]_{A_2(b)}\,. \tag{9}$$

We also need the identity

$$\|Sg\|_{L^2(Sw)} = \frac{1}{\sqrt{b}}\,\|g\|_{L^2(w)}\,. \tag{10}$$

Combine the above observations into an estimate of M_Λ on $L^2(w)$:

$$\begin{aligned}\|M_\Lambda\|_{B(L^2(w))} &\overset{(8)}{=} \|S^{-1}\circ M \circ S\|_{B(L^2(w))} \overset{(10)}{=} \|M\|_{B(L^2(Sw))} \\ &\overset{(7)}{\leqslant} C[Sw]_{A_2} \overset{(9)}{\leqslant} C[w]_{A_2(b)} \overset{(5)}{\leqslant} C\varphi(b)^2[w]_{A_2}\,.\end{aligned}$$

The next step is the averaging. As in [8, Section 3.2], with the distinction that now we are working in $L^2(w)$, for fixed $n \in \mathbb{N}, b > 0$ we get $T^n = C_b(n)\cdot T'$, where T' is an "average" of M_Λ, meaning that T' is an operator from the weakly closed convex hull of $\{M_\Lambda\}$, the point here being that its upper norm estimate is the same as for any of the M_Λ. The constant $C_b(n)$ is the same as in [8, p. 505]:

$$C_b(n) = \frac{(-1)^n n}{\pi \hat{k}_b(-2n)},$$

where k_b is, essentially, the average of the kernels involved in the process. The justification as to why the averaging process for T^n also works in weighted Lebesgue spaces is to be found in [9]; it was there that the same kind of representation was done for T alone. Again we do not write the proof for T^n since what it takes is reading already published results [9, 8].

Therefore, we proved that there is $C > 0$ such that for all $b > 0$ and $w \in A_2$,

$$\|T^n\|_{B(L^2(w))} \leqslant C\left|\frac{n}{\hat{k}_b(-2n)}\right|\varphi(b)^2[w]_{A_2}\,. \tag{11}$$

A lower estimate for $\hat{k}_b$ was also worked out in [8]. It was shown (see the proof of Proposition 3 there) that for some $\lambda, \varepsilon > 0$ and $n_0 \in \mathbb{N}$ we have

$$|\hat{k}_{\lambda/n}(-2n)| > \varepsilon \tag{12}$$

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for all $n \geqslant n_0$, while for n up to n_0 use that $\sup_{b>0} |\hat{k}_b(-2n)| > 0$. We find it appropriate to mention that the estimate (12) was the reason to consider more general "rectangular" Haar systems; cf. [8, Remark on p. 506]. Now choose $b = \lambda/n$ and apply (12) to (11). Note that $\varphi(\lambda/n) \leqslant Cn$ for some $C > 0$ and all $n \in \mathbb{N}$. We get $C > 0$ such that for all $n \in \mathbb{N}$ and all $w \in A_2$,

$$\|T^n\|_{B(L^2(w))} \leqslant C|n|^3 [w]_{A_2} \,. \tag{13}$$

The inverse T^{-1} exists on $L^p(w)$ and acts on test functions as $f \mapsto \overline{T\overline{f}}$. This quickly implies that (13) also holds for negative integers. Finally we extend (13) to arbitrary $p > 1$ by means of the sharp version of the Rubio-de-Francia extrapolation theorem, established in [7, Theorem 1]. □

Before proving Corollary 1 we need to invoke some basic notions and facts. Let $\mathcal{S}$ denote the Schwartz class on $\mathbb{C}$. We are still working in $L^p(w)$ with $p > 1$ and $w \in A_p$ fixed. Let $\rho(T) = \mathbb{C} \backslash \sigma(T)$ be the resolvent set and $r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}$ the spectral radius of T. Recall that $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$.

Proof of Corollary 1. It follows straight from Theorem 1 that $\max\{r(T), r(T^{-1})\} \leqslant 1$. Hence $\sigma(T) \cup \sigma(T^{-1}) \subset \overline{\Delta}$. By the spectral mapping theorem, $\sigma(T^{-1}) = [\sigma(T)]^{-1}$. Thus we conclude that $\sigma(T) \subset \partial\Delta$.

Now let us verify the inclusion in the opposite direction. The author is indebted to Michael Cowling for showing him how to do that.

Take $\lambda \in \partial\Delta$ and write $\lambda = e^{-2i\vartheta}$. Our intention is to show that λ is an approximate eigenvalue for T on $L^p(w)$. For $n \in \mathbb{Z}$ and a function $g = g(z)$ write $(S_n g)(z) = e^{2n\pi i \Re(e^{-i\vartheta} z)} g(z)$. Take $f \in \mathcal{S}$ such that $\|f\|_{L^p(w)} = 1$ and $\operatorname{supp} \widehat{f} \subset \Delta$. We claim that $\lim_{n\to\infty} (T - \lambda I)(S_n f) = 0$ in $L^p(w)$. This is equivalent to

$$\lim_{n\to\infty} (T_n - \lambda I) f = 0 \quad \text{ in } L^p(w)\,, \tag{14}$$

where $T_n = S_n^{-1} T S_n$. It suffices to demonstrate that the convergence from (14) holds in $\mathcal{S}$, i.e., that $\widehat{T_n f} \to \lambda \widehat{f}$ in $\mathcal{S}$. One calculates

$$\widehat{T_n f}(\xi) = e^{-2i \arg(\xi + n e^{i\vartheta})} \widehat{f}(\xi) \,.$$

Owing to the assumption on $\operatorname{supp} \widehat{f}$ it is now enough to verify that $\arg(\cdot\, e^{-i\vartheta} + n)$ tends to 0 in $C^\infty(\Delta)$, which emerges after a short computation. □

Remark 1. With $U_\zeta \varphi(z) = \varphi(\zeta z)$ for $\zeta \in \partial\Delta$ and $z \in \mathbb{C}$ we have

$$T - \zeta^2 I = \zeta^2 U_\zeta (T - I) U_\zeta^{-1} \,. \tag{15}$$

When w is rotation-invariant this identity quickly implies that $\sigma(T)$ is also rotation-invariant; hence $\sigma(T) \subset \partial\Delta$ immediately gives $\sigma(T) = \partial\Delta$. Kari Astala pointed out (personal communication) that a similar argument is valid in many other function spaces, e.g., Besov, Hölder, Triebel-Lizorkin.

Proof of Theorem 2. It suffices to consider the case $p \geqslant 2$. Indeed, since the same proofs as for T also work for T^* and since the spaces L^p and L^q are mutually dual, the case $1 < p < 2$ follows from the well-known fact that, for any bounded operator A on a Banach space X, $\operatorname{Im} A$ is dense in X if and only if $\operatorname{Ker} A^* = \{0\}$. Furthermore, by (15) it is enough to see that $1 \in \sigma_c(T)$.

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Thus assume $p \geqslant 2$ and write $T_1 = T - I$. First we will show that $\operatorname{Ker} T_1 = \{0\}$ on L^p. If $p = 2$ this follows directly from (1). Now suppose $p > 2$. For every $h \in L^p$ we define

$$(Ph)(z) := -\frac{1}{\pi}\int_{\mathbb{C}} h(\zeta)\left(\frac{1}{\zeta - z} - \frac{1}{\zeta}\right) dA(\zeta)\,.$$

We will need a couple of well-known properties of the operator P. They can be found in [1, Chapter V.A], for example. First of all, if $p > 2$, then $g = Ph$ is well defined on L^p and satisfies

$$h = \partial_{\bar z} g \qquad \text{and} \qquad Th = \partial_z g \tag{16}$$

in the distributional sense, with test functions from C_c^1. Here $\partial_{\bar z} = (\partial_x + i\partial_y)/2$ and $\partial_z = (\partial_x - i\partial_y)/2$, as usual. If we additionally assume that $h \in \operatorname{Ker} T_1$, then $\partial_y g = 0$ in the same distributional sense. We want to show that $\partial_y g$ exists and is equal to zero in the usual sense, as well.

Since g is Hölder continuous with the exponent $1 - 2/p$, it defines a tempered distribution, denoted by the same letter. The space C_c^∞ is dense in $\mathcal{S}$, equipped with the usual topology induced by the Schwartz seminorms; therefore $\partial_y g = 0$ is also valid for test functions from $\mathcal{S}$. By taking the Fourier transform we get $\xi_2\,\hat g = 0$, where (ξ_1, ξ_2) are coordinates in the Fourier domain. From here we conclude that $\hat g = w \otimes \delta$, where $w \in \mathcal{S}'(\mathbb{R})$ and δ is the Dirac distribution, i.e., the evaluation at 0. Repeated application of the Fourier transform yields $g = \widetilde w \otimes 1$ for some $\widetilde w \in \mathcal{S}'(\mathbb{R})$. This implies that g, now again understood as a function, only depends on the real part, i.e., $g(z) = g(\operatorname{Re} z)$ for all $z \in \mathbb{C}$, as desired. In particular, $Qh \equiv 0$, where $(Qh)(z) = g(z) - g(z+i)$. By means of (16) we obtain $[\partial_{\bar z}(Qh)](z) = h(z) - h(z+i)$ in the distributional sense. Thus we proved that $h(z) = h(z+i)$ a.e. $z \in \mathbb{C}$, which for $h \in L^p$ is only possible if $h \equiv 0$ a.e. $\mathbb{C}$. This confirms that $\operatorname{Ker} T_1$ is trivial.

We are left with proving that $\operatorname{Im} T_1$ is dense in L^p. Take $f \in L^p$ and $\varepsilon > 0$. Our goal is to find $g \in \mathcal{S}$ such that $\|f - T_1 g\|_p < \varepsilon$. We can assume that $f \in \mathcal{S}$. Take any $\varphi \in \mathcal{S}$ such that $\|\varphi\|_q = 1$. By using (1) and the Plancherel identity compute, for a generic g which is to be determined later, $\langle f - T_1 g, \varphi\rangle = \langle \hat f - H\hat g, \hat\varphi\rangle$, where $H(\xi) = e^{-2i\arg\xi} - 1$ for $\xi \in \mathbb{C}\backslash\{0\}$. Since $1 < q \leqslant 2$, the Hausdorff-Young and the Cauchy-Schwarz inequalities imply that $|\langle f - T_1 g, \varphi\rangle| \leqslant \|\hat f - H\hat g\|_q$. Therefore, given $F \in \mathcal{S}$, our problem reduces to finding $G \in \mathcal{S}$ such that $\|F - HG\|_q < \varepsilon$. We may assume that $F \in C_c^\infty(\mathbb{C})$. Choose $R > 0$ so that $\operatorname{supp} F \subset \{z \in \mathbb{C} : |z| \leqslant R\} =: K_R$. For small $\delta > 0$ define $D_\delta := \{(x,y) \in K_R : |y| \geqslant \delta\}$ and $E_\delta := K_R \backslash D_\delta$. There exists a Urysohn function $u_\delta \in C_c^\infty(\mathbb{C})$ such that

- (i) $u_\delta \in [0,1]$ everywhere on $\mathbb{C}$,
- (ii) $u_\delta \equiv 1$ on D_δ,
- (iii) $u_\delta \equiv 0$ on $E_{\delta/2}$.

By letting $G_\delta(\xi) := 0$ for $\xi \in \mathbb{R}$ and $G_\delta := F \cdot u_\delta / H$ otherwise, we see that $G_\delta \equiv 0$ on $E_{\delta/2}$ and so $G_\delta \in C_c^\infty(\mathbb{C})$. Now clearly $\|F - HG_\delta\|_q \to 0$ as $\delta \to 0$. □

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