

SEQUENTIALLY S_r SIMPLICIAL COMPLEXES AND SEQUENTIALLY S_2 GRAPHS

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ABSTRACT. We introduce sequentially S_r modules over a commutative graded ring and sequentially S_r simplicial complexes. This generalizes two properties for modules and simplicial complexes: being sequentially Cohen-Macaulay, and satisfying Serre's condition S_r . In analogy with the sequentially Cohen-Macaulay property, we show that a simplicial complex is sequentially S_r if and only if its pure i -skeleton is S_r for all i . For $r = 2$, we provide a more relaxed characterization. As an algebraic criterion, we prove that a simplicial complex is sequentially S_r if and only if the minimal free resolution of the ideal of its Alexander dual is componentwise linear in the first r steps. We apply these results for a graph, i.e., for the simplicial complex of the independent sets of vertices of a graph. We characterize sequentially S_r cycles showing that the only sequentially S_2 cycles are odd cycles and, for $r \geq 3$, no cycle is sequentially S_r with the exception of cycles of length 3 and 5. We extend certain known results on sequentially Cohen-Macaulay graphs to the case of sequentially S_r graphs. We prove that a bipartite graph is vertex decomposable if and only if it is sequentially S_2 . We provide some more results on certain graphs which in particular implies that any graph with no chordless even cycle is sequentially S_2 . Finally, we propose some questions.

1. INTRODUCTION

Let $R = k[x_1, \dots, x_n]$ be the polynomial ring over a field k . For finitely generated graded R -modules, Stanley has defined the sequentially Cohen-Macaulay property [16, Chapter III, Definition 2.9] and has studied the corresponding simplicial complexes. Here we consider sequentially S_r graded modules, i.e., finitely generated graded R -modules which satisfy Serre's S_r condition sequentially. Then we study the corresponding simplicial complexes, sequentially S_r simplicial complexes. Duval has shown that a simplicial complex is sequentially Cohen-Macaulay if and only if its pure i -skeleton is Cohen-Macaulay for all i [5, Theorem 3.3]. We prove the analogue result for sequentially S_r simplicial complexes (see Theorem 2.6). For $r = 2$, we show that a simplicial complex is sequentially S_2 if and only if its pure i -skeletons are connected for all $i \geq 1$ and the link of every singleton is sequentially

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S_2 (see Theorem 2.9). A major result of Eagon and Reiner states that a simplicial complex is Cohen-Macaulay if and only if the Stanley-Reisner ideal of its Alexander dual has a linear resolution [6, Theorem 3]. Later, Herzog and Hibi generalized this result by proving that a simplicial complex is sequentially Cohen-Macaulay if and only if the minimal free resolution of the Stanley-Reisner ideal of its Alexander dual is componentwise linear [10, Theorem 2.9]. The result of Eagon and Reiner has been generalized in another direction by Yanagawa (with N. Terai), showing that a simplicial complex is S_r if and only if the minimal free resolution of its Alexander dual is linear in the first r steps [22, Corollary 3.7]. We adopt the above two results to show that a simplicial complex is sequentially S_r if and only if the minimal free resolution of the Stanley-Reisner ideal of its Alexander dual is componentwise linear in the first r steps (see Corollary 3.3).

As the first application of our results, we characterize sequentially S_r cycles. It is known that the only cycles which are sequentially Cohen-Macaulay are C_3 and C_5 [8, Proposition 4.1] and that the only cycles which are S_2 , are C_3 , C_5 and C_7 [9, Proposition 1.6]. We extend these results by showing that C_n is sequentially S_2 if and only if n is odd and that the only sequentially S_3 cycles are C_3 and C_5 , i.e., the Cohen-Macaulay cycles (see Theorem 4.1 and Proposition 4.2).

Van Tuyl [19, Theorem 2.10] has recently proved that a bipartite graph is vertex decomposable if and only if it is sequentially Cohen-Macaulay. We prove that a bipartite graph is vertex decomposable if and only if it is sequentially S_2 (see Theorem 4.5). This also generalizes the result [9, Theorem 1.3] which states that a bipartite graph is Cohen-Macaulay if and only if it is S_2 .

Van Tuyl and Villarreal [20] have studied sequentially Cohen-Macaulay graphs. We extend some of their results and generalize a result of Francisco and Hà [7, Theorem 4.1] on graphs with whiskers (see Corollary 4.7). We also provide some graph theoretic criterion for graphs to be sequentially S_2 and for bipartite graphs to be vertex decomposable (see Theorem 4.8 and Corollary 4.9).

Woodrooffe [21, Theorem 1.1] has proved that a graph with no chordless cycles other than cycles of length 3 and 5 is sequentially Cohen-Macaulay. We extend this result for the S_2 property (see Theorem 4.10). This in particular implies that any graph with no chordless even cycle is sequentially S_2 (see Corollary 4.11).

At the end of this paper we propose two questions on sequentially the S_r property of the join of two simplicial complexes and the topological invariance of S_r , respectively.

The motivation behind our work is the general philosophy that Serre's S_r condition plays an important role, not only in algebraic geometry and commutative algebra, but also in algebraic combinatorics (e.g. see [15], [22], [17]).

2. CRITERIA FOR SEQUENTIALLY S_r SIMPLICIAL COMPLEXES

In this section we give some basic definitions and criteria for sequentially S_r property on simplicial complexes. We prove that a simplicial complex is sequentially S_r if and only if its pure skeletons are all S_r , a generalization of Duval's result on sequentially Cohen-Macaulay simplicial complexes [5, Theorem 3.3]. We show that a simplicial complex is sequentially S_2 if and only if its pure i -skeletons are connected for all $i \geq 1$ and the link of every singleton is sequentially S_2 .

Recall that a finitely generated graded module M over a Noetherian graded k -algebra R is said to satisfy the Serre’s condition S_r if

$$\text{depth } M_{\mathfrak{p}} \geq \min(r, \dim M_{\mathfrak{p}}),$$

for all $\mathfrak{p} \in \text{Spec}(R)$.

First we introduce the concept of sequentially S_r modules.

Definition 2.1. Let M be a finitely generated \mathbb{Z} -graded module over a standard graded k -algebra R where k is a field. For a positive integer r we say that M is sequentially S_r if there exists a finite filtration

$$0 = M_0 \subset M_1 \subset \dots \subset M_t = M$$

of M by graded submodules M_i satisfying the following two conditions:

- (a) Each quotient M_i/M_{i-1} satisfies the S_r condition of Serre.
- (b) $\dim(M_1/M_0) < \dim(M_2/M_1) < \dots < \dim(M_t/M_{t-1})$.

We say that a simplicial complex Δ on $[n] = \{1, \dots, n\}$ is sequentially S_r (over a field k) if its face ring $k[\Delta] = k[x_1, \dots, x_n]/I_{\Delta}$, as a module over $R = k[x_1, \dots, x_n]$, is sequentially S_r .

This is a natural generalization of an S_r simplicial complex, i.e., when $k[\Delta]$ satisfies the S_r condition of Serre.

Since $k[\Delta]$ is a reduced ring, it always satisfies the S_1 condition. Thus, throughout this paper we will always deal with S_r for $r \geq 2$.

Using a result of Schenzel [15, Lemma 3.2.1] and Hochster’s formula on local cohomology modules, N. Terai has formulated the analogue of Reisner’s criterion for Cohen-Macaulay simplicial complexes in the case of S_r simplicial complexes [17, page 4, following Theorem 1.7]. According to this formulation, a simplicial complex Δ of dimension $d - 1$ is S_r if and only if for all $-1 \leq i \leq r - 2$ and all $F \in \Delta$ (including $F = \emptyset$) with $\#F \leq d - i - 2$ we have $\tilde{H}_i(\text{lk}_{\Delta} F; k) = 0$, where $\text{lk}_{\Delta} F = \{G \in \Delta : F \cup G \in \Delta, F \cap G = \emptyset\}$. For $i = -1$ the vanishing condition is equivalent to purity of Δ , and for $i = 0$ it is equivalent to the connectedness of $\text{lk}_{\Delta} F$ [17, page 4 and 5].

By this characterization of S_r simplicial complexes it follows that the S_r property carries over links.

Lemma 2.2. *Let Δ be a $(d - 1)$ -dimensional simplicial complex which satisfies the S_r condition. Then for each $F \in \Delta$ the simplicial complex $\text{lk}_{\Delta} F$ also satisfies S_r .*

Proof. Let $\#F = j$; then $\dim \text{lk}_{\Delta} F \leq d - j - 1$. By the above characterization of S_r simplicial complexes, it suffices to show that for all $i \leq r - 2$ and every $G \in \text{lk}_{\Delta} F$, with $\#G \leq d - j - i - 2$, the reduced homology module $\tilde{H}_i(\text{lk}_{\text{lk}_{\Delta} F} G; k)$ is zero. This follows from the facts that $\text{lk}_{\text{lk}_{\Delta} F}(G) = \text{lk}_{\Delta}(F \cup G)$, $\#(F \cup G) \leq d - i - 2$ and Δ is S_r . □

Recall that a relative simplicial complex is a pair of simplicial complexes (Δ, Γ) where Γ is a subcomplex of Δ . For a relative simplicial complex (Δ, Γ) define $I_{\Delta, \Gamma}$ to be the ideal in $k[\Delta]$ generated by the monomials $x_{i_1} x_{i_2} \dots x_{i_s}$ with $F = \{i_1, \dots, i_s\} \in \Delta \setminus \Gamma$. A relative simplicial complex is said to be S_r if $I_{\Delta, \Gamma}$ is S_r as a module over $R = k[x_1, \dots, x_n]$. Let Δ_i^* be the subcomplex of Δ generated by its i -dimensional facets. Following [3, Appendix II], it turns out that Δ is sequentially S_r if and only if the relative simplicial complex $(\Delta_i^*, \Delta_i^* \cap (\Delta_{i+1}^* \cup \dots \cup \Delta_{\dim(\Delta)}^*))$ is S_r for all i .

For a relative simplicial complex (Δ, Γ) , let $\tilde{H}_i(\Delta, \Gamma; k)$ denote the i th reduced relative homology group of the pair (Δ, Γ) over k (see [16, Chapter III, §7]). Reisner’s criterion for Cohen-Macaulayness of a relative simplicial complex is similar to the one for a simplicial complex [16, Chapter III, Theorem 7.2]. Likewise, in an exact analogy, Terai’s formulation for an S_r simplicial complex carries over for the relative case. In other words, a relative simplicial complex (Δ, Γ) is S_r if and only if $\tilde{H}_i(\text{lk}_\Delta F, \text{lk}_\Gamma F; k) = 0$ for all $-1 \leq i \leq r - 2$ and all $F \in \Delta$ (including $F = \emptyset$) with $\#F \leq d - i - 2$, where $d - 1 = \dim(\Delta)$.

For a relative simplicial complex (Δ, Γ) , as an R -module, $I_{\Delta, \Gamma}$ only depends on the difference $\Delta \setminus \Gamma$ (see the remarks following [16, Chapter III, Theorem 7.3]). In particular, if $\Delta^{(i)}$ is the i -skeleton of Δ and $\Delta^{[i]}$ is the pure i -skeleton of Δ , then

$$\Delta_i^* \setminus (\Delta_i^* \cap \Delta_{i+1}^* \cup \dots \cup \Delta_{\dim(\Delta)}^*) = \Delta^{[i]} \setminus (\Delta^{[i+1]})^{(i)}.$$

Duval makes the above observation and proves that the relative simplicial complex $\Delta^{[i]} \setminus (\Delta^{[i+1]})^{(i)}$ is Cohen-Macaulay for all i if and only if every pure i -skeleton $\Delta^{[i]}$ is Cohen-Macaulay [5, Section 3]. We follow his proof step by step to show that the same result is true if we replace the Cohen-Macaulay property with S_r . To do this we need some preliminary results.

It is known that if Δ is a Cohen-Macaulay simplicial complex, then so is $\Delta^{(i)}$, the i -skeleton of Δ . We generalize this result for the property S_r .

Proposition 2.3. *If Δ satisfies Serre’s condition S_r , then $\Delta^{(i)}$ satisfies this condition ($2 \leq r \leq i + 1$).*

Proof. We check Terai’s criterion for S_r simplicial complexes. To prove the assertion for $\Delta^{(i)}$ we use induction on $r \geq 2$. Assume that Δ satisfies Serre’s condition S_2 . Then Δ is pure and hence $\Delta^{(i)}$ is pure. Furthermore, for $F \in \Delta$ with $\#F \leq d - 2$, $\text{lk}_\Delta F$ is connected. Let $F \in \Delta^{(i)}$ and $\#F \leq i - 1$. It is enough to show that $\text{lk}_{\Delta^{(i)}} F$ is connected, or equivalently, path connected. Let $\{u\}, \{v\} \in \text{lk}_{\Delta^{(i)}} F$. Then $\{u\}, \{v\} \in \text{lk}_\Delta F$, which is connected. Hence, there exists a sequence of vertices of Δ , $u_0 = u, u_1, \dots, u_t = v$, such that $\{u_j, u_{j+1}\} \in \text{lk}_\Delta F$, $j = 0, \dots, t - 1$. Thus, $\{u_j, u_{j+1}\} \cap F = \emptyset$ and $\{u_j, u_{j+1}\} \cup F \in \Delta$. Since $\#(\{u_j, u_{j+1}\} \cup F) \leq i + 1$, $\{u_j, u_{j+1}\} \cup F \in \Delta^{(i)}$ and hence $\{u_j, u_{j+1}\} \in \text{lk}_{\Delta^{(i)}} F$.

Now assume that Δ satisfies Serre’s condition S_r for $r > 2$. Then Δ satisfies Serre’s condition S_j for $j = 1, \dots, r$. Thus by induction hypothesis $\Delta^{(i)}$ satisfies Serre’s condition S_j for $j = 1, \dots, r - 1$. Therefore, for $q \leq r - 3$ and $F \in \Delta^{(i)}$ with $\#F \leq i - q - 1$, $\tilde{H}_q(\text{lk}_{\Delta^{(i)}} F; k) = 0$. Thus it remains to show that for $\#F \leq i - r + 1$, $\tilde{H}_{r-2}(\text{lk}_{\Delta^{(i)}} F; k) = 0$. To prove this, since $\text{lk}_{\Delta^{(i)}} F \subset \text{lk}_\Delta F$ it is enough to show that for $q \leq r - 1$, any q -dimensional face H of $\text{lk}_\Delta F$ lies in $\text{lk}_{\Delta^{(i)}} F$. But $\#(H \cup F) \leq i + 1$, and hence $H \cup F \in \Delta^{(i)}$, and consequently $H \in \text{lk}_{\Delta^{(i)}} F$. \square

Now we adopt Duval’s results for the case of sequentially S_r simplicial complexes.

Lemma 2.4 (see [5, Lemma 3.1]). *Let F be a face of a $(d - 1)$ -dimensional simplicial complex Δ and let Γ be either the empty simplicial complex or a S_r simplicial complex of the same dimension as Δ . Then*

$$\tilde{H}_i(\text{lk}_\Delta F; k) = \tilde{H}_i(\text{lk}_\Delta F; \text{lk}_\Gamma F; k)$$

for all $i \leq r - 2$ and all $F \in \Delta$ with $\#F \leq d - i - 2$.

Proof. The proof is the same as the proof of the similar lemma of Duval [5, Lemma 3.1]. If $\text{lk}_\Gamma F$ is an empty set, then the equality is obvious. Otherwise one only needs to change the range of the index i with the one given above, impose the condition on $\#F$ and replace Cohen-Macaulay property with S_r . \square

Corollary 2.5 (see [5, Corollary 3.2]). *Let Δ be a simplicial complex, and let Γ be either the empty simplicial complex or a S_r simplicial complex of the same dimension as Δ . Then Δ is S_r if and only if (Δ, Γ) is relative S_r .*

Proof. Similar to the corresponding corollary by Duval [5, Corollary 3.2], it follows from Lemma 2.4. \square

Theorem 2.6 (see [5, Theorem 3.3]). *Let Δ be a $(d - 1)$ -dimensional simplicial complex. Then Δ is sequentially S_r if and only if its pure i -skeleton $\Delta^{[i]}$ is S_r for all $-1 \leq i \leq d - 1$.*

Proof. The proof is the same as the one given by Duval [5, Theorem 3.3]. The only item needed is that each i -skeleton of an S_r simplicial complex is again S_r for all i . But this is proved in Proposition 2.3. \square

The following is an immediate bi-product of this theorem.

Corollary 2.7. *A simplicial complex Δ is S_r if and only if it is sequentially S_r and pure.*

Proof. Since the S_r condition implies purity, one implication is clear. Assume that Δ is pure and $\dim \Delta = d - 1$. Then $\Delta^{[d-1]} = \Delta$, and the assertion follows by Theorem 2.6. \square

Remark 2.8. Some authors define a simplicial complex to be sequentially Cohen-Macaulay if its pure i -skeleton is Cohen-Macaulay for all i . Likewise, we might take a similar statement as the definition of sequentially S_r simplicial complexes. But we preferred to begin with the algebraic definition given in this section and prove that both definitions are equivalent.

We end this section with the following characterization of sequentially S_2 simplicial complexes which will be used in the last section.

Theorem 2.9. *Let Δ be a simplicial complex with vertex set V . Then Δ is sequentially S_2 if and only if the following conditions hold:*

- (i) $\Delta^{[i]}$ is connected for all $i \geq 1$.
- (ii) $\text{lk}_\Delta(x)$ is sequentially S_2 for all $x \in V$.

Proof. Let Δ be sequentially S_2 . Then $\Delta^{[i]}$ is S_2 for all $-1 \leq i \leq d - 1$. Thus $\Delta^{[i]}$ is connected for all $i \geq 1$. On the other hand $\text{lk}_{\Delta^{[i]}}(x)$ is S_2 for all $-1 \leq i \leq d - 1$, and so $\text{lk}_{\Delta^{[i+1]}}(x) = (\text{lk}_\Delta(x))^{[i]}$ is S_2 for all $-1 \leq i \leq d - 2$. Therefore $\text{lk}_\Delta(x)$ is sequentially S_2 .

Now let Δ satisfy conditions (i) and (ii). Since $\text{lk}_\Delta(x)$ is sequentially S_2 for all $x \in V$, we have that $(\text{lk}_\Delta(x))^{[i]} = \text{lk}_{\Delta^{[i+1]}}(x)$ is S_2 for all $-1 \leq i \leq d - 2$, and so $\text{lk}_{\Delta^{[i]}}(x)$ is S_2 for all $-1 \leq i \leq d - 1$. Now the connectedness of $\Delta^{[i]}$ for $i \geq 1$ implies that $\Delta^{[i]}$ is S_2 for all $-1 \leq i \leq d - 1$. Indeed, for $F \neq \emptyset$ and $x \in F$, $\text{lk}_{\Delta^{[i]}F} = \text{lk}_{\text{lk}_{\Delta^{[i]}(x)}G}$ for $G = F \setminus \{x\}$. Therefore Δ is sequentially S_2 . \square

3. ALEXANDER DUAL OF SEQUENTIALLY S_r SIMPLICIAL COMPLEXES

In this section we show that a simplicial complex is sequentially S_r if and only if the minimal free resolution of the Stanley-Reisner ideal of its Alexander dual is componentwise linear in the first r steps. This result resembles a result of Herzog and Hibi [10, Proposition 1.5] on sequentially Cohen-Macaulay simplicial complexes. Also our proof would be a modification of the sequentially Cohen-Macaulay case together with an application of Theorem 2.6.

We first adopt the following definitions from [22, Definition 3.6] and [10, §1].

Consider $R = k[x_1, \dots, x_n]$ with $\deg(x_i) = 1$ for all i . If I is a homogenous ideal of R and $r \geq 1$, then I is said to be linear in the first r steps, if for some integer d , $\beta_{i, i+t}(I) = 0$ for all $0 \leq i < r$ and $t \neq d$. We write $I_{(j)}$ for the ideal generated by all homogenous polynomials of degree j belonging to I . We say that a homogenous ideal $I \subset R$ is componentwise linear if $I_{(j)}$ has a linear resolution for all j . The ideal I is said to be *componentwise linear in the first r steps* if for all $j \geq 0$, $I_{(j)}$ is linear in the first r steps. A simplicial complex Δ on $[n]$ is said to be linear in the first r steps, componentwise linear and componentwise linear in the first r steps if I_Δ satisfies either of these properties, respectively.

Now let $I \subset R$ be an ideal generated by squarefree monomials. Then for each degree j we write $I_{[j]}$ for the ideal generated by the squarefree monomials of degree j belonging to I . We say that I is squarefree componentwise linear if $I_{[j]}$ has linear resolution for all j . The ideal I is said to be *squarefree componentwise linear in the first r steps* if $I_{[j]}$ has a resolution which is linear in the first r steps for all j .

Below we adopt a result of Herzog and Hibi [10, Proposition 1.5] for the case of componentwise linearity in the first r steps.

Proposition 3.1. *Let I be a squarefree monomial ideal in R . Then I is componentwise linear in the first r steps if and only if I is squarefree componentwise linear in the first r steps.*

Proof. The proof is the same as [10, Proposition 1.5] with just a restriction on the index i used in the proof of Herzog and Hibi. Here we need to assume that $i < r$. Also, one needs to observe that when I has a linear resolution in the first r steps, the ideal $\mathfrak{m}I$ has a linear resolution in the first r steps too. Here $\mathfrak{m} = (x_1, \dots, x_n)$ is the irrelevant maximal ideal. \square

We may now generalize a result of Herzog and Hibi [10, Theorem 2.1]. As we already mentioned, Yanagawa and Terai proved that Δ is S_r if and only if I_Δ^\vee is linear in the first r steps.

Theorem 3.2. *The Stanley-Reisner ideal of Δ on $[n]$ is componentwise linear in the first r steps if and only if Δ^\vee , the Alexander dual of Δ , is sequentially S_r .*

Proof. The proof is an adaptation of the proof of part (a) of [10, Theorem 2.1] with the following additional remarks: Let $I = I_\Delta$. Then by Proposition 3.1, I is squarefree componentwise linear in the first r steps if and only if I is componentwise linear in the first r steps. By [22, Corollary 3.7] for every j , $I_{[j]}$ is linear in the first r steps if and only if $(\Delta^\vee)^{[n-j-1]}$ is S_r . Therefore, I is componentwise linear in the first r steps if and only if $(\Delta^\vee)^{[q]}$ is S_r for every q . But by Theorem 2.6, this is equivalent to the sequentially S_r property for Δ^\vee . \square

Van Tuyl and Villarreal [20, Theorem 3.8 (a)] state the dual version of [10, Theorem 2.1] for sequentially Cohen-Macaulay simplicial complexes. Dualizing the statement of the above theorem, we get a similar generalization for sequentially S_r simplicial complexes.

Corollary 3.3. *A simplicial complex Δ is sequentially S_r if and only if the Stanley-Reisner ideal of the Alexander dual of Δ is componentwise linear in the first r steps.*

4. SOME CHARACTERIZATIONS OF SEQUENTIALLY S_r CYCLES
AND SEQUENTIALLY S_2 BIPARTITE GRAPHS

In this section, we provide some applications of the results of the previous sections. We first classify sequentially S_r cycles and show that a cycle C_n is sequentially S_2 if and only if n is odd and no cycles are sequentially S_3 except those which are Cohen-Macaulay, i.e., C_3 and C_5 . This generalizes a result of Francisco and Van Tuyl [8, Proposition 4.1]. Then we generalize a result of Van Tuyl [19, Theorem 2.10], who proves that a bipartite graph is vertex decomposable if and only if it is sequentially Cohen-Macaulay. We prove that a bipartite graph is vertex decomposable if and only if it is sequentially S_2 . Then we generalize some results of Van Tuyl and Villarreal [20] for sequentially S_r graphs. We extend a result of Francisco and Hà [7, Theorem 4.1] on graphs with whiskers. We also provide some graph theoretic criteria for S_2 property on graphs and for vertex decomposability on bipartite graphs. Woodroofe [21, Theorem 1] proved that a graph with no chordless cycles other than cycles of length 3 and 5 is sequentially Cohen-Macaulay. We provide some results which extend this statement for the S_2 property. In particular, they imply that any graph with no chordless even cycle is sequentially S_2 .

At the end of this section we pose two questions on sequentially S_r property of the join of two simplicial complexes and topological invariance of S_r , respectively.

Recall that to a simple graph G one associates a simplicial complex Δ_G on $V(G)$, the set of vertices of G whose faces correspond to the independent sets of vertices of G . A graph G is said to be S_r if Δ_G is a S_r simplicial complex. Likewise, G is Cohen-Macaulay, sequentially Cohen-Macaulay and shellable if Δ_G satisfies either of these properties, respectively. We adopt the definition of shellability in the nonpure sense of Björner-Wachs [1].

We also recall the definition of a vertex decomposable simplicial complex. A simplicial complex Δ is recursively defined to be vertex decomposable if it is either a simplex or else has some vertex v so that

- (1) both $\Delta \setminus \{v\}$ and $\text{lk}_\Delta(v)$ are vertex decomposable and
- (2) no face of $\text{lk}_\Delta(v)$ is a facet of $\Delta \setminus \{v\}$.

The notion of vertex decomposability was introduced in the pure case by Provan and Billera [13] and was extended to nonpure complexes by Björner and Wachs [1].

Sequentially Cohen-Macaulay cycles have been characterized by Francisco and Van Tuyl [8, Proposition 4.1]. They are just C_3 and C_5 . Woodroofe has given a more geometric proof for this result [21, Theorem 3.1]. In [9, Proposition 1.6] it is shown that the only S_2 cycles are C_3 , C_5 and C_7 . We now generalize this result and prove that the odd cycles are sequentially S_2 and that they are the only sequentially S_2 cycles.

Theorem 4.1. *The cycle C_n is sequentially S_2 if and only if n is odd.*

Proof. Let $n = 2k$. Then $\Delta = \Delta_{C_n}$ has only two facets of dimension $2k - 1$ ($= \dim \Delta$), namely, $\{1, 3, \dots, 2k - 1\}$ and $\{2, 4, \dots, 2k\}$. Thus $\Delta^{[k-1]}$ is the union of two disjoint $(k - 1)$ -simplices. In particular, it is disconnected, contradicting Terai's criterion for $\Delta^{[k-1]}$ to be S_2 .

Let n be an odd integer. The assertion is trivially valid for $n \leq 3$. Thus we may assume that $n \geq 5$. To show that C_n is sequentially S_2 , by Theorem 2.9, we need to show that $\Delta^{[i]}$ is connected for all $i \geq 1$ and $\text{lk}_\Delta(x)$ is sequentially S_2 for all $\{x\} \in \Delta$. Observe that $\text{lk}_\Delta(x)$ is the simplicial complex of an $(n - 3)$ -path, P_{n-3} , and so it is sequentially Cohen-Macaulay, e.g., by [8, Theorem 3.2].

To prove the connectedness of the pure i -skeleton of Δ we first claim that for each $E \in \Delta$ with $\dim E = 1$ there exists $F \in \Delta$ such that $\dim F = \dim \Delta$ and $E \subseteq F$. We know that

$$1 + \dim \Delta = \max\{\#F \mid F \text{ is an independent set of } C_n\} = (n - 1)/2.$$

We may assume that $E = \{1, m\}$, $m \geq 3$. If m is odd, we take F to be the set of odd integers less than $n - 1$. If m is even, then F may be taken to be the union of the set of odd integers less than $m - 1$ with the set of even integers e , $m \leq e \leq n - 1$. In both cases we have $\#F = (n - 1)/2$ and $E \subseteq F$. Therefore the claim holds.

Let $E \in \Delta$ with $\dim E = 1$. Then there exists $F \in \Delta$ such that $E \subseteq F$ and $\dim F = \dim \Delta$. Thus for each $i \geq 1$ there exists $H \in \Delta$ such that $\dim H = i$ and $E \subseteq H \subseteq F$. Therefore $E \subseteq H \in \Delta^{[i]}$. Thus we have shown that $E \in \Delta$ with $\dim E = 1$ if and only if $E \in \Delta^{[i]}$ for all $i \geq 1$. Therefore $(\Delta^{[i]})^{[1]} = \Delta^{[1]}$.

On the other hand $\Delta^{[i]}$ is connected if and only if $(\Delta^{[i]})^{[1]}$ is connected. Therefore it is enough to show that $\Delta^{[1]}$ is connected. Let x and y be two elements in V . If x is not adjacent to y , then $\{x, y\} \in \Delta$. If x is adjacent to y , then there exists $z \in V$ such that x is not adjacent to z and y is not adjacent to z . Therefore $\{x, z\}$ and $\{y, z\}$ belong to Δ , and hence $\Delta^{[1]}$ is connected. \square

The following result completes the characterization of cycles with respect to the property S_r . We will give two proofs using some extra data from the algebraic proof of [8, Proposition 4.1] and the geometric proof of [21, Theorem 3.1].

Proposition 4.2. *The cycle C_n is sequentially S_3 if and only if $n = 3, 5$, i.e., if and only if C_n is Cohen-Macaulay.*

Proof. The first proof: Let $\Delta = \Delta_{C_n}$. Let $n = 2t + 1 > 7$. By the proof of [8, Theorem 4.1], if J is the Alexander dual of I_Δ , then $J_{[t+1]}$ fails to have linear resolution in the first 3 steps. Hence C_n is not sequentially S_3 by Corollary 3.3.

For $n = 7$, again we follow the proof of [8, Theorem 4.1]. In this case, the ideal J is generated in degree 4. Thus the resolution of J is the same as the resolution of $J_{[4]}$. Moreover, the resolution of J is linear in the first 2 steps but not in step 3. Thus by Corollary 3.3, C_7 is not sequentially S_3 .

The second proof: Let $n = 2t + 1 \geq 7$. By [21, Lemma 3.2], $\Delta^{[t-1]}$ is homotopic to the circle S^1 . Thus $\tilde{H}_1(\Delta^{[t-1]}; k) = k$. By Hochster's formula on Betti numbers of simplicial complexes, we have $\text{depth}(k[\Delta^{[t-1]}]) \leq 2$. Thus $k[\Delta^{[t-1]}]$ does not satisfy S_3 . Hence C_n is not sequentially S_3 . \square

We now outline the analogue statements of [20, Section 3] to conclude the sequentially S_r counterparts of some of the results there.

Lemma 4.3 (see [20, Lemma 3.9]). *Let G be a bipartite graph with bipartition $\{x_1, \dots, x_m\}$ and $\{y_1, \dots, y_n\}$. If G is sequentially S_r , then there is a vertex $v \in V(G)$ with $\deg(v) = 1$.*

Proof. The only modification needed in the proof of [20, Lemma 3.9] is to justify that the kernel of the linear map f used in that proof is generated by linear syzygies. But under the hypothesis of the lemma, this is proved in Proposition 3.1. \square

For a graph G and a vertex $x \in V_G$, the set of neighbors of x will be denoted by $N_G(x)$. For $F \in \Delta_G$ we set

$$N_G(F) = \bigcup_{x \in F} N_G(x).$$

We also use the following notation:

$$\begin{aligned} N_G[x] &= \{x\} \cup N_G(x), \\ N_G[F] &= F \cup N_G(F). \end{aligned}$$

The following lemma gives a recursive procedure to check if a graph fails to be sequentially S_r . It is a sequentially S_r version of [20, Theorem 3.3].

Lemma 4.4. *Let G be graph and x a vertex of G . Let $G' = G \setminus N_G[x]$. If G is sequentially S_r , then G' is also sequentially S_r .*

Proof. The proof is identical with that of [20, Lemma 3.3]. We only need to use Theorem 2.6 and Lemma 2.2 instead of their sequentially Cohen-Macaulay and Cohen-Macaulay counterparts, respectively. \square

Van Tuyl [19, Theorem 2.10] has proved that a bipartite graph is vertex decomposable if and only if it is sequentially Cohen-Macaulay. We now generalize this result. Observe that our result also generalizes the authors' result, which states that a bipartite graph is Cohen-Macaulay if and only if it is S_2 [9, Theorem 1.3].

Theorem 4.5. *Let G be a bipartite graph. The following conditions are equivalent:*

- (i) G is vertex decomposable.
- (ii) G is shellable.
- (iii) G is sequentially Cohen-Macaulay.
- (iv) G is sequentially S_2 .

Proof. (i) \Rightarrow (ii): Follows from [2, Theorem 11.3].

(ii) \Rightarrow (iii): Follows from [16, Chap. III, §2].

(iii) \Rightarrow (iv): This is trivial.

(iii) \Rightarrow (i): Follows from [19, Theorem 2.10].

(iv) \Rightarrow (ii): The proof is by induction on the number of vertices of G . Now let G be a sequentially S_2 graph. By Lemma 4.3 there exists a degree one vertex $x_1 \in V(G)$. Assume that $N_G(x_1) = \{y_1\}$. Let $G_1 = G \setminus N_G[x_1]$ and $G_2 = G \setminus N_G[y_1]$. By Lemma 4.4 both of these graphs are sequentially S_2 ; hence by the induction hypothesis they are both shellable. Therefore, by [20, Theorem 2.9] G is shellable. \square

Lemma 4.4 could be extended further.

Corollary 4.6. *Let G be a graph which is sequentially S_r . Let F be an independent set in G . Then the graph $G' = G \setminus N_G[F]$ is sequentially S_r .*

Proof. This follows by repeated applications of Lemma 4.4. \square

The following generalizes a result of Francisco and Hà [7, Theorem 4.1], which is also proved by Van Tuyl and Villarreal by a different method (see [20, Corollary 3.5]).

Recall that for a subset $S = \{y_1, \dots, y_m\}$ of a graph G , the graph $G \cup W(S)$ is obtained from G by adding an edge (whisker) $\{x_i, y_i\}$ to G for all $i = 1, \dots, m$, where x_1, \dots, x_m are new vertices.

Corollary 4.7. *Let $S \subset V(G)$ and suppose that the graph $G \cup W(S)$ is sequentially S_r ; then $G \setminus S$ is sequentially S_r .*

Proof. This also follows by repeated applications of Lemma 4.4. \square

The following result gives some graph theoretic criteria which implies sequentially S_2 property.

Theorem 4.8. *Let $G = (V, E)$ be a graph. Suppose $H = G \setminus N_G[F]$ satisfies one of the conditions (i), (ii), and (iii) for any $F \in \Delta_G$ which is not a facet:*

- (i) H has no chordless even cycle.
- (ii) H has a simplicial vertex; i.e., for some $z \in V(H)$, $N_H[z]$ is a complete graph.
- (iii) For some $t \geq 2$, H has a chordless $(2t + 1)$ -cycle which has t independent vertices of degree 2 in H .

Then G is sequentially S_2 .

Proof. We prove the theorem by induction on n , the number of vertices of G . The assertion holds for $n \leq 3$. Now we assume that $n \geq 4$. Set $\Delta = \Delta_G$ and let $x \in V(G)$. Observe that $G' = G \setminus N_G[x]$ satisfies the statement of the theorem. Thus it is sequentially S_2 by the induction hypothesis. Hence by [20, Lemma 2.5], $\text{lk}_G(x) = \Delta_{G'}$ is sequentially S_2 . Thus by Theorem 2.9 it is enough to show that $\Delta^{[i]}$ is connected for $1 \leq i \leq \dim \Delta$. We show that for any $X, Y \in \Delta$ with $\dim X = \dim Y = i$, there is a chain $X = X_0, X_1, \dots, X_s = Y$ of i -faces of Δ such that $X_{j-1} \cap X_j \neq \emptyset$ for $j = 1, 2, \dots, s$. We may assume that $X \cap Y = \emptyset$. For simplicity we set $X = \{x_1, x_2, \dots, x_{i+1}\}$ and $Y = \{y_1, y_2, \dots, y_{i+1}\}$.

We assume that condition (i) is satisfied for G . Set $B = G_{X \cup Y}$, the restriction of G to $X \cup Y$. Then B is a bipartite graph on the partition $X \cup Y$. Since B is bipartite, B has no odd cycle. Since B has no (chordless) even cycle by condition (i), B is a forest. Then there exists a vertex with degree 0 or 1 in B . We may assume that x_1 is such a vertex and that x_1 is adjacent at most to y_1 . Set $X_1 = \{x_1, y_2, \dots, y_{i+1}\}$. Then X, X_1, Y is a desired chain.

We assume that condition (ii) is satisfied for G . Then using the hypotheses for $F = \emptyset$, there is a simplicial vertex z in G . Assume $z \notin X \cup Y$. Since z is simplicial, z is adjacent to at most one vertex in X . We may assume that z is not adjacent to x_2, \dots, x_{i+1} . Similarly, we may assume that z is not adjacent to y_2, \dots, y_{i+1} . Set $X_1 = \{z, x_2, \dots, x_{i+1}\}$ and $X_2 = \{z, y_2, \dots, y_{i+1}\}$. Then X, X_1, X_2, Y is a desired chain. Assume $z \in X \cup Y$. We may assume $z = y_1 \in Y$. Then X, X_1, Y is a desired chain.

Next we assume that condition (iii) is satisfied for G . Then for some $t \geq 2$ there exists a chordless $(2t + 1)$ -cycle C which has t vertices of degree 2 in G which are independent in G . Let $\{z_1, z_2, \dots, z_t\} \subset V(C)$ be an independent set of vertices of G such that $\deg_G z_j = 2$ for $j = 1, 2, \dots, t$.

Case I. $X \cup Y \subset V(C)$. As in the case that condition (i) is satisfied, $B = G_{X \cup Y}$ is a bipartite graph on the partition $X \cup Y$. Since C has no chord, B is a disjoint union of paths. Then we can construct a desired chain as in the above case.

Case II. $X \subset V(C)$, and $Y \cap (V(G) \setminus V(C)) \neq \emptyset$. We may assume that $y_1 \in V(G) \setminus V(C)$. Note that $i + 1 \leq t$. Set $Y_1 = \{y_1, z_2, \dots, z_{i+1}\}$ and $Z = \{z_1, z_2, \dots, z_{i+1}\}$. Then Y, Y_1, Z is a chain with $Y \cap Y_1 \neq \emptyset, Y_1 \cap Z \neq \emptyset$. Between Z and X we have a desired chain as in Case I. Hence we have a desired chain between X and Y .

Case III. $X \cap (V(G) \setminus V(C)) \neq \emptyset$ and $Y \subset V(C)$. As in Case II.

Case IV. $X \cap (V(G) \setminus V(C)) \neq \emptyset$ and $Y \cap (V(G) \setminus V(C)) \neq \emptyset$. As in Case II, we can construct desired chains between X and Z and between Y and Z . Thus we have a desired chain between X and Y via Z . □

By the following result one can detect whether a bipartite graph is vertex decomposable/shellable/sequentially Cohen-Macaulay/sequentially S_2 directly from the graph theoretic properties.

Corollary 4.9. *Let G be a bipartite graph. Then G is vertex decomposable if and only if for any $F \in \Delta_G$ (including \emptyset) which is not a facet, there is a vertex $v \in G \setminus N_G[F]$ such that $\deg_{G \setminus N_G[F]}(v) \leq 1$.*

Proof. Assume that G is vertex decomposable. Then G is sequentially Cohen-Macaulay by Theorem 4.5. For $F \in \Delta_G$ which is not a facet, set $H = G \setminus N_G[F]$. If H has no edge, every vertex $v \in V(H)$ satisfies $\deg_H(v) = 0$. Therefore, we may assume that the bipartite graph H has an edge. Applying [20, Theorem 3.3] repeatedly, we know H is sequentially Cohen-Macaulay. Hence by [20, Lemma 3.9] there is a vertex $v \in V(H)$ such that $\deg_H(v) = 1$.

Conversely, assume that for any $F \in \Delta_G$ which is not a facet, there is a vertex $v \in H := G \setminus N_G[F]$ such that $\deg_H(v) \leq 1$. Hence $N_H[v] = \{v\}$ or is an edge $\{v, w\}$; thus in either case it is a complete graph. This means v is a simplicial vertex in H . By Theorem 4.8 (ii), G is sequentially S_2 . Thus by Theorem 4.5, G is vertex decomposable. □

In [21, Theorem 1.1] it is shown that a graph G with no chordless cycles of length other than 3 or 5 is sequentially Cohen-Macaulay. In the following we extend this result on a larger class of graphs for the sequentially S_2 property.

Theorem 4.10. *Let G be a graph. Suppose that a vertex in each chordless even cycle in G has a whisker. Then G is sequentially S_2 .*

Proof. We prove the theorem by induction on n , the number of vertices of G . The assertion holds for $n \leq 3$. Now we assume $n \geq 4$. Set $\Delta = \Delta_G$ and let $x \in V$. Since $G \setminus N_G[x]$ satisfies the condition of the theorem, it is sequentially S_2 by the induction hypothesis. Hence by Theorem 2.9 it is enough to show that $\Delta^{[i]}$ is connected for $1 \leq i \leq \dim \Delta$. We show that for any $X, Y \in \Delta$ with $\dim X = \dim Y = i$, there is a chain $X = X_0, X_1, \dots, X_s = Y$ of i -faces of Δ such that $X_{j-1} \cap X_j \neq \emptyset$ for $j = 1, 2, \dots, s$. We may assume that $X \cap Y = \emptyset$. For simplicity we set $X = \{x_1, x_2, \dots, x_{i+1}\}$ and $Y = \{y_1, y_2, \dots, y_{i+1}\}$. Let x_1 have a whisker; that is, there exists $z \in V(G)$ such that $\deg z = 1$ and $\{x_1, z\} \in E$. Assume $z \notin Y$. Set $X_1 = \{z, x_2, \dots, x_{i+1}\}$ and $X_2 = \{z, y_2, \dots, y_{i+1}\}$. Then X, X_1, X_2, Y is a desired chain. Assume $z = y_1 \in Y$. Then X, X_1, Y is a desired chain.

Hence we may assume that no vertex in $X \cup Y$ has a whisker in G . Set $B = G_{X \cup Y}$, the restriction of G to $X \cup Y$. Then B is a bipartite graph on the partition $X \cup Y$. Since B is bipartite, B has no odd cycle. Since any vertices in $X \cup Y$ do not have a whisker in G , B has no (chordless) even cycle. Hence B is a forest. Then there exists a vertex with degree 0 or 1 in B . We may assume that x_1 is such a vertex and that x_1 is connected at most to y_1 . Set $X_1 = \{y_1, x_2, \dots, x_{i+1}\}$. Then X, X_1, Y is a desired chain. \square

Corollary 4.11. *If G is a graph with no chordless even cycle, then G is sequentially S_2 .*

Remark 4.12. Corollary 4.11 gives an alternative proof for the fact that any odd cycle is sequentially S_2 , the significant part of Theorem 4.1.

We end this section by proposing two questions.

Let Δ and Γ be two simplicial complexes over disjoint vertex sets. In [14] it is shown that $\Delta * \Gamma$ is sequentially Cohen-Macaulay if and only if Δ and Γ are both sequentially Cohen-Macaulay. By [18, Theorem 6] it follows that for $r \leq t$, if Δ is S_r but not S_{r+1} and Γ is S_t , then $\Delta * \Gamma$ is S_r but not S_{r+1} . One may study similar questions for sequentially S_2 complexes.

Question 4.13. Let Δ and Γ be two simplicial complexes. Is it true that $\Delta * \Gamma$ is sequentially S_2 if and only if Δ and Γ are both sequentially S_2 ?

In particular, it is tempting to show that the join of the simplicial complexes of two disjoint odd cycles is sequentially S_2 .

Munkres [12, Theorem 3.1] showed that Cohen-Macaulayness of a simplicial complex is a topological property. Stanley [16, Chap. III, Proposition 2.10] proved that sequentially Cohen-Macaulayness is also a topological property. Recently, Yanagawa [23, Theorem 4.5(d)] proved that Serre's condition S_r is a topological property as well. Therefore it is natural to pose the following question.

Question 4.14. Is sequentially S_r a topological property on simplicial complexes?

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