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ON MULTIPLE APPELL POLYNOMIALS

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Abstract. In this paper, we first define the multiple Appell polynomials and find several equivalent conditions for this class of polynomials. Then we give a characterization theorem that if multiple Appell polynomials are also multiple orthogonal, then they are the multiple Hermite polynomials.

1. Introduction

For a sequence $\{P_n(x)\}_{n=0}^{\infty}$ of Appell polynomials ([3]), which is a sequence of polynomials satisfying

$$P_n'(x) = nP_{n-1}(x), \quad n \geq 1,$$

tremendous properties are well known. Among them, the most important characterizations of Appell polynomials may be the following equivalent conditions ([12, 24]).

Theorem A. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials. Then the following are all equivalent:*

(a) *$\{P_n(x)\}_{n=0}^{\infty}$ is a sequence of Appell polynomials.*
(b) *$\{P_n(x)\}_{n=0}^{\infty}$ has a generating function of the form*

$$A(t)e^{xt} = \sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!},$$

where $A(t)$ is a formal power series independent of x with $A(0) \neq 0$.
(c) *There exists a sequence $\{a_n\}_{n=0}^{\infty}$ with $a_0 \neq 0$ such that*

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_{n-k}x^k.$$

(d) *There exists a sequence $\{a_n\}_{n=0}^{\infty}$ with $a_0 \neq 0$ such that*

$$P_n(x) = \left(\sum_{k=0}^{\infty} \frac{a_k}{k!}D^k\right) x^n,$$

where $D = \frac{d}{dx}$.

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(e) $\{P_n(x)\}_{n=0}^{\infty}$ *satisfies*

$$P_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} P_{n-k}(x) y^k, \quad n = 0, 1, 2, \cdots .$$

The typical examples of Appell polynomials besides the trivial example $\{x^n\}_{n=0}^{\infty}$ are the Bernoulli polynomials, the Euler polynomials, and the Hermite polynomials. In particular, a Hermite polynomial is a unique sequence of Appell polynomials that is also orthogonal with respect to a positive measure.

The literature on orthogonal polynomials contains many papers dealing with characterization theorems which specify a class of orthogonal polynomials satisfying a special property. Al-Salam gave a very nice survey on characterization theorems of orthogonal polynomials. See [1] and the references therein. In [17], which is also a nice reference, Ismail mentioned open characterization problems. One of the simplest of the characterization problems is the following ([1]).

Problem 1. Find all Appell polynomial sets which are also orthogonal.

As mentioned by Al-Salam, this problem was first solved by Angelesco and later by many authors. (See Carlitz [10], Hahn [16], Meixner [20], Shohat [23], Toscano [25], and Webster [27].)

Recently multiple orthogonal polynomials were extensively studied in a special function theory, in number theory, in approximation theory, and so on ([6, 7, 11, 13, 14, 19, 21, 22]). But there do not exist characterization theorems to specify a class of multiple orthogonal polynomials (in the general case), as far as the author knows. In the meantime, there exist characterization theorems for d-orthogonal polynomials which can be regarded as a special case of multiple orthogonal polynomials. See [4] or Chapter 23 written by Walter Van Assche in Ismail's book [17]. Douak obtained a characterization by introducing a more general Appell character, which is an example of Problem 1 for d-orthogonal polynomials ([15]). The characterizations of d-orthogonal polynomials were studied by many authors such as Douak, Maroni, Ben Cheikh, and others. We recommend references [8, 9, 28] and therein for a good survey of d-orthogonal polynomials.

In this paper, we first define the multiple Appell polynomials and find several equivalent conditions for this class of polynomials (Theorem 2.1). Then we solve the following characterization problem, which is a generalization of Problem 1, in the context of multiple orthogonal polynomials.

Problem 2. Find all multiple Appell polynomial sets which are also multiple orthogonal.

The author actually proves that if a set of Appell polynomials is also multiple orthogonal, then it is a set of multiple Hermite polynomials (Theorem 2.3). This result may be viewed as the only known characterization theorem for multiple orthogonal polynomials until now.

2. Main results

By a multiple polynomial system (multiple PS) we mean a set of polynomials $\{P_{n_1,n_2}(x)\}_{n_1,n_2=0}^{\infty}$ with $\deg(P_{n_1,n_2}) = n_1 + n_2$ for $n_1, n_2 \geq 0$.

A multiple PS is called a multiple orthogonal polynomial system (multiple OPS) if there exist positive weight functions w_1 and w_2 such that

$$\int_{-\infty}^{\infty} x^k P_{n_1,n_2}(x) w_i(x)dx = 0 \qquad \text{for} \quad k = 0, 1, 2, \cdots, n_i - 1 \quad (i = 1, 2).$$

In this case, (w_1, w_2) is called a pair of orthogonalizing weight functions for multiple OPS $\{P_{n_1,n_2}(x)\}_{n_1,n_2=0}^{\infty}$. In particular, if $w_i(x) = e^{\frac{\delta}{2}x^2+\alpha_i x}$, where $\delta < 0$ and $\alpha_1 \neq \alpha_2$, then the multiple OPS is called the multiple Hermite polynomials.

The origin of multiple OPS goes back to Angelesco's paper dealing with the simultaneous Padé approximation ([2]). The multiple Hermite polynomials actually appeared and were treated as an example of multiple OPS in the papers [5, 26], which generalize the properties of classical orthogonal polynomials to a multiple OPS for the case of classical weights.

In [18], the author obtained generating functions for classical multiple OPS's including multiple Hermite polynomials. In particular, the author showed that for the multiple Hermite polynomial $\{H_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x)\}_{n_1,n_2=0}^{\infty}$ $(\alpha_1 \neq \alpha_2)$,

$$e^{\frac{\delta}{2}(t_1+t_2)^2+\delta(t_1+t_2)x+\alpha_1 t_1+\alpha_2 t_2} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} H_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x)\frac{t_1^{n_1}\, t_2^{n_2}}{n_1!\, n_2!} \qquad (\delta < 0),$$

from which many properties of multiple Hermite polynomials are obtained. Note that if we take $t_2 = 0$, it is the generating function for the classical Hermite polynomials.

From the generating function of multiple Hermite polynomials, we can deduce how to define multiple Appell polynomials. A multiple PS $\{P_{n_1,n_2}(x)\}_{n_1,n_2=0}^{\infty}$ is called multiple Appell if there exists a generating function of the form

$$A(t_1, t_2)e^{x(t_1+t_2)} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} P_{n_1,n_2}(x)\frac{t_1^{n_1}\, t_2^{n_2}}{n_1!\, n_2!}, \tag{2.1}$$

where A has a series expansion

$$A(t_1, t_2) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} a_{n_1,n_2}\frac{t_1^{n_1}\, t_2^{n_2}}{n_1!\, n_2!} \tag{2.2}$$

with $A(0, 0) = a_{0,0} \neq 0$.

Then we can easily generalize Theorem A to the following.

Theorem 2.1. *Let $\{P_{n_1,n_2}(x)\}_{n_1,n_2=0}^{\infty}$ be a multiple PS. Then the following are all equivalent:*

(a) $\{P_{n_1,n_2}(x)\}_{n_1,n_2=0}^{\infty}$ *is a set of multiple Appell polynomials.*
(b) *There exists a sequence $\{a_{n_1,n_2}\}_{n_1,n_2=0}^{\infty}$ with $a_{0,0} \neq 0$ such that*

$$P_{n_1,n_2}(x) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} \binom{n_1}{k_1}\binom{n_2}{k_2} a_{n_1-k_1,n_2-k_2} x^{k_1+k_2}.$$

(c) *For every $n_1 + n_2 \geq 1$, we have*

$$P'_{n_1,n_2}(x) = n_1 P_{n_1-1,n_2}(x) + n_2 P_{n_1,n_2-1}(x). \tag{2.3}$$

(d) *There exists a sequence* $\{a_{n_1,n_2}\}_{n_1,n_2=0}^{\infty}$ *with* $a_{0,0} \neq 0$ *such that*

(2.4)
$$P_{n_1,n_2}(x) = \left\{ \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \binom{n_1}{k_1} \binom{n_2}{k_2} \frac{(n_1+n_2-k_1-k_2)!}{(n_1+n_2)!} a_{k_1,k_2} D^{k_1+k_2} \right\} x^{n_1+n_2},$$

where $D = \frac{d}{dx}$.

(e) $P_{n_1,n_2}(x)$ *satisfies that for* $n_1, n_2 = 0, 1, 2, \cdots$,

$$P_{n_1,n_2}(x+y) = \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \binom{n_1}{k_1} \binom{n_2}{k_2} P_{n_1-k_1,n_2-k_2}(x) y^{k_1+k_2}. \tag{2.5}$$

Proof. (a)⇒(b). Since $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n b_k = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{n-k} b_k$, we have from equations (2.1) and (2.2),

(2.6)
$$\begin{aligned} A(t_1,t_2) e^{x(t_1+t_2)} &= \left(\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} a_{n_1,n_2} \frac{t_1^{n_1}}{n_1!} \frac{t_2^{n_2}}{n_2!} \right) \left(\sum_{k=0}^{\infty} \frac{x^k}{k!} (t_1+t_2)^k \right) \\ &= \sum_{n_1=0}^{\infty} \sum_{k_1=0}^{\infty} \left(\sum_{n_2=0}^{\infty} \sum_{k_2=0}^{\infty} a_{n_1,n_2} \frac{x^{k_2}}{k_2!} \frac{t_2^{k_2+n_2}}{n_2!} \right) \frac{x^{k_1}}{k_1!} \frac{t_1^{k_1+n_1}}{n_1!} \\ &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left\{ \sum_{k_2=0}^{n_2} \sum_{k_1=0}^{n_1} \binom{n_1}{k_1} \binom{n_2}{k_2} a_{n_1-k_1,n_2-k_2} x^{k_1+k_2} \right\} \frac{t_1^{n_1}}{n_1!} \frac{t_2^{n_2}}{n_2!}. \end{aligned}$$

Hence, we obtain (c) by comparing the coefficients of (2.1) and (2.6).

(b)⇒(c). From the definition of $P_{n_1,n_2}(x)$ and the identity

$$(k_1+k_2) \binom{n_1}{k_1} \binom{n_2}{k_2} = n_1 \binom{n_1-1}{k_1-1} \binom{n_2}{k_2} + n_2 \binom{n_2-1}{k_2-1} \binom{n_1}{k_1}, \quad k_1, k_2 \geq 1,$$

we have

$$\begin{aligned} P'_{n_1,n_2}(x) &= \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} (k_1+k_2) \binom{n_1}{k_1} \binom{n_2}{k_2} a_{n_1-k_1,n_2-k_2} x^{k_1+k_2-1} \\ &= \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \left\{ n_1 \binom{n_1-1}{k_1-1} \binom{n_2}{k_2} + n_2 \binom{n_2-1}{k_2-1} \binom{n_1}{k_1} \right\} a_{n_1-k_1,n_2-k_2} x^{k_1+k_2-1} \\ &= n_1 \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2} \binom{n_1-1}{k_1} \binom{n_2}{k_2} a_{(n_1-1)-k_1,n_2-k_2} x^{k_1+k_2} \\ &\quad + n_2 \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2-1} \binom{n_2-1}{k_2} \binom{n_1}{k_1} a_{n_1-k_1,(n_2-1)-k_2} x^{k_1+k_2} \\ &= n_1 P_{n_1-1,n_2}(x) + n_2 P_{n_1,n_2-1}(x), \end{aligned}$$

where $\binom{n}{a} = 0$ if $a < 0$.

(c)⇒(a). Assume that $P'_{n_1,n_2}(x) = n_1 P_{n_1-1,n_2}(x) + n_2 P_{n_1,n_2-1}(x)$ for $n_1+n_2 \geq 1$ and

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} P_{n_1,n_2}(x) \frac{t_1^{n_1}}{n_1!} \frac{t_2^{n_2}}{n_2!} = A(x; t_1, t_2) e^{x(t_1+t_2)}.$$

Taking a differentiation on both sides in the variable x, we have

$$\begin{aligned}\left(\frac{\partial A}{\partial x}+(t_1+t_2)A(x;t_1,t_2)\right)e^{x(t_1+t_2)} &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}P'_{n_1,n_2}(x)\frac{t_1^{n_1}\,t_2^{n_2}}{n_1!\,n_2!}\\ &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(n_1P_{n_1-1,n_2}(x)+n_2P_{n_1,n_2-1}(x)\right)\frac{t_1^{n_1}\,t_2^{n_2}}{n_1!\,n_2!}\\ &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(t_1P_{n_1,n_2}(x)+t_2P_{n_1,n_2}(x)\right)\frac{t_1^{n_1}\,t_2^{n_2}}{n_1!\,n_2!}\\ &= (t_1+t_2)A(x;t_1,t_2)e^{x(t_1+t_2)},\end{aligned}$$

where $P_{n_1,n_2}(x)=0$ if $n_1<0$ or $n_2<0$. Hence, $\frac{\partial A}{\partial x}=0$, which implies that A is independent of x so that $A(x;t_1,t_2)=A(t_1,t_2)$.

(b)⇔(d). This immediately follows from

$$\begin{aligned}P_{n_1,n_2}(x) &= \left\{\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{n_1}{k_1}\binom{n_2}{k_2}\frac{(n_1+n_2-k_1-k_2)!}{(n_1+n_2)!}a_{k_1,k_2}D^{k_1+k_2}\right\}x^{n_1+n_2}\\ &= \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{n_1}{k_1}\binom{n_2}{k_2}a_{n_1-k_1,n_2-k_2}x^{k_1+k_2}.\end{aligned}$$

(a)⇔(e). Using the generating function, we have

(2.7)

$$\begin{aligned}A(t_1,t_2)e^{(x+y)(t_1+t_2)} &= \left(\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}P_{n_1,n_2}(x)\frac{t_1^{n_1}\,t_2^{n_2}}{n_1!\,n_2!}\right)\left(\sum_{k_1=0}^{\infty}\sum_{j=0}^{k_1}\binom{k_1}{j}\frac{y^{k_1}}{k_1!}t_1^{j}t_2^{k_1-j}\right)\\ &= \left(\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}P_{n_1,n_2}(x)\frac{t_1^{n_1}\,t_2^{n_2}}{n_1!\,n_2!}\right)\left(\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}y^{k_1+k_2}\frac{t_1^{k_1}\,t_2^{k_2}}{k_1!\,k_2!}\right)\\ &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}P_{n_1,n_2}(x)y^{k_1+k_2}\frac{t_1^{n_1+k_1}\,t_2^{n_2+k_2}}{k_1!\,k_2!\,n_1!\,n_2!}\\ &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left\{\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{n_1}{k_1}\binom{n_2}{k_2}P_{n_1-k_1,n_2-k_2}(x)y^{k_1+k_2}\right\}\frac{t_1^{n_1}\,t_2^{n_2}}{n_1!\,n_2!}.\end{aligned}$$

By definition of $P_{n_1,n_2}(x+y)$ with a generating function, the equation (2.5) is an immediate consequence of (2.7). □

By the definition of multiple Appell polynomials, the multiple Hermite polynomials are a subset of the multiple Appell polynomials. Hence, by Theorem 2.1, the multiple Hermite polynomials $\{H_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x)\}_{n_1,n_2=0}^{\infty}$ satisfy the equation (2.3) which was already proved in [18].

Remark 2.2. If $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ are Appell polynomials with generating functions

$$A(t)e^{xt}=\sum_{n=0}^{\infty}P_n(x)\frac{t^n}{n!}\quad\text{and}\quad B(t)e^{xt}=\sum_{n=0}^{\infty}Q_n(x)\frac{t^n}{n!},$$

then we can make a new set of multiple Appell polynomials in two ways. Let $R_{n_1,n_2}(x) = P_{n_1+n_2}(x)$ and $S_{n_1,n_2}(x) = P_{n_1}(x)Q_{n_2}(x)$. By the Leibniz rule we can easily prove that $R_{n_1,n_2}(x)$ and $S_{n_1,n_2}(x)$ satisfy

$$R'_{n_1,n_2}(x) = P'_{n_1+n_2}(x) = (n_1+n_2)P_{n_1+n_2-1}(x) = n_1R_{n_1-1,n_2} + n_2R_{n_1,n_2-1}$$

and

$$S'_{n_1,n_2}(x) = (P_{n_1}(x)Q_{n_2}(x))' = n_1S_{n_1-1,n_2}(x) + n_2S_{n_1,n_2-1}(x),$$

which are the equation (2.3) in Theorem 2.1. Hence, $\{R_{n_1,n_2}(x)\}_{n_1,n_2=0}^{\infty}$ and $\{S_{n_1,n_2}(x)\}_{n_1,n_2=0}^{\infty}$ are sets of multiple Appell polynomials. In this case, the generating functions of $\{R_{n_1,n_2}(x)\}_{n_1,n_2=0}^{\infty}$ and $\{S_{n_1,n_2}(x)\}_{n_1,n_2=0}^{\infty}$ are $A(t_1+t_2)e^{x(t_1+t_2)}$ and $A(t_1)B(t_2)e^{x(t_1+t_2)}$, respectively. More precisely, by a direct calculation and by the definition of $\{R_{n_1,n_2}(x)\}_{n_1,n_2=0}^{\infty}$, we have

$$\begin{aligned}
A(t_1+t_2)e^{x(t_1+t_2)} &= \sum_{n=0}^{\infty} P_n(x)\frac{(t_1+t_2)^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}P_n(x)\frac{t_1^k\,t_2^{n-k}}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n+k}{k}P_{n+k}(x)\frac{t_1^k\,t_2^n}{(n+k)!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n+k}{k}R_{n,k}(x)\frac{t_1^k\,t_2^n}{(n+k)!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}R_{n,k}(x)\frac{t_1^k\,t_2^n}{k!n!}.
\end{aligned}$$

Similarly, we can prove the case of $\{S_{n_1,n_2}(x)\}_{n_1,n_2=0}^{\infty}$. Hence, any product of Appell polynomials generates a new multiple Appell polynomial.

However, the multiple Hermite polynomials are not such a trivial type of multiple Appell polynomials.

We proved by Theorem 2.1 that the multiple Hermite polynomials form a sequence of multiple Appell polynomials as well as a multiple OPS. Here, we prove the converse.

Theorem 2.3. *If $\{P_{n_1,n_2}(x)\}_{n_1,n_2=0}^{\infty}$ is a multiple Appell PS and also a multiple OPS, then $\{P_{n_1,n_2}(x)\}_{n_1,n_2=0}^{\infty}$ are multiple Hermite polynomials.*

Proof. Since $\{P_{n_1,n_2}(x)\}_{n_1,n_2=0}^{\infty}$ is multiple Appell, we have by Theorem 2.1 that

$$A(t_1,t_2)e^{x(t_1+t_2)} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}P_{n_1,n_2}(x)\frac{t_1^{n_1}\,t_2^{n_2}}{n_1!\,n_2!}.$$

Multiplying (2.1) by $x^k w_i(x)$ on both sides, where (w_1,w_2) is a pair of orthogonalizing weight funtions for $\{P_{n_1,n_2}(x)\}_{n_1,n_2=0}^{\infty}$, and then applying an integration, we obtain

(2.8)
$$A(t_1,t_2)\int_{-\infty}^{\infty}x^k e^{x(t_1+t_2)}w_i(x)dx = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\int_{-\infty}^{\infty}x^kP_{n_1,n_2}(x)w_i(x)dx\frac{t_1^{n_1}\,t_2^{n_2}}{n_1!\,n_2!}.$$

By the orthogonality of multiple OPS, the right-hand side of (2.8) becomes a polynomial of degree k in the variable t_i. In particular, if we take $k = 0, 1$, we get

$$A(t_1,t_2)\int_{-\infty}^{\infty} x^k e^{x(t_1+t_2)} w_i(x)dx = \begin{cases} \gamma(t_2) & k=0, i=1, \\ \alpha(t_2)t_1+\beta(t_2) & k=1, i=1, \\ \tilde{\gamma}(t_1) & k=0, i=2, \\ \tilde{\alpha}(t_1)t_2+\tilde{\beta}(t_1) & k=1, i=2. \end{cases}$$

Let

$$f_i(t_1,t_2) = \int_{-\infty}^{\infty} e^{x(t_1+t_2)} w_i(x)dx \qquad (i=1,2).$$

Then $f_i(t_1,t_2) = f_i(t_2,t_1)$ and f_i is a function of t_1+t_2. By a series expansion

$$e^{x(t_1+t_2)} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} x^{n_1+n_2}\frac{t_1^{n_1}\,t_2^{n_2}}{n_1!\,n_2!},$$

we have

$$f_i(t_1,t_2) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \mu_{n_1+n_2}^{(i)}\frac{t_1^{n_1}\,t_2^{n_2}}{n_1!\,n_2!},$$

where the $\mu_n^{(i)}$'s are the moments of the weight function w_i. Note that

$$\begin{aligned}\int_{-\infty}^{\infty} x^k e^{x(t_1+t_2)} w_i(x)dx &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \mu_{n_1+n_2+k}^{(i)}\frac{t_1^{n_1}\,t_2^{n_2}}{n_1!\,n_2!} \\ &= \sum_{n_1=k}^{\infty}\sum_{n_2=0}^{\infty} \frac{n_1!}{(n_1-k)!}\mu_{n_1+n_2}^{(i)}\frac{t_1^{n_1-k}\,t_2^{n_2}}{n_1!\,n_2!} \\ &= \frac{\partial^k f_i}{\partial t_1^k}(t_1,t_2)\end{aligned}$$

and similarly

$$\frac{\partial^k f_i}{\partial t_1^k}(t_1,t_2) = \int_{-\infty}^{\infty} x^k e^{x(t_1+t_2)} w_i(x)dx = \frac{\partial^k f_i}{\partial t_2^k}(t_1,t_2).$$

Thus, we obtain

$$\begin{aligned} A(t_1,t_2)f_1(t_1,t_2) &= \gamma(t_2), \\ A(t_1,t_2)\frac{\partial f_1}{\partial t_1}(t_1,t_2) &= \alpha(t_2)t_1+\beta(t_2), \\ A(t_1,t_2)f_2(t_1,t_2) &= \tilde{\gamma}(t_1), \\ A(t_1,t_2)\frac{\partial f_2}{\partial t_2}(t_1,t_2) &= \tilde{\alpha}(t_1)t_2+\tilde{\beta}(t_1), \end{aligned} \tag{2.9}$$

and so $\gamma(t_2)\frac{\partial f_1}{\partial t_1}(t_1,t_2) = (\alpha(t_2)t_1+\beta(t_2))f_1(t_1,t_2)$. Hence,

$$f_1(t_1,t_2) = e^{\frac{1}{2}a(t_2)t_1^2+b(t_2)t_1+c(t_2)},$$

where a, b, c are functions determined by α, β, γ. Similarly,

$$f_2(t_1,t_2) = e^{\frac{1}{2}\tilde{a}(t_1)t_2^2+\tilde{b}(t_1)t_2+\tilde{c}(t_1)}, \tag{2.10}$$

where $\tilde{a}, \tilde{b}, \tilde{c}$ are functions determined by $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$. Since $f_1(t_1,t_2) = f_1(t_2,t_1)$, we have

$$\frac{1}{2}a(t_2)t_1^2 + b(t_2)t_1 + c(t_2) = \frac{1}{2}a(t_1)t_2^2 + b(t_1)t_2 + c(t_1), \tag{2.11}$$

from which we obtain that $a(t)$ and $b(t)$ are quadratic polynomials of t, and $c(t) = \frac{1}{2}a(0)t^2 + b(0)t + c(0)$. Let us write

$$a(t) = \frac{1}{2}a''(0)t^2 + a'(0)t + a(0),$$
$$b(t) = \frac{1}{2}b''(0)t^2 + b'(0)t + b(0).$$

By comparing the coefficients of (2.11), we have $a'(0) = b''(0)$ so that

$$f_1(t_1,t_2) = e^{\frac{1}{4}a''(0)t_1^2t_2^2 + \frac{1}{2}a'(0)t_1t_2(t_1+t_2) + \frac{1}{2}a(0)(t_1^2+t_2^2) + b'(0)t_1t_2 + b(0)(t_1+t_2) + c(0)}.$$

Since f_1 is a function of t_1+t_2, we have $a''(0) = a'(0) = 0$ and $b'(0) = a(0)$. Hence,

$$f_1(t_1,t_2) = e^{\frac{1}{2}a(0)(t_1+t_2)^2 + b(0)(t_1+t_2) + c(0)}.$$

By the same process with $f_2(t_1,t_2) = f_2(t_2,t_1)$ and (2.10), we have

$$f_2(t_1,t_2) = e^{\frac{1}{2}\tilde{a}(0)(t_1+t_2)^2 + \tilde{b}(0)(t_1+t_2) + \tilde{c}(0)}.$$

Hence, we have by (2.9),

$$\begin{aligned} A(t_1,t_2) &= \gamma(t_2)e^{-\frac{1}{2}a(0)(t_1+t_2)^2 - b(0)(t_1+t_2) - c(0)} \\ &= \tilde{\gamma}(t_1)e^{-\frac{1}{2}\tilde{a}(0)(t_1+t_2)^2 - \tilde{b}(0)(t_1+t_2) - \tilde{c}(0)}. \end{aligned}$$

By comparing the coefficient of t_1t_2, we have $a(0) = \tilde{a}(0)$. Letting $t_1 = 0$, and $t_2 = 0$, respectively, and then comparing the coefficients we can easily see that

$$\gamma(t_2) = \gamma(0)e^{(b(0)-\tilde{b}(0))t_2},$$
$$\tilde{\gamma}(t_1) = \tilde{\gamma}(0)e^{(\tilde{b}(0)-b(0))t_1}$$

so that

$$A(t_1,t_2) = \gamma(0)e^{-\frac{1}{2}a(0)(t_1+t_2)^2 - b(0)t_1 - \tilde{b}(0)t_2 - c(0)}.$$

Hence, we can write $A(t_1,t_2)$ by

$$A(t_1,t_2) = e^{\frac{\delta}{2}(t_1+t_2)^2 + \alpha_1t_1 + \alpha_2t_2},$$

where $\delta = -a(0)$, $\alpha_1 = -b(0)$, $\alpha_2 = -\tilde{b}(0)$ are constants. By comparing generating functions of $\{P_{n_1,n_2}(x)\}_{n_1,n_2=0}^{\infty}$ and $\{H_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x)\}_{n_1,n_2=0}^{\infty}$, we conclude that $P_{n_1,n_2}(x) = H_{n_1,n_2}(\frac{x}{\delta})$. □

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