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ON VARIETIES OF ALMOST MINIMAL DEGREE II: A RANK-DEPTH FORMULA

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Abstract. Let $X \subset \mathbb{P}^r_K$ denote a variety of almost minimal degree other than a normal del Pezzo variety. Then X is the projection of a rational normal scroll $\tilde{X} \subset \mathbb{P}^{r+1}_K$ from a point $p \in \mathbb{P}^{r+1}_K \setminus \tilde{X}$. We show that the arithmetic depth of X can be expressed in terms of the rank of the matrix $M'(p)$, where M' is the matrix of linear forms whose 3×3 minors define the secant variety of $\tilde{X}$.

1. Introduction

Let $X \subset \mathbb{P}^r_K$ denote an irreducible and reduced projective variety over an algebraically closed field K. In the following we always assume that X is nondegenerate, i.e. that X is not contained in any hyperplane. Then it is well known (see for instance [H]) that there is the inequality $\deg X \geq \operatorname{codim} X + 1$ between the degree and the codimension of X. Varieties satisfying the equality are called varieties of *minimal degree*. See [H] for a classification of these varieties.

Varieties of almost minimal degree are those for which $\deg X = \operatorname{codim} X + 2$. The results of [F1], [F2] and [BS] imply that a variety $X \subset \mathbb{P}^r$ of almost minimal degree is either a normal del Pezzo variety or a linear projection of a variety of minimal degree $\tilde{X} \subset \mathbb{P}^{r+1}$ from a point $p \in \mathbb{P}^{r+1}_K \setminus \tilde{X}$. In the latter case, $X \subset \mathbb{P}^r_K$ is either smooth and not linearly normal or else nonnormal, depending on the location of p with respect to $\tilde{X}$. The arithmetic depth of X is defined as the depth of the coordinate ring of A_X and is denoted by $\operatorname{depth} X$. It is an important homological invariant. In the case of a smooth rational normal scroll $\tilde{X}$ we have

$$\operatorname{depth} X = \dim \Sigma_p(\tilde{X}) + 2 \leq 4,$$

where $\Sigma_p(\tilde{X})$ denotes the secant locus of $\tilde{X}$ with respect to p; see [BS, Theorem 7.5].

The secant variety $\operatorname{Sec} \tilde{X}$ of a smooth rational normal scroll $\tilde{X} \subset \mathbb{P}^{r+1}_K$ is described (see [C]) as the variety $V_3(M')$ defined by the ideal generated by the 3×3 minors of a certain $3 \times s$ matrix M' associated to the matrix defining the scroll $\tilde{X} \subset \mathbb{P}^{r+1}_K$. Let $M'(p)$ denote the matrix M' with the entries given by $p \in \mathbb{P}^{r+1}_K$. Although p is defined up to a scalar the rank of $M'(p)$ is well defined.

The particular case that $X \subset \mathbb{P}^r_K$ and $\tilde{X} \subset \mathbb{P}^{r+1}_K$ are isomorphic (by means of our projection) holds if and only if $\operatorname{depth} X = 1$, i.e. if and only if $p \notin \operatorname{Sec} \tilde{X}$. In

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terms of the matrix M' this means that $\operatorname{rank} M'(p) = 3$. The main result of the present paper is an extension of this phenomenon to the general situation.

Theorem 1.1. *With the previous notation let $X \subset \mathbb{P}^r_K$ denote a variety of almost minimal degree obtained as a linear projection of a rational normal scroll $\tilde{X} \subset \mathbb{P}^{r+1}$ from a point $p \in \mathbb{P}^{r+1}_K \setminus \tilde{X}$. Then* $\operatorname{depth} X = 4 - \operatorname{rank} M'(p)$.

The advantage of this theorem is an intrinsic description of the arithmetic depth without knowing the secant locus of $\tilde{X}$ with respect to p. Our main result is proved in Section 3 of the present paper. It is a consequence of the first two authors' work on the secant stratification of $\tilde{X}$; see [BP]. Theorem 1.1 also admits a straightforward generalization to scrolls which are not necessarily smooth; see Corollary 3.6.

In the following we shall give an illustration of Theorem 1.1.

Example 1.2. Let $\tilde{X} \subset \mathbb{P}^8_K$ be the rational normal scroll defined by the vanishing of the 2×2 minors of the matrix

$$M = \left(\begin{array}{c|c|cccc} x_0 & x_2 & x_4 & x_5 & x_6 & x_7 \\ x_1 & x_3 & x_5 & x_6 & x_7 & x_8 \end{array} \right).$$

We consider the following four points $p_i \in \mathbb{P}^8_K \setminus \tilde{X}, i = 1, \ldots, 4:$

$$p_1 = (0:0:0:0:0:0:1:0:0),\ p_2 = (0:0:0:0:0:1:0:0:0),$$
$$p_3 = (0:0:0:1:1:0:0:0:0),\ p_4 = (0:1:1:0:0:0:0:0:0).$$

Let $X_{p_i} \subset \mathbb{P}^7_K$ denote the image of $\tilde{X} \subset P^8_K$ under the linear projection map $\pi_{p_i} : \mathbb{P}^8_K \setminus \{p_i\} \to \mathbb{P}^7_K$.

By [C] the secant variety $\operatorname{Sec} \tilde{X} \subset \mathbb{P}^8_K$ is given by the vanishing of the determinant of the matrix

$$M' = \begin{pmatrix} x_4 & x_5 & x_6 \\ x_5 & x_6 & x_7 \\ x_6 & x_7 & x_8 \end{pmatrix}.$$

By the definition of $M'(p)$ it is easily seen that $\operatorname{rank} M'(p_i) = 4 - i$ for $i = 1, 2, 3, 4$. Therefore, $\operatorname{depth} X_{p_i} = i, i = 1, 2, 3, 4$, by Theorem 1.1. So all the possible values for the arithmetic depth of the projection occur.

2. Preliminaries

We first fix some notation which we shall keep for the whole paper.

Notation and Remark 2.1. (A) Let $r \geq 2$ be an integer and let K be an algebraically closed field. Let

$$\tilde{X} = S(\underbrace{1, \ldots, 1}_{k}, \underbrace{2, \ldots, 2}_{l}, a_1, \ldots, a_{n-k-l}) = S(\underline{1}, \underline{2}, \underline{a}) \subset \mathbb{P}^{r+1}_K$$

be the smooth rational normal scroll of type $(1, \ldots, 1, 2, \ldots, 2, a_1, \ldots, a_{n-k-l})$ with $3 \leq a_1 \leq \ldots \leq a_{n-k-l}$. So, we have

$$\dim(\tilde{X}) = n,\ \deg(\tilde{X}) = k + 2l + \sum_{j=1}^{n-k-l} a_j = r + 2 - n.$$

(B) We consider the polynomial ring

$$K[\underline{x}, \underline{y}, \underline{z}] = K[\{x_{h,u}\}_{h=1,\ldots k, u=0,1}, \{y_{i,s}\}_{i=1,\ldots,l, s=0,1,2}, \{z_{j,t}\}_{j=1,\ldots,n-k-l, t=0,\ldots,a_j}]$$

and its subrings $K[\underline{x}], K[\underline{y}], K[\underline{z}]$ and $K[\underline{y}, \underline{z}] \subseteq K[\underline{x}, \underline{y}, \underline{z}]$. We write

$$\mathbb{P}_K^{r+1} = \operatorname{Proj}(K[\underline{x}, \underline{y}, \underline{z}])$$

and consider the four spaces

$$\mathbb{P}_K^{2k-1} = \operatorname{Proj}(K[\underline{x}]), \quad \mathbb{P}_K^{3l-1} = \operatorname{Proj}(K[\underline{y}]),$$
$$\mathbb{P}_K^{r+1-2k-3l} = \operatorname{Proj}(K[\underline{z}]), \quad \mathbb{P}_K^{r+1-2k} = \operatorname{Proj}(K[\underline{y}, \underline{z}])$$

canonically as subspaces of $\mathbb{P}_K^{r+1}$.

Using this notation we now may define the following subscrolls of $\tilde{X}$:

$$\begin{aligned} S(\underline{1}) =& \tilde{X} \cap \operatorname{Proj}(K[\underline{x}]) \subseteq \langle S(\underline{1}) \rangle = \operatorname{Proj}(K[\underline{x}]); \\ S(\underline{2}) =& \tilde{X} \cap \operatorname{Proj}(K[\underline{y}]) \subseteq \langle S(\underline{2}) \rangle = \operatorname{Proj}(K[\underline{y}]); \\ S(\underline{a}) =& \tilde{X} \cap \operatorname{Proj}(K[\underline{z}]) \subseteq \langle S(\underline{a}) \rangle = \operatorname{Proj}(K[\underline{z}]); \\ S(\underline{2}, \underline{a}) =& \tilde{X} \cap \operatorname{Proj}(K[\underline{y}, \underline{z}]) \subseteq \langle S(\underline{2}, \underline{a}) \rangle = \operatorname{Proj}(K[\underline{y}, \underline{z}]). \end{aligned}$$

(C) Let $\varphi : \tilde{X} \to \mathbb{P}_K^1$ denote the canonical projection map which turns $\tilde{X}$ in a variety ruled by the linear subspaces $\mathbb{P}_K^{n-1} = \mathbb{L}(x) =: \varphi^{-1}(x)$ with $x \in \mathbb{P}^1$.

Let $x = (\lambda : \mu)$. Then $\mathbb{L}(x)$ consists precisely of the points given by

$$(\ldots : u_a\mu : u_a\lambda : \ldots : v_b\mu^2 : v_b\mu\lambda : v_b\lambda^2 : \ldots$$
$$\ldots : w_c\mu^{a_c} : \ldots : w_c\mu^{a_c-1}\lambda : \ldots : w_c\mu\lambda^{a_c-1} : w_c\lambda^{a_c} : \ldots)$$

with $(u_1, \ldots, u_k, v_1, \ldots, v_l, w_1, \ldots, w_{n-k-l}) \in K^n \setminus \{\underline{0}\}$ and integers satisfying $1 \leq a \leq k, 1 \leq b \leq l, 1 \leq c \leq n-k-l$.

Notation and Remark 2.2. (A) Let N be an $(m \times n)$-matrix whose entries are linear forms in the polynomial ring $K[\underline{t}] = K[t_0, \ldots, t_s]$ and let e be a nonnegative integer. We then write $I_e(N)$ for the ideal generated by all $(e \times e)$-minors of N and $V_e(N) := V(I_e(N)) \subseteq \mathbb{P}_K^s$ for the projective variety defined by these minors.

(B) Consider the $(2 \times (k + l + \sum_{j=1}^{n-k-l} a_j))$-matrix

$$\left(\begin{array}{c|c|c|cc|c|ccc|c} \cdots & x_{a,0} & \cdots & y_{b,0} & y_{b,1} & \cdots & z_{c,0} & \cdots & z_{c,a_c-1} & \cdots \\ \cdots & x_{a,1} & \cdots & y_{b,1} & y_{b,2} & \cdots & z_{c,1} & \cdots & z_{c,a_c} & \cdots \end{array} \right)$$

with entries in $K[\underline{x}, \underline{y}, \underline{z}]$, so that $I_2(M) \subseteq K[\underline{x}, \underline{y}, \underline{z}]$ is the homogeneous vanishing ideal of $\tilde{X}$ and hence

$$\tilde{X} = V_2(M).$$

Notation and Reminder 2.3. (A) Keep the previous notation and let $p \in \mathbb{P}_K^{r+1}$. We consider a linear projection $\mathbb{P}^{r+1} \setminus \{p\} \to \mathbb{P}_K^r$, denote the image of $\tilde{X}$ under this projection by X_p and consider the induced finite morphism $\pi_p : \tilde{X} \to X_p$ which is birational. Moreover X_p is of almost minimal degree, that is, $\deg X_p = \operatorname{codim} X_p + 2$.

(B) We introduce the *secant cone* and the *secant locus*

$$\operatorname{Sec}_p(\tilde{X}) = \bigcup_{x \in \tilde{X}, \text{closed}, \#(\langle p,x \rangle \cap \tilde{X}) > 1} \langle p, x \rangle \text{ and } \Sigma_p(\tilde{X}) = \operatorname{Sec}_p(\tilde{X}) \cap \tilde{X}.$$

Then, by [BS, Theorem 3.1] we know that $\operatorname{Sec}_p(\tilde{X}) = \mathbb{P}_K^{t-1}$ and $\Sigma_p(\tilde{X}) \subseteq \operatorname{Sec}_p(\tilde{X})$ is a hyperquadric, where $t = \operatorname{depth} X_p$ is the *arithmetic depth* of X_p.

As $\tilde{X}$ is smooth, we also have $1 \leq \operatorname{depth} X_p \leq 4$ by [BS, Corollary 7.6].

We also consider the submatrices $M_{\underline{x}}, M_{\underline{y}}, M_{\underline{z}}$ and $M_{\underline{y},\underline{z}}$ of M which consist of the columns whose entries are the indeterminates indicated by the index. Then clearly $I_2(M_{\underline{x}}) \subseteq K[\underline{x}], I_2(M_{\underline{y}}) \subseteq K[\underline{y}]$, and so on.

(C) Next, we also consider the $(3 \times (k + 2l - n + \sum_{j=1}^{n-k-l} a_j))$-matrix

$$M' := \left(\begin{array}{c|c|c|cccc|c} \cdots & y_{i,0} & \cdots & z_{j,0} & z_{j,1} & \cdots & z_{j,a_j-2} & \cdots \\ \cdots & y_{i,1} & \cdots & z_{j,1} & z_{j,2} & \cdots & z_{j,a_j-1} & \cdots \\ \cdots & y_{i,2} & \cdots & z_{j,2} & z_{j,3} & \cdots & z_{j,a_j} & \cdots \end{array} \right)$$

with entries in $K[\underline{y}, \underline{z}]_1$. This matrix allows us to describe the secant variety $\operatorname{Sec}(\tilde{X})$ of $\tilde{X}$ by

$$V_3(M') = \operatorname{Sec}(\tilde{X}) := \overline{\bigcup_{p,q,\in \tilde{X}, p \neq q} \langle p, q \rangle}$$

(see [C]). Similarly as above we now define the submatrices $M'_{\underline{y}}$ and $M'_{\underline{z}}$.

It is easy to see that

$$I_2(M'_{\underline{y}}) \subseteq I_2(M_{\underline{y}}) \subseteq K[\underline{y}] \text{ and } I_2(M') \subseteq I_2(M_{\underline{y},\underline{z}}) \subseteq K[\underline{y}, \underline{z}].$$

In particular, in view of the observations made in part (B) we get:

$$S(\underline{2}, \underline{a}) \subseteq V_2(M') \cap \langle S(\underline{2}, \underline{a}) \rangle \text{ and } S(\underline{a}) = V_2(M'_{\underline{z}}) \cap \langle S(\underline{a}) \rangle = V_2(M') \cap \langle S(\underline{a}) \rangle.$$

(D) Next, we consider the Segre embedding

$$\sigma : \mathbb{P}^2_K \times \mathbb{P}^{l-1}_K \hookrightarrow \langle S(\underline{2}) \rangle = \mathbb{P}^{3l-1}_K,$$
$$((u_0 : u_1 : u_2), (v_1 : \ldots : v_l)) \mapsto (\ldots : u_i v_j : \ldots), i = 0, 1, 2, j = 1, \ldots, l,$$

and set

$$\Delta := \operatorname{Im}(\sigma).$$

Then it is well known that Δ is defined by the 2×2 minors of $M'_{\underline{y}}$; thus (see [S, §5])

$$\Delta = V_2(M'_{\underline{y}}) \cap \langle S(\underline{2}) \rangle = V_2(M') \cap \langle S(\underline{2}) \rangle.$$

3. The rank-depth formula

We keep the hypotheses and notation of the previous section. Moreover we continue with some further definitions.

Notation 3.1. If $p = [\overline{p}] \in \mathbb{P}^{r+1}_K$ with

$$\overline{p} = (\ldots, a_{s,0}, a_{s,1}, \ldots, b_{i,0}, b_{i,1}, b_{i,2}, \ldots, c_{j,0}, c_{j,1}, \ldots, c_{j,a_j}, \ldots) \in K^{r+1} \setminus \{\underline{0}\},$$

we allow ourselves to write

$$M'(p) := \left(\begin{array}{c|c|c|cccc|c} \cdots & b_{i,0} & \cdots & c_{j,0} & c_{j,1} & \cdots & c_{j,a_j-2} & \cdots \\ \cdots & b_{i,1} & \cdots & c_{j,1} & c_{j,2} & \cdots & c_{j,a_j-1} & \cdots \\ \cdots & b_{i,2} & \cdots & c_{j,2} & c_{j,3} & \cdots & c_{j,a_j} & \cdots \end{array} \right),$$

although this matrix is determined by p only up to a nonzero scalar multiple.

The aim of this section is to relate the rank of the matrix $M'(p)$ (which is obviously well defined) with the arithmetic depth of the projected image X_p of $\tilde{X}$. We begin with two auxiliary results.

Lemma 3.2. $\mathrm{Join}(S(\underline{1}), \tilde{X}) = \mathrm{Join}(\langle S(\underline{1})\rangle, S(\underline{2}, \underline{a}))$.

Proof. "$\subseteq$": This containment is easy to see.

"$\supseteq$": Let $z'' \in \langle S(\underline{1})\rangle$ and $z' \in S(\underline{2}, \underline{a})$. It suffices to show that the line $\langle z', z''\rangle$ is contained in $\mathrm{Join}(S(\underline{1}), \tilde{X})$. So, let $z \in \langle z', z''\rangle$. Let $x \in \mathbb{P}^1_K \setminus \varphi(z')$. Then

$$\mathbb{L}(x) \cap \langle S(\underline{1})\rangle \text{ and } \mathbb{L}(\varphi(z')) \cap \langle S(\underline{1})\rangle$$

are two disjoint $(k-1)$-dimensional subspaces of $\langle S(\underline{1})\rangle = \mathbb{P}^{2k-1}_K$. So, these two subspaces span $\langle S(\underline{1})\rangle$. Hence there are points $u \in \mathbb{L}(\varphi(z')) \cap \langle S(\underline{1})\rangle$ and $v \in \mathbb{L}(x) \cap \langle S(\underline{1})\rangle$ such that $z'' \in \langle u, v\rangle$.

Observe that $\langle u, z'\rangle \subseteq \mathbb{L}(\varphi(z')) \subseteq \tilde{X}$ and that $v \in S(\underline{1})$. Moreover the four points u, v, z, z' are coplanar so that the lines $\langle v, z\rangle$ and $\langle u, z'\rangle$ intersect and the lines $\langle v, z\rangle$ and $\langle u, z'\rangle$ intersect in some point $\overline{z}$. It follows that $z \in \langle v, \overline{z}\rangle \subseteq \mathrm{Join}(S(\underline{1}), \tilde{X})$. □

Lemma 3.3. *Assume that $k = 0$. Then $V_2(M') \setminus \Delta \subseteq S(\underline{2}, \underline{a})$.*

Proof. Let $q \in \mathbb{P}^r_K$ be a point with $q \in V_2(M') \setminus \Delta$ such that

$$q = (\ldots : b_{i,0} : b_{i,1} : b_{i,2} : \ldots : c_{j,0} : c_{j,1} : \ldots : c_{j,a_j} : \ldots).$$

Therefore the matrix $M'(q) = (B_1 \ldots B_l C_1 \ldots C_{n-l})$ has rank one, where

$$B_i := \begin{pmatrix} b_{i,0} \\ b_{i,1} \\ b_{i,2} \end{pmatrix} \text{ and } C_j = \begin{pmatrix} c_{j,0} & c_{j,1} & \cdots & c_{j,a_j-2} \\ c_{j,1} & c_{j,2} & \cdots & c_{j,a_j-1} \\ c_{j,2} & c_{j,3} & \cdots & c_{j,a_j} \end{pmatrix}$$

for $i = 1, \ldots, l$, and $j = 1, \ldots, n-l$.

Clearly some of the entries $c_{j,t}$ do not vanish, as otherwise we would have $q \in \langle S(\underline{2})\rangle$ in contradiction to $q \in V_2(M') \cap \langle S(\underline{2})\rangle = \Delta$ (see Notation and Remark 2.2 (D)). So, we find a largest index j such that the block matrix C_j does not vanish.

Assume first that $c_{j,0} \neq 0$. Then, the fact that the columns of C_j are linearly dependent shows that there is some $\lambda \in K$ such that

$$C_j = \begin{pmatrix} c_{j,0}\lambda^0 & c_{j,0}\lambda^1 & \cdots & c_{j,0}\lambda^{a_j-2} \\ c_{j,0}\lambda^1 & c_{j,0}\lambda^2 & \cdots & c_{j,0}\lambda^{a_j-1} \\ c_{j,0}\lambda^2 & c_{j,0}\lambda^3 & \cdots & c_{j,0}\lambda^{a_j} \end{pmatrix}.$$

Now, by the linear dependence of the columns in $M'(p)$ the above formula holds for all blocks C_j with the same element λ, and moreover all columns B_i have the shape

$$B_i = \begin{pmatrix} b_{i,0}\lambda^0 \\ b_{i,0}\lambda^1 \\ b_{i,0}\lambda^2 \end{pmatrix}.$$

Setting $b_i := b_{i,0}$ for $i = 1, \ldots, l$ and $c_j = c_{j,0}$ for $j = 1, \ldots, n-l$ we get

$$b_{i,s} = b_i\lambda^s, \text{ for } i = 1, \ldots, l, \text{ and } s = 0, 1, 2,$$
$$c_{j,t} = c_j\lambda^t, \text{ for } j = 1, \ldots, n-l, \text{ and } t = 0, 1, \ldots, a_j.$$

But this implies that $q \in S(\underline{2}, \underline{a})$.

Assume now that $c_{j,0} = 0$. As $\mathrm{rank}(C_j) = 1$ it follows immediately that

$$C_j = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ \vdots & & & 0 \\ 0 & \cdots & 0 & c_{j,a_j} \end{pmatrix}$$

with $c_{j,a_j} \neq 0$. By the linear dependence of columns in $M'(q)$ it follows easily that all blocks C_j must have this shape (with $c_{j,a_j} = 0$, possibly) and that all columns B_i have the shape

$$B_i = \begin{pmatrix} 0 \\ 0 \\ b_{i,2} \end{pmatrix}.$$

This implies again that $p \in S(\underline{2}, \underline{a})$. This completes the proof. □

Theorem 3.4. *For each point $p \in \mathbb{P}_K^{r+1} \setminus \tilde{X}$ it follows that*

$$\operatorname{depth}(X_p) = \dim(\Sigma_p(\tilde{X})) + 2 = 4 - \operatorname{rank}(M'(p)).$$

Proof. The first equality follows by the observations made in Notation and Reminder 2.3 (B).

Now, set

$$A := \langle S(\underline{1}) \rangle, B := \operatorname{Join}(S(\underline{1}), \tilde{X}) \text{ and } U := \operatorname{Join}(A, \Delta).$$

Then, by the secant stratification of $\tilde{X}$ (see [BP, Theorem 4.2]) we have the following four cases:

1: $\dim \Sigma_p(\tilde{X}) = 2$ if and only if $p \in A \setminus \tilde{X}$.
2: $\dim \Sigma_p(\tilde{X}) = 1$ if and only if $p \in (U \cup B) \setminus (A \cup \tilde{X})$.
3: $\dim \Sigma_p(\tilde{X}) = 0$ if and only if $p \in \operatorname{Sec}(\tilde{X}) \setminus (U \cup B)$.
4: $\dim \Sigma_p(\tilde{X}) = -1$ if and only if $p \in \mathbb{P}_K^{r+1} \setminus \operatorname{Sec}(\tilde{X})$.

Clearly $p \in A \setminus \tilde{X}$ is equivalent to $M'(p) = 0$, whereas $p \in \mathbb{P}_K^{r+1} \setminus \operatorname{Sec}(\tilde{X})$ is equivalent to $p \notin V_3(M')$ (see Notation and Remark 2.2 (C)), whence to the fact that $\operatorname{rank}(M'(p)) = 3$. So, we are in case 1 precisely if the matrix $M'(p)$ has rank 0 and in case 4 precisely if this matrix has rank 3. It remains to show that we are in case 2 precisely if $M'(p)$ is of rank 1 and in case 3 precisely if $M'(p)$ is of rank 2. By exclusion, it suffices to prove the first of these equivalences. It thus remains to show that for our point $p \in \mathbb{P}_K^{r+1} \setminus \tilde{X}$ we have $\operatorname{rank}(M'(p)) = 1$ if and only if $p \in (U \cup B) \setminus A$.

Assume first that $\operatorname{rank}(M'(p)) = 1$. Then $p \in V_2(M')$ and $p \notin A$. Now suppose first that $p \in \langle S(\underline{2}, \underline{a}) \rangle$. Assume for the moment that $p \notin \Delta$. Then, by Lemma 3.3 applied to the scroll $S(\underline{2}, \underline{a}) = \tilde{X} \cap \langle S(\underline{2}, \underline{a}) \rangle \subset \langle S(\underline{2}, \underline{a}) \rangle$ we get the contradiction that $p \in S(\underline{2}, \underline{a}) \subset \tilde{X}$. Therefore we must have $p \in \Delta$ and hence $p \in U$ in this case.

Suppose now that $p \notin \langle S(\underline{2}, \underline{a}) \rangle$. As $M'(p) \neq 0$ we cannot have $p \in A$ (see Notation 3.1). Therefore we can write $p \in \langle t, q \rangle$ with $t \in A$ and $q \in \langle S(\underline{2}, \underline{a}) \rangle$. Observe that by definition of M', the matrix $M'(q)$ must be a nontrivial scalar multiple of the matrix $M'(p)$ (see Notation 3.1), so that $q \in V_2(M')$. Since $q \in S(\underline{2}, \underline{a})$ we have $p \in \operatorname{Join}(A, S(\underline{2}, \underline{a})) = B$ (see Lemma 3.2). So, if $\operatorname{rank}(M'(p)) = 1$, we have indeed $p \in (U \cup B) \setminus A$.

Assume now conversely that $p \in (U \cup B) \setminus A$. As $p \notin A$ we must then have $\operatorname{rank}(M'(p)) \geq 1$. If $p \in U$, we write $p = \langle t, q \rangle$ with $t \in A$ and $q \in \Delta \subseteq \langle S(\underline{2}) \rangle$. In view of Notation and Remark 2.2 (D) it follows that $q \in V_2(M')$, whence $p \in V_2(M')$ so that $\operatorname{rank}(M'(p)) = 1$. If $p \in B = \operatorname{Join}(A, S(\underline{2}, \underline{a}))$, we write $p \in \langle t, q \rangle$ with $t \in A$ and $q \in S(\underline{2}, \underline{a})$. By Notation and Remark 2.2 (C) it follows that $q \in V_2(M')$, whence $p \in V_2(M')$ so that $\operatorname{rank}(M'(p)) = 1$. □

Finally, we wish to extend our rank-depth formula to the case of possibly singular scrolls. We first give a few preparatory remarks.

Notation and Reminder 3.5. (A) Let h be an integer ≥ -1. Consider the polynomial ring $K[\underline{w}, \underline{x}, \underline{y}, \underline{z}] = K[w_0, .., w_h, \underline{x}, \underline{y}, \underline{z}]$ and the possibly singular scroll

$$\begin{aligned}\tilde{X} &= S(0, \dots, 0, 1, \dots, 1, 2, \dots, 2, a_1, \dots, a_{n-k-l}) \\ &= S(\underline{0}, \underline{1}, \underline{2}, \underline{a}) \subset \mathbb{P}^{r+h+2}_K = \mathrm{Proj}(K[\underline{w}, \underline{x}, \underline{y}, \underline{z}]).\end{aligned}$$

Define the matrix M' as in Notation and Reminder 2.3 (A).

(B) Observe that $\tilde{X}$ is a cone with vertex

$$\mathrm{Vert}(\tilde{X}) = \mathbb{P}^h_K = \mathrm{Proj}(K[\underline{w}, \underline{x}, \underline{y}, \underline{z}]/(\underline{x}, \underline{y}, \underline{z}) = \mathrm{Proj}(K[\underline{w}])$$

over the smooth rational normal scroll

$$\tilde{X}_0 = S(\underline{1}, \underline{2}, \underline{a}) \subset \mathbb{P}^{r+1}_K = \mathrm{Proj}(K[\underline{w}, \underline{x}, \underline{y}, \underline{z}]/(\underline{w})) = \mathrm{Proj}(K[\underline{x}, \underline{y}, \underline{z}])$$

defined in Notation and Remark 2.1 (A).

Now, let $p \in \mathbb{P}^{r+h+2}_K \setminus \tilde{X}$ and let p_0 be the point obtained by projecting p from $\mathrm{Vert}(\tilde{X}) = \mathbb{P}^h_K$ to the span $\langle \tilde{X}_0 \rangle = \mathbb{P}^{r+1}_K$. Then $p_0 \in \mathbb{P}^{r+1}_K \setminus \tilde{X}_0$. Moreover if $X_p \subset \mathbb{P}^{r+h+1}_K$ and $(X_0)_p \subset \mathbb{P}^r_K$ respectively are projections of $\tilde{X}$ from p and of $\tilde{X}_0$ from p_0, we have (see [BP, Remark 5.4]) that

$$\mathrm{depth}(X_p) = \dim(\Sigma_p(\tilde{X})) + 2 = \dim(\Sigma_p(\tilde{X}_0)) + h + 2.$$

Now, in the previous notation we obtain:

Corollary 3.6. *For each point $p \in \mathbb{P}^{r+h+2}_K \setminus \tilde{X}$ it follows that*

$$\mathrm{depth}\, X_p = \dim(\Sigma_p(\tilde{X})) + 2 = 5 + h - \mathrm{rank}\, M'(p).$$

Proof. The claim is immediate by Theorem 3.4 and Notation and Reminder 3.5. □

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