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A NOTE ON PROJECTIVE NORMALITY

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Abstract. Let G be any finite group, $G \to GL(V)$ be a representation of G, where V is a finite-dimensional vector space over an algebraically closed field k. Theorem. Assume that either $char\, k = 0$ or $char\, k = p > 0$ with $p \nmid |G|$. Then the quotient variety $\mathbb{P}(V)/G$ is projectively normal with respect to the line bundle $\mathcal{L}$, where $\mathcal{L}$ is the descent of $\mathcal{O}(1)^{\otimes m}$ from $\mathbb{P}(V)$ with $m = |G|!$. This partially solves a question raised in the paper of Kannan, Pattanayak and Sardar [Proc. Amer. Math. Soc. 137 (2009), 863–867].

1. Introduction

Let k be an algebraically closed field, G be a finite group, $G \to GL(V)$ be a representation of G to $GL(V)$, where V is a finite-dimensional vector space over k. Then G acts on V and $\mathbb{P}(V)$ (which is the projective space $\mathbb{P}^{r-1}$ if $\dim_k V = r$). Thus $\mathcal{O}(1)^{\otimes |G|}$ descends to the quotient variety $\mathbb{P}(V)/G$, and so $\mathcal{O}(1)^{\otimes |G|!}$ also descends to the quotient variety (see [KKV] for details). In [KPS], the projective normality of $\mathbb{P}(V)/G$ with respect to the descent of $\mathcal{O}(1)^{\otimes |G|}$ is considered. The following result is proved there.

Theorem 1.1 (Kannan, Pattanayak and Sardar [KPS, Theorem 3.1])**.** *Let $\mathcal{L}$ be the line bundle on $\mathbb{P}(V)/G$ which is the descent of $\mathcal{O}(1)^{\otimes m}$ from $\mathbb{P}(V)$, where $m = |G|$. Assume that either $char\, k = 0$ or $char\, k = p > 0$ with $p \nmid |G|$. If G is a solvable group or a pseudo-reflection group, then $\mathbb{P}(V)/G$ is projectively normal with respect to $\mathcal{L}$.*

The reader is referred to [KP] for progress made on this question for Weyl group representations. What we will prove in this article is the following theorem, which holds for any finite group.

Theorem 1.2. *For any finite group G, let $\mathcal{L}$ be the line bundle on $\mathbb{P}(V)/G$ which is the descent of $\mathcal{O}^{\otimes m}$ from $\mathbb{P}(V)$, where $m = |G|!$. Assume that either $char\, k = 0$ or $char\, k = p > 0$ with $p \nmid |G|$. Then $\mathbb{P}(V)/G$ is projectively normal with respect to $\mathcal{L}$.*

Before stating a key lemma for proving Theorem 1.2, we give the definition of the "restricted product".

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Definition 1.3. Let $a_1, \ldots, a_n$ be positive integers (it is allowed that $a_i = a_j$ for some $i \neq j$). The usual product is simply $a_1 a_2 \cdots a_n$ (for short, $\prod_i a_i$). We will define $\prod'_i a_i$, the restricted product, as $\prod'_i a_i = \prod_{1 \leq j \leq e} b_j$ if $\{a_1, a_2, \ldots, a_n\} = \{b_1, b_2, \ldots, b_e\}$, where $b_i \neq b_j$ if $i \neq j$.

Lemma 1.4. *Let k be a field, $A = A_0 \oplus A_1 \oplus \cdots \oplus A_l \oplus \cdots$ be a commutative graded algebra, where $A_0 = k$ and A_l is the k-vector space of homogeneous elements of degree l in A. Let $A = k[f_1, \ldots, f_n]$ for some homogeneous elements $f_1, \ldots, f_n$ with $\deg f_i = a_i \geq 1$. Define $m = \prod'_i a_i$ to be the restricted product of $a_1, a_2, \ldots, a_n$. Then $k[A_m] = A_0 \oplus A_m \oplus A_{2m} \oplus \cdots \oplus A_{dm} \oplus \cdots$, where $k[A_m] = k[g_1, \ldots, g_r]$ if $A_m = \sum_{1 \leq j \leq r} k \cdot g_j$.*

We remark that the above lemma was known to Zariski when A is the integral closure of some graded k-algebra $B = B_0 \oplus B_1 \oplus \cdots \oplus B_l \oplus \cdots$ (with $B_0 = k$) such that $\mathrm{Proj}(B)$ is a normal variety and B is generated by B_1 over k (see [Za]; [La], Proposition 11, p. 141; [Ha], Exercise 5.14, p. 126).

2. Proof

We will denote by $\mathbb{N}$ the set of all positive integers, and by $\mathbb{N}_{\geq 0}$ the set $\mathbb{N} \cup \{0\}$.

Let $a_1, a_2, \ldots, a_n$ be positive integers (it is allowed that $a_i = a_j$ for some $i \neq j$). Define

$$N_d = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n_{\geq 0} : \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n = d\}.$$

By $N_d + N_e$, we mean the set $\{\lambda + \mu : \lambda \in N_d, \mu \in N_e\}$. Thus $2N_d = N_d + N_d$. Similarly $lN_d = N_d + \cdots + N_d$ (the sum of l copies of N_d). Recall a result in [KPS] which will be used in the proof of Lemma 1.4.

Lemma 2.1 ([KPS, Lemma 2.1])**.** *Define $m = \prod_{1 \leq i \leq n} a_i$. Then $N_{dm} = dN_m$.*

Proof of Lemma 1.4. Recall that $m = \prod'_i a_i$ is the restricted product of $a_1, \ldots, a_n$.

Write $\{a_1, \ldots, a_n\} = \{b_1, \ldots, b_e\}$, where $b_i \neq b_j$ if $i \neq j$. We may arrange $a_1, \ldots, a_n$ so that $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 < b_2 < \cdots < b_e$. Moreover, find integers $l(1), \ldots, l(e)$ such that $1 \leq l(1) < l(2) < \cdots < l(e) = n$ and $a_1 = a_2 = \cdots = a_{l(1)} = b_1$, $a_{l(1)+1} = a_{l(1)+2} = \cdots = a_{l(2)} = b_2$, ..., $a_{l(e-1)+1} = a_{l(e-1)+2} = \cdots = a_{l(e)} = b_e$.

In order to prove that $k[A_m] = A_0 \oplus A_m \oplus \cdots \oplus A_{dm} \oplus \cdots$, we will prove by induction that $A_{dm} = A^d_m$, where A^d_m is the k-vector space generated by elements of the form $h_1 h_2 \cdots h_d$ with each $h_j \in A_m$ for $1 \leq j \leq d$.

Each element in A_{dm} is a linear combination of monomials of the form $f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_n^{\lambda_n}$, where $\lambda_1 a_1 + \cdots + \lambda_n a_n = dm$ (note that $f_j \in A_{a_j}$ for $1 \leq j \leq n$). It suffices to show that these monomials $f_1^{\lambda_1} \cdots f_n^{\lambda_n} \in A^d_m$.

For a monomial $f = f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_n^{\lambda_n}$ with $\lambda_1 a_1 + \cdots + \lambda_n a_n = dm$, define $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n_{\geq 0}$. Define $\mu = (\mu_1, \ldots, \mu_e) \in \mathbb{N}^e_{\geq 0}$ by

$$\mu_i = \sum_{l(i-1)+1 \leq j \leq l(i)} \lambda_j \tag{1}$$

for $1 \leq i \leq e$, where it is understood that $l(0) = 0$.

Define

$$N'_d = \{\eta = (\eta_1, \ldots, \eta_e) \in \mathbb{N}^e_{\geq 0} : \eta_1 b_1 + \cdots + \eta_e b_e = d\}.$$

It follows that $\mu \in N'_{dm}$. Apply Lemma 2.1 to N'_{dm}. We find there $\mu = \nu^{(1)} + \nu^{(2)} + \cdots + \nu^{(d)}$ for some $\nu^{(1)}, \ldots, \nu^{(d)} \in N'_m$.

Write $\nu = \nu^{(1)} = (\nu_1, \nu_2, \ldots, \nu_e)$. It follows that $\mu_i \geq \nu_i$ for $1 \leq i \leq e$. By Formula (1), we find that $\sum_{l(i-1)+1 \leq j \leq l(i)} \lambda_j \geq \nu_i$ for $1 \leq i \leq e$. Hence it is possible to find $\lambda'_1, \ldots, \lambda'_n \in \mathbb{N}_{\geq 0}$ satisfying the conditions (i) $\lambda'_j \leq \lambda_j$ for $1 \leq j \leq n$, and (ii) $\nu_i = \sum_{l(i-1)+1 \leq j \leq l(i)} \lambda'_j$ for $1 \leq i \leq e$.

Define $g \in A_m$ and $h \in A_{(d-1)m}$ by

$$g = \prod_{1 \leq j \leq n} f_j^{\lambda'_j}, \quad h = \prod_{1 \leq j \leq n} f_j^{\lambda_j - \lambda'_j}.$$

We find that $f = g \cdot h$. Since $h \in A_{(d-1)m} = A_m^{d-1}$ by induction, we find that $f \in A_m^d$. □

Before proving Theorem 1.2, we recall a theorem about Noether's bound due to Fleischmann and Fogarty independently.

Theorem 2.2 (Fleischmann, Fogarty [Fl], [Fo]). *Let G be a finite group, k be a field, and V be an r-dimensional vector space over k. Suppose that $G \to GL(V)$ is a representation of G. Let $x_1, \ldots, x_r$ be a basis of the dual space V^* of V such that $k[V]^G$ is a k-subalgebra of the graded algebra $k[V] = k[x_1, \ldots, x_r]$, where each x_i is of degree one.*

Assume that either $\operatorname{char} k = 0$ or $\operatorname{char} k = p > 0$ with $p \nmid |G|$. Then $k[V]^G$ is generated over k by invariant polynomials of degree less than or equal to $|G|$.

Proof of Theorem 1.2. Apply Theorem 2.2 to get a set of generators of $A = k[V]^G$ and then use Lemma 1.4.

If there is a non-zero invariant homogeneous polynomial of degree d for all $1 \leq d \leq |G|$, then apply Theorem 2.2 and Lemma 1.4. Note that the integer m in Lemma 1.4 is $\prod_{1 \leq d \leq r} d = |G|!$. Hence we get $k[A_m] = A_0 \oplus A_m \oplus \cdots \oplus A_{dm} \oplus \cdots$.

Suppose that there is no non-zero invariant homogeneous polynomial of degree d for some d with $1 \leq d \leq |G|$. We just throw away these integers d and proceed as above. Thus we get an integer m' such that $k[A_{m'}] = A_0 \oplus A_{m'} \oplus \cdots \oplus A_{dm'} \oplus \cdots$. Since $|G|!$ is a multiple of m', it is easy to see that $k[A_m] = A_0 \oplus A_m \oplus \cdots \oplus A_{dm} \oplus \cdots$ if we define $m = |G|!$.

The remaining proof is the same as the proof of Theorem 3.1 in [KPS]. □

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