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WEIGHTED BERGMAN PROJECTIONS AND KERNELS: L^p REGULARITY AND ZEROS

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ABSTRACT. We investigate L^p regularity of weighted Bergman projections and zeros of weighted Bergman kernels for the weights that are radially symmetric and comparable to 1 on the unit disc.

1. INTRODUCTION

1.1. **Setup and notation.** Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}^1$ and let $\lambda(r)$ be a non-negative function on $[0,1)$ that is comparable to 1.

Definition 1.1. A function $\lambda(r)$ on $[0,1)$ is said to be comparable to 1 and denoted by $\lambda \sim 1$ if there exists $C > 0$ such that $\frac{1}{C} \leq \lambda(r) \leq C$ for all $r \in [0,1)$.

We consider λ as a radial weight on $\mathbb{D}$ by setting $\lambda(z) = \lambda(|z|)$. We denote the Lebesgue measure on $\mathbb{C}$ by $dA(z)$ and the space of square integrable functions on $\mathbb{D}$ with respect to the measure $\lambda(z)dA(z)$ by $L^2(\lambda)$. This is a Hilbert space with the inner product and the norm defined by

$$\langle f, g\rangle_\lambda = \int_{\mathbb{D}} f(z)\overline{g(z)}\lambda(z)dA(z) \quad \text{and} \quad ||f||_\lambda^2 = \int_{\mathbb{D}} |f(z)|^2\lambda(z)dA(z).$$

The space of holomorphic functions that are in $L^2(\lambda)$ is denoted by $A^2(\lambda)$. The Bergman inequality (see the first page of [6]) shows that $A^2(\lambda)$ is a closed subspace of $L^2(\lambda)$. The orthogonal projection between these two spaces is called *the weighted Bergman projection* and is denoted by $\mathbf{B}_\lambda$, i.e.

$$\mathbf{B}_\lambda : L^2(\lambda) \to A^2(\lambda). \tag{1.2}$$

It follows from the Riesz representation theorem that $\mathbf{B}_\lambda$ is an integral operator. The kernel is called *the weighted Bergman kernel* and denoted by $B_\lambda(z,w)$; i.e. for any $f \in L^2(\lambda)$,

$$\mathbf{B}_\lambda f(z) = \int_{\mathbb{D}} B_\lambda(z,w) f(w)\lambda(w)dA(w). \tag{1.3}$$

For a radial weight λ as above, the monomials $\{z^n\}_{n=0}^\infty$ form an orthogonal basis for $A^2(\lambda)$ and after normalization the weighted Bergman kernel is given by $B_\lambda(z,w) = \sum_{n=0}^\infty \alpha_n (z\bar{w})^n$, where $\alpha_n = \frac{1}{2\pi\int_0^1 r^{2n+1}\lambda(r)dr}$.

The general theory and details can be found in [7] and [11].

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1.2. **Questions and results.** Both the weighted Bergman projection and the weighted Bergman kernel are canonical objects on the weighted space $(\mathbb{D}, \lambda)$. It is a fundamental problem as to how the perturbations of λ affect the analytic properties of these canonical objects. In this paper, we are particularly interested in the following questions:

I. For a given radial weight λ on $\mathbb{D}$ as above, does the weighted kernel $B_\lambda(z, w)$ have zeros in $\mathbb{D} \times \mathbb{D}$?

II. For a given radial weight λ on $\mathbb{D}$ as above, for which values of $p \in (1, \infty)$ is the weighted projection $\mathbf{B}_\lambda$ bounded from $L^p(\lambda)$ to $L^p(\lambda)$?

The answers for both of these questions are known for $\lambda(r) \equiv 1$. In this case, a direct computation gives that

$$B_1(z, w) = \frac{1}{\pi} \sum_{n=0}^{\infty} (n+1)(z\bar{w})^n = \frac{1}{\pi(1 - z\bar{w})^2}. \tag{1.4}$$

This closed form immediately tells us that $B_1(z, w)$ never vanishes inside $\mathbb{D} \times \mathbb{D}$. Moreover, this explicit form and Schur's lemma prove that $\mathbf{B}_1$ is bounded from $L^p(1)$ to $L^p(1)$ for all $p \in (1, \infty)$.

For an arbitrary radial weight, in general, we do not have such a closed form for the kernel, and it requires more work to answer the questions above. In the next two sections, we prove the following two theorems that answer questions **I** and **II**.

Theorem 1.5. *There exists a radial weight λ on $\mathbb{D}$, comparable to* 1*, such that the kernel function $B_\lambda(z, w)$ has zeros.*

Theorem 1.6. *Suppose λ is a non-negative radial weight on $\mathbb{D}$ that is comparable to* 1*. Then the weighted Bergman projection $\mathbf{B}_\lambda$ is bounded from $L^p(\lambda)$ to $L^p(\lambda)$ for all $p \in (1, \infty)$.*

In the sprit of the problem above, these two theorems say that a comparable perturbation of the radial weight on $\mathbb{D}$ might change the vanishing properties of the kernel but does not alter the L^p mapping properties of the projection.

2. Zeros of weighted Bergman kernels

Question **I** recalls the well-known Lu Qi-Keng problem that asks about the zeros of Bergman kernels as the underlying domain changes. A detailed survey of this problem can be found in [2].

In this section, after the proof of Theorem 1.5, we use the Forelli-Rudin formula from [7] (also called the inflation principle in [3]) to explore zeros of Bergman kernels in higher dimensions.

Proof of Theorem 1.5. Define $\lambda(r) = \begin{cases} 18, & 0 \le r \le \frac{1}{4}, \\ 1, & \frac{1}{4} < r \le 1. \end{cases}$ For this weight, we explicitly compute the coefficients α_n and get

$$\alpha_n = \frac{1}{2\pi} \frac{16^{n+1}(2n+2)}{16^{n+1} + 17}.$$

We split the product $(1-z\bar{w})^2 B_\lambda(z,w)$ into two parts as

$$
\begin{aligned}
&(1-z\bar{w})^2 \sum_{k=0}^{\infty} \alpha_k (z\bar{w})^k \\
&\qquad = \alpha_0 + (\alpha_1 - 2\alpha_0) z\bar{w} + \sum_{k=2}^{\infty} (\alpha_k - 2\alpha_{k-1} + \alpha_{k-2})(z\bar{w})^k \\
&\qquad := L(t) + S(t),
\end{aligned}
$$

where $L(t)$ denotes the linear part, $S(t)$ denotes the series part and t stands for $z\bar{w}$. Note that $L(t)$ has a zero in $\mathbb{D}$ by explicitly computing α_0 and α_1. Indeed, $\alpha_0 = \frac{16}{33\pi}$ and $\alpha_1 = \frac{512}{273\pi}$, so $L(t)$ vanishes at $t = \frac{-91}{170}$.

Next, we show that

$$\min_{|t|=1-\epsilon} |L(t)| > \max_{|t|=1-\epsilon} |S(t)| \tag{2.1}$$

for small enough $\epsilon > 0$.

It is clear that $\max_{|t|=1-\epsilon} |S(t)| < \sum_{k=2}^{\infty} |\alpha_k - 2\alpha_{k-1} + \alpha_{k-2}|$, so it is enough to show that

$$\min_{|t|=1-\epsilon} |L(t)| > \sum_{k=2}^{\infty} |\alpha_k - 2\alpha_{k-1} + \alpha_{k-2}|. \tag{2.2}$$

By using the explicit formula for α_k, we get for $k \geq 2$,

$$
\begin{aligned}
\alpha_k - \alpha_{k-1} &= \frac{32 \times 16^{2k} + 34 \times 16^k (15k+16)}{(16^k + 17)(16^{k+1} + 17)}, \\
\alpha_{k-1} - \alpha_{k-2} &= \frac{2 \times 16^{2k-1} + 34 \times 16^{k-1}(15k+1)}{(16^k + 17)(16^{k-1} + 17)}.
\end{aligned}
$$

By comparing right hand sides, we note that $\alpha_k - \alpha_{k-1} < \alpha_{k-1} - \alpha_{k-2}$. Therefore, we get $\alpha_k - 2\alpha_{k-1} + \alpha_{k-2} < 0$ for $k \geq 2$, and hence the sum on the right hand side of (2.2) is telescoping and converges to

$$\alpha_1 - \alpha_0 - \lim_{k\to\infty} (\alpha_k - \alpha_{k-1}) = (\alpha_1 - \alpha_0) - 2.$$

On the other hand, by a direct calculation, for small enough $\epsilon > 0$,

$$\min_{|t|=1-\epsilon} |L(t)| = (\alpha_1 - 3\alpha_0) - \epsilon(\alpha_1 - 2\alpha_0).$$

Thus, it remains to show that

$$(\alpha_1 - 3\alpha_0) - \epsilon(\alpha_1 - 2\alpha_0) > (\alpha_1 - \alpha_0) - 2,$$

for small enough ϵ, in order to get the inequality (2.2). The last inequality is verified by plugging in the actual values of α's, and we get the inequality (2.1).

Once the inequality (2.1) is obtained, we use Rouche's theorem: $L(t)$ has a zero in $|t| < 1-\epsilon$ and $|L(t)| > |S(t)|$ on $|t| = 1-\epsilon$ for small enough ϵ; therefore, the sum $L(t) + S(t) = (1-t)^2 \sum_{k=0}^{\infty} \alpha_k t^k$ has a zero in $\mathbb{D}$. Since $(1-t)^2$ does not vanish in $\mathbb{D}$ we conclude that $B_\lambda(z,w)$ has zeros in $\mathbb{D} \times \mathbb{D}$. □

Remark. The same argument can be applied to more weights. We can show that by choosing $A, x > 0$ suitably, if we define $\lambda(r) = \begin{cases} A, & 0 \leq r \leq x, \\ 1, & x < r \leq 1, \end{cases}$ then again the weighted Bergman kernel has zeros in $\mathbb{D} \times \mathbb{D}$.

Furthermore, we can modify these discontinuous functions so that we get smooth radial weight functions for which the weighted Bergman kernels have zeros.

For higher dimensional application, let us take the particular weight λ in the proof of Theorem 1.5 and consider the following Reinhardt domain in $\mathbb{C}^2$:

$$\Omega = \{(z,w) \in \mathbb{C}^2 : z \in \mathbb{D} \ \text{ and } |w|^2 < \lambda(z)\}.$$

Let $B_\Omega[(z,w),(t,s)]$ denote the unweighted Bergman kernel of the domain Ω and $B_\lambda(z,t)$ be the weighted Bergman kernel of $\mathbb{D}$. We have the following relation between these kernels (see [7], [8], [3]):

$$B_\Omega[(z,0),(t,0)] = \frac{1}{\pi} B_\lambda(z,t). \tag{2.3}$$

Thus, when the weighted kernel has a zero so does the unweighted kernel in higher dimension.

Remark. H.P. Boas observed that Theorem 1.5 can be also proven by using Ramadanov's approximation theorem in [9]. A direct computation shows that the weighted Bergman kernel for the weight $(1 + k\delta_0)$, where δ_0 is the Dirac mass at $z = 0$ and $k > 0$, has a zero in $\mathbb{D} \times \mathbb{D}$. We approximate $(1 + k\delta_0)$ by smooth functions ϕ_n that are all comparable to 1. Then Ramadanov's theorem and Hurwitz's theorem together say that the weighted kernels $B_{\phi_n}(z,w)$ must have zeros in $\mathbb{D} \times \mathbb{D}$ after a certain value of n. See [1] for the details of this method.

3. L^p mapping properties

Before we prove Theorem 1.6, we generalize question **II** to the following setup. For a given sequence of complex numbers $\{\beta_n\}$, define a Bergman type integral operator as

$$Tf(z) = \int_{\mathbb{D}} K(z,w) f(w) \lambda(w) dA(w), \quad \text{where} \quad K(z,w) = \sum_{n=0}^{\infty} \beta_n (z\bar{w})^n, \tag{3.1}$$

and investigate the L^p boundedness of these operators. The following necessary condition is easy to prove.

Proposition 3.2. *If the operator T, defined in* (3.1)*, is bounded from $L^p(1)$ to $L^p(1)$ for some $p \in (1,\infty)$, then $\limsup_{n\to\infty} \frac{|\beta_n|}{n}$ is finite.*

Unfortunately, it turns out that this necessary condition is not sufficient. Namely, there exists a sequence $\{\gamma_n\}$ such that $\limsup_{n\to\infty} \frac{|\gamma_n|}{n}$ is finite but the associated operator by (3.1) is not bounded from $L^{p_0}(1)$ to $L^{p_0}(1)$ for some $p_0 \in (1,\infty)$. See the third section of [4].

Schur's lemma is one of the most commonly used arguments to prove boundedness of integral operators, and in this section we need this lemma. The proof can be found in [11].

Lemma 3.3 (Schur)**.** *Let (X, A, μ) be a sigma finite measure space and $K(x,y)$ be a positive measurable function on $X \times X$. For $p \in (1,\infty)$ suppose there exists a positive measurable function $h(x)$ on X and a finite number $C > 0$ such that*

$$\int_X K(x,y) h(x)^p d\mu(x) \leq C h(y)^p \ \textit{for a.e. } y \in X$$

and

$$\int_X K(x,y)h(y)^q d\mu(y) \leq Ch(x)^q \text{ for a.e. } x \in X,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Then the operator $Of(y) = \int_X K(x,y)f(x)d\mu(x)$ *is bounded on* $L^p(X,\mu)$.

We prove the next lemma by invoking Schur's lemma.

Lemma 3.4. *Let* $\{\beta_n\}$ *be a bounded sequence of complex numbers. Then the operator defined by*

$$Sf(z) = \int_{\mathbb{D}} \left| \sum_{n=0}^{\infty} \beta_n (z\bar{w})^n \right|^2 f(w)dA(w) \tag{3.5}$$

is bounded from $L^p(1)$ *to* $L^p(1)$ *for all* $p \in (1,\infty)$.

Proof. It is enough to show that the inequalities in (3.3) are satisfied with the correct choice of auxiliary function $h(x)$. Particularly, we take $h(w) = (1-|w|^2)^\epsilon$ and prove that for any $-1 < \epsilon < 0$, there exists $C_\epsilon > 0$ such that for any $z \in \mathbb{D}$,

$$I(\epsilon, z) := \int_{\mathbb{D}} \left| \sum_{n=0}^{\infty} \beta_n (z\bar{w})^n \right|^2 (1-|w|^2)^\epsilon dA(w) \leq C_\epsilon (1-|z|^2)^\epsilon.$$

We first use the orthogonality of monomials to get

$$\begin{aligned} I(\epsilon, z) &= \int_{\mathbb{D}} \left(\sum_{n=0}^{\infty} \beta_n (z\bar{w})^n \right) \left(\sum_{n=0}^{\infty} \overline{\beta_n} (\bar{z}w)^n \right) (1-|w|^2)^\epsilon dA(w) \\ &= \sum_{n=0}^{\infty} |\beta_n|^2 |z|^{2n} \int_{\mathbb{D}} |\bar{w}|^{2n} (1-|w|^2)^\epsilon dA(w). \end{aligned}$$

Using boundedness of β_n's and adding up the geometric series we obtain

$$\begin{aligned} I(\epsilon, z) &\lesssim \int_{\mathbb{D}} \sum_{n=0}^{\infty} |z|^{2n} |\bar{w}|^{2n} (1-|w|^2)^\epsilon dA(w) \\ &= \int_{\mathbb{D}} \frac{(1-|w|^2)^\epsilon dA(w)}{(1-|zw|^2)} \\ &= \pi \int_0^1 \frac{(1-r)^\epsilon}{(1-|z|^2 r)} dr. \end{aligned}$$

We break up the last integral into two pieces and estimate separately (recall that $\epsilon \in (-1,0)$)

$$\begin{aligned}
I(\epsilon, z) &= \pi \int_0^{|z|^2} \frac{(1-r)^\epsilon}{(1-|z|^2 r)} dr + \pi \int_{|z|^2}^1 \frac{(1-r)^\epsilon}{(1-|z|^2 r)} dr \\
&\leq \pi \int_0^{|z|^2} \frac{(1-r)^\epsilon}{(1-r)} dr + \pi \frac{1}{1-|z|^2} \int_{|z|^2}^1 (1-r)^\epsilon dr \\
&= \pi \left(\frac{1}{\epsilon} - \frac{(1-|z|^2)^\epsilon}{\epsilon} \right) + \pi \frac{(1-|z|^2)^{\epsilon+1}}{(1-|z|^2)(\epsilon+1)} \\
&\leq \frac{-\pi}{\epsilon}(1-|z|^2)^\epsilon + \frac{\pi}{\epsilon+1}(1-|z|^2)^\epsilon \\
&= \left(\frac{-\pi}{\epsilon} + \frac{\pi}{\epsilon+1} \right)(1-|z|^2)^\epsilon.
\end{aligned}$$

Therefore, we conclude that there exists $C_\epsilon > 0$ such that for any $z \in \mathbb{D}$,

$$I(\epsilon, z) \leq C_\epsilon (1-|z|^2)^\epsilon.$$

Since the kernel is symmetric, we similarly get the second inequality in (3.3). Now for given $p \in (1,\infty)$ choose $\epsilon = \frac{-1}{pq}$ to finish the proof. □

By using this lemma, we obtain the following sufficient condition.

Proposition 3.6. *If $\{\beta_n\}$ is a sequence such that the difference sequence $\{\beta_{n+1} - \beta_n\}$ is bounded, then the associated operator T, by* (3.1)*, is bounded from $L^p(1)$ to $L^p(1)$ for all $p \in (1,\infty)$.*

Remark. If $\lambda \equiv 1$, then $\beta_n = \frac{1}{\pi}(n+1)$, so the difference sequence is not only bounded, but it is constant $\beta_{n+1} - \beta_n = \frac{1}{\pi}$.

Proof. We start with the decomposition of the kernel function $K(z,w)$:

$$\begin{aligned}
K(z,w) &= \sum_{n=0}^{\infty} \beta_n (z\bar{w})^n = \frac{1}{1-z\bar{w}} \sum_{n=0}^{\infty} b_n (z\bar{w})^n \\
&= \sum_{n=0}^{\infty} (z\bar{w})^n \sum_{n=0}^{\infty} b_n (z\bar{w})^n \quad \text{for a sequence } \{b_n\}.
\end{aligned}$$

It is easy to see that $\beta_n = \sum_{k=0}^{n} b_k$, which implies $b_n = \beta_n - \beta_{n-1}$ (set $\beta_{-1} = 0$). Hence, by the assumption in the statement of the proposition, $\{b_n\}$ is a bounded sequence. Furthermore,

$$\begin{aligned}
|Tf(z)| &= \left| \int_{\mathbb{D}} K(z,w) f(w) dA(w) \right| \\
&\leq \int_{\mathbb{D}} \left| \sum_{n=0}^{\infty} (z\bar{w})^n \sum_{n=0}^{\infty} b_n (z\bar{w})^n \right| |f(w)| dA(w) \\
&\leq \left(\int_{\mathbb{D}} \left| \sum_{n=0}^{\infty} (z\bar{w})^n \right|^2 |f(w)| dA(w) \right)^{\frac{1}{2}} \left(\int_{\mathbb{D}} \left| \sum_{n=0}^{\infty} b_n (z\bar{w})^n \right|^2 |f(w)| dA(w) \right)^{\frac{1}{2}} \\
&:= |S_1 f(z)|^{\frac{1}{2}} |S_2 f(z)|^{\frac{1}{2}},
\end{aligned}$$

where S_1 and S_2 are respective integral operators. We integrate both sides with respect to z and apply the Cauchy-Schwarz inequality

$$\int_{\mathbb{D}} |Tf(z)|^p dA(z) \leq \int_{\mathbb{D}} |S_1 f(z)|^{\frac{p}{2}} |S_2 f(z)|^{\frac{p}{2}} dA(z),$$
$$||Tf||_p^{2p} \leq ||S_1 f||_p^p ||S_2 f||_p^p.$$

Note that S_1 and S_2 are operators of the type in (3.4), and we know such operators are bounded from $L^p(1)$ to $L^p(1)$. Therefore, we get

$$||Tf||_p^{2p} \leq ||S_1 f||_p^p ||S_2 f||_p^p \leq C_1 ||f||_p^p C_2 ||f||_p^p \leq C||f||_p^{2p}.$$

This finishes the proof of Proposition 3.6. □

We now give the proof of Theorem 1.6.

Proof. We know that $\mathbf{B}_\lambda$ is an integral operator of the form (3.1). Since $\lambda \sim 1$, it is enough to show $\mathbf{B}_\lambda$ is bounded from $L^p(1)$ to $L^p(1)$. We use Proposition 3.6 to do this. Namely, we show that the coefficients α_n of $B_\lambda(z,w)$ satisfy the condition in (3.6):

$$\begin{aligned}
\alpha_{n+1} - \alpha_n &= \frac{1}{\int_{\mathbb{D}} |z|^{2n+2}\lambda(z) dA(z)} - \frac{1}{\int_{\mathbb{D}} |z|^{2n}\lambda(z) dA(z)} \\
&= \frac{\int_{\mathbb{D}} (1-|z|^2)|z|^{2n}\lambda(z) dA(z)}{\int_{\mathbb{D}} |z|^{2n+2}\lambda(z) dA(z) \int_D |z|^{2n}\lambda(z) dA(z)} \\
&\sim \frac{\int_{\mathbb{D}} (1-|z|^2)|z|^{2n} dA(z)}{\int_{\mathbb{D}} |z|^{2n+2} dA(z) \int_{\mathbb{D}} |z|^{2n} dA(z)} \quad \text{since } \lambda \sim 1 \\
&\sim \frac{\int_0^1 (1-r^2) r^{2n+1} dr}{\int_0^1 r^{2n+3} dr \int_0^1 r^{2n+1} dr} \quad \text{after switching to polar coordinates} \\
&= (2n+4)(2n+2) \int_0^1 (1-r^2) r^{2n+1} dr \\
&= 2.
\end{aligned}$$

We obtain that the sequence $\{\alpha_{n+1} - \alpha_n\}$ is bounded, and this finishes the proof of Theorem 1.6. □

Remark. An alternative way of proving Theorem 1.6 is to show that $B_\lambda(z,w)$ is a standard kernel in the sense of Coifman-Weiss [5] on the space of homogeneous type $(\mathbb{D}, |.|, \lambda)$, where $|.|$ is the Euclidean distance.

Remark. A generalization and an alternative proof of Theorem 1.6 appear in [10].

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