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SPECIAL SYSTEMS THROUGH DOUBLE POINTS ON AN ALGEBRAIC SURFACE

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Abstract. Let S be a smooth projective algebraic surface satisfying the following property: $H^i(S,B)=0$ for $i>0$, for any irreducible and reduced curve B of S. The aim of this paper is to provide a characterization of special linear systems on S which are singular along a set of double points in very general position. As an application, the dimension of such systems is evaluated in case S is a simple Abelian surface, a $K3$ surface which does not contain elliptic curves or an anticanonical rational surface.

Introduction

In what follows S will be a smooth projective algebraic surface defined over the complex numbers.

Let H be an integral divisor of S. The problem of determining the dimension of the non-complete linear subsystem of $|H|$ made by curves through $r+1$ double points, i.e. singular at those points, in very general position on S is strictly connected with the problem of evaluating the dimension of the r-secant variety of S by Terracini's Lemma [14, Lemma 3.4.28]. The subject and its generalizations have been studied by many authors (see for example [4, 5, 7, 16]), and the main results are about classifying the *defective surfaces*, i.e. surfaces whose r-secant variety does not have the expected dimension. In this case H is assumed to be very ample, and even under this hypothesis it is not easy to determine the numerical characters of the special pairs (S,H). Trying to fill this gap, this paper is mainly devoted to the study of linear systems through double points on those surfaces S which have the following property:

$$H^i(S,B)=0 \quad \text{for} \quad i>0 \tag{0.1}$$

for any integral curve B of S. As an application a complete characterization of special linear systems of this type on simple abelian surfaces, $K3$ surfaces which do not contain elliptic curves and anticanonical rational surfaces is given.

The paper is organized as follows: in Section 1 we introduce some preliminary material about linear systems and in Proposition 1.4 give a partial classification of surfaces satisfying (0.1). Section 2 deals with the main part of the paper, where the characterization of these special systems is stated and proved. As an application, in

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Section 3, special linear systems on simple abelian surfaces and $K3$ surfaces which do not contain elliptic curves are completely classified. As a consequence none of these surfaces is defective. Finally, Section 4 focuses on the proof of the Gimigliano-Harbourne-Hirschowitz-Segre conjecture [9, 11, 13, 15] for linear systems of $\mathbb{P}^2$ with nine points of any multiplicity and r double points. The complete list of defective blow-ups of $\mathbb{P}^2$ at most nine very general points is given.

1. Notation and preliminaries

In what follows S will be a smooth algebraic surface defined over $\mathbb{C}$ with canonical bundle K_S. A divisor L and its associated line bundle will be denoted by the same letter. We adopt the notation $h^i(L) := \dim H^i(S, L)$ for the dimension of the cohomology groups. A compact notation like the one used in the formula $\chi = p_g - q + 1$ will be adopted in what follows for denoting the main invariants of a surface. The arithmetic genus of a curve B of S will be denoted by $p_a(B) := \frac{1}{2}(B^2 + B \cdot K_S) + 1$. We recall that by Riemann-Roch, the Euler characteristic $\chi(B) := h^0(B) - h^1(B) + h^2(B)$ of the line bundle $\mathcal{O}_S(B)$ is equal to

$$\chi(B) = \frac{1}{2}(B^2 - B \cdot K_S) + \chi(\mathcal{O}_S). \tag{1.1}$$

See [1, 3] for the main properties of these invariants. The base locus of a linear system $|L|$ is denoted by $\mathrm{Bs}\,|L|$. A divisor L is *special* if

$$h^0(L) \cdot h^1(L) > 0.$$

Let $p_1, \ldots, p_r$ be points in very general position on S and let $|H - \sum_i 2p_i|$ be the linear systems of divisors of $|H|$ which are singular at all the p_i's. We say that the linear system $|H - \sum_i 2p_i|$ is *special* if

$$\dim |H - \sum_i 2p_i| > \max\{-1, \dim |H| - 3r\}.$$

Proposition 1.2. *Let $\phi : S_r \to S$ be the blow-up at all the p_i's with exceptional divisors E_i. If $h^l(H) = 0$ for $l > 0$, then $L_r := \phi^* H - \sum_i 2E_i$ is special if and only if $|H - \sum_i 2p_i|$ is special.*

Proof. Since ϕ has connected fibers, then $\phi_* \mathcal{O}_{S_r} = \mathcal{O}_S$ by the Zariski connectedness theorem. This, the projection formula [12, II, Exercise 5.1 (d)] and $R^l \phi_* \phi^* H = 0$ for $l > 0$ imply the equalities $h^l(\phi^* H) = h^l(H)$ for any l. Since $\phi^* H \cdot E_i = 0$ for all i and $\chi(2E_i) = -3$, the Riemann-Roch theorem and what was proved before give

$$\chi(L_r) = \chi(\phi^* H) - 3r = \chi(H) - 3r = h^0(H) - 3r,$$

where the last equality is by hypothesis. Let $E := \sum_i 2E_i$ and consider the exact sequence of sheaves:

$$0 \longrightarrow \mathcal{O}_{S_r}(D - E_i) \longrightarrow \mathcal{O}_{S_r}(D) \longrightarrow \mathcal{O}_{E_i}(D) \longrightarrow 0.$$

If $D \cdot E_i \geq 0$ and $h^2(D) = 0$, then taking cohomology of the exact sequence and using $h^1(D_{|E_i}) = 0$, we deduce that $h^2(D - E_i) = 0$. Taking D to be $\phi^* H$, $\phi^* H - E_1, \phi^* H - 2E_1, \ldots, \phi^* H - \sum_i 2E_i = L_r$ we deduce that $h^2(L_r) = 0$. Thus, by what was proved before, L_r is special if and only if $h^0(L_r) > \max\{0, h^0(H) - 3r\}$. We conclude by observing that an element of $|L_r|$ is the strict transform of an element of $|H - \sum_i 2p_i|$ so that the dimensions of the two linear systems are equal. $\square$

We recall that an abelian surface S is *simple* if it does not contain 1-dimensional subgroups. In particular S is simple if it does not contain elliptic curves.

Definition 1.3. In what follows a ***neat surface*** is a smooth algebraic projective surface which satisfies property (0.1).

Proposition 1.4. *If S is a neat surface, then it is one of the following:*

p_g	q	χ	*Type of surface*
0	0	1	
1	0	2	$K3$
1	2	0	*Simple abelian*

Proof. Assume that K_S is effective. Then either $K_S \sim \mathcal{O}_S$ or $K_S - C$ is effective for some integral curve C of S. In the second case by Serre duality $h^2(C) = h^0(K_S - C) > 0$, which is a contradiction. This implies that $K_S \sim \mathcal{O}_S$, so by [1, Theorem VIII.2] S is either a $K3$ or an abelian surface. If S is abelian and C is an elliptic curve on it, then $C^2 = 0$ by adjunction, so $h^0(C) - h^1(C) = \chi(C) = 0$ gives $h^1(C) > 0$, which is a contradiction. This implies that S is simple.

Now assume K_S is not effective, hence $p_g = 0$. If $q(S) > 0$, then by [1, Proposition V.15] the Albanese morphism $\alpha : S \to \mathrm{Alb}(S)$ has connected 1-dimensional fibers. By [3, Theorem 20.1] the general fiber F of α is smooth. The Riemann-Roch theorem and $F^2 = 0$ give

$$p_a(F) + q - 1 = \frac{1}{2}(F \cdot K_S) + q = 1 - \chi(F) \leq 0,$$

where the last inequality is due to $h^i(F) = 0$ for $i > 0$ and $h^0(F) > 0$. This leaves us with the case $p_a(F) = 0$ and $q = 1$. By [1, Chapter VI] the minimal model of S is a ruled surface $S_{\min}$ whose basis $B = \alpha(S)$ is a smooth elliptic curve. Let $\alpha = \phi \circ \alpha_{\min}$, where $\alpha_{\min} : S_{\min} \to B$ is obtained by blowing down all the (-1)-curves contained in the fibers of α. By [12, Proposition 2.9], $\alpha_{\min}$ has a section C with $C^2 \leq 0$. Let $\tilde{C} \subset S$ be the strict transform of C through ϕ. Thus $\tilde{C}^2 \leq C^2 \leq 0$. Taking the exact sequence of $\tilde{C}$,

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(\tilde{C}) \longrightarrow \mathcal{O}_{\tilde{C}}(\tilde{C}) \longrightarrow 0,$$

one obtains that $h^1(\tilde{C}) > 0$, which is a contradiction. So if $p_g(S) = 0$, then $q(S) = 0$. $\square$

Proposition 1.5. *Let S be a neat surface and let B be an integral curve such that $h^0(B) \geq 2$. Then either $h^1(2B) = 0$ or S is a $K3$ surface, $h^1(2B) = 1$ and $B^2 = 0$.*

Proof. If $p_g = 0$, consider the exact sequence

$$0 \longrightarrow \mathcal{O}_S((n-1)B) \longrightarrow \mathcal{O}_S(nB) \longrightarrow \mathcal{O}_B(nB) \longrightarrow 0.$$

When $n = 1$ the equalities $h^1(B) = h^2(\mathcal{O}_S) = 0$ imply that $h^1(B_{|B}) = 0$, so that $h^1(nB_{|B}) = 0$ for any positive n. Taking $n = 2$ we deduce that $h^1(2B) = 0$.

If $p_g > 0$, then $K_S \sim \mathcal{O}_S$ by Proposition 1.4. If $B^2 > 0$, then $2B$ is a nef and big divisor so that $h^1(2B) = 0$ by Kawamata-Viehweg vanishing. If $B^2 = 0$, then $\chi(\mathcal{O}_S) = \chi(B) = h^0(B) \geq 2$, where the first equality is due to the Riemann-Roch theorem and the second to the hypothesis. Thus S is a $K3$ surface and $\chi(\mathcal{O}_S) = 2$, by Proposition 1.4. Since $\chi(2B) = 2$ and $h^0(2B) = 3$, by Riemann-Roch we deduce $h^1(2B) = 1$. $\square$

2. Linear systems through double points on surfaces

Let S be a neat surface and let $\phi : S_r \to S$ be the blow-up map at r points in very general position. Let H be an integral curve of S and

$$L_r := \phi^* H - 2E_1 - \cdots - 2E_r, \tag{2.1}$$

where $E_i = \phi^{-1}(p_i)$ are the exceptional divisors.

Proposition 2.2. *Let S be a neat surface and let L_r be as in* (2.1). *If L_r is non-special and L_{r+1} is special, then*

$$L_r \sim F + nD,$$

where F is the fixed part of $|L_r|$ and $H^0(nD) = \operatorname{Sym}^n H^0(D)$, where $n > 1$.

Proof. By hypothesis $h^1(L_r) = 0$, so by Proposition 1.2 and the fact that L_{r+1} is special, we have $\dim |L_r - 2p| > \max\{-1, \dim |L_r| - 3\}$ for a point $p \in S_r$ in very general position. Let $L_r \sim F + M$, where F is the fixed part of $|L_r|$. Since p can be chosen to lie outside F, then

$$\dim |M - 2p| > \max\{-1, \dim |M| - 3\}. \tag{2.3}$$

Let $\varphi : S \dashrightarrow \mathbb{P}^N$ be the rational map defined by the linear system $|M|$, let $C := \varphi(S)$ and let $q := \varphi(p)$. Observe that a hyperplane H of $\mathbb{P}^N$ contains the tangent space $T_q C$ if and only if $\varphi^{-1}(H) \in |M|$ is singular at p. Thus the elements of $|M - 2p|$ are in one to one correspondence with hyperplanes H such that $H \supset T_q C$. From (2.3) we deduce that $T_q C$ imposes less than 3 conditions on the hyperplanes containing it, and this implies that $\dim T_q C < 2$. If $\dim T_q C = 0$, i.e. C is a point, then $\dim |M| = 1$ so that $\dim |M - 2p| = -1$, a contradiction. Thus $\dim T_q C = 1$ and C is a curve. Consider the following diagram of maps:

$$\begin{tikzcd} S_r \arrow[d, dashed, "\varphi"'] & \tilde{S}_r \arrow[l, "\pi"'] \arrow[d, "\tilde{\varphi}"] \arrow[dr, "\beta"] & \\ C & \tilde{C} \arrow[l, "\eta"] & B \arrow[l, "\rho"] \end{tikzcd}$$

where π is a blow-up map, η is a normalization map, $\tilde{\varphi}$ is the lifting of the resolution of indeterminacy of φ to $\tilde{C}$ and $\rho \circ \beta$ is the Stein factorization of $\tilde{\varphi}$, i.e. β has connected fibers and ρ is a finite map. Observe that on the bottom line of the diagram we have curves and on the top line we have surfaces.

Assume that B is non-rational and let E be a (-1)-curve of $\tilde{S}_r$. Since E is rational, $\beta(E)$ is a point. Thus β descends to a morphism $\beta_S : S \to B$ which pulls back all the non-trivial holomorphic 1-forms of B to corresponding 1-forms on S so that $q(S) > 0$. Then S is an abelian surface by Proposition 1.4. If $C_q := \beta_S^{-1}(q)$ for some $q \in B$, then $C_q^2 = 0$ so that $h^0(C_q) - h^1(C_q) = \chi(C_q) = 0$ by Riemann-Roch, Serre's duality and $K_S \sim \mathcal{O}_S$. Thus $h^1(C_q) > 0$, which is a contradiction. We proved that B is rational so that if $\tilde{D}$ is a fiber of β, then

$$H^0(a\tilde{D}) \cong \operatorname{Sym}^a H^0(\tilde{D}).$$

A general element Z of $|M|$ is the closure of $\varphi^{-1}(H \cap C)$, where H is a hyperplane of $\mathbb{P}^N$ which avoids the singularities of C. Thus $Z = \pi(n\tilde{D})$, where $n = \deg(\rho)\deg(C)$. This implies that $H^0(M) = \operatorname{Sym}^n H^0(D)$, where $D := \pi(\tilde{D})$. $\square$

Following the lines of the last proof, it is easy to observe that even if S does not satisfy property (0.1), the fixed part of system $|L_r - 2p|$ contains a double curve through p. That is why we have the following well-known result (see [8, Theorem 4.1] or [16]).

Corollary 2.4. *Let S be a smooth projective algebraic surface and let L_k be defined as in* (2.1). *If L_k is special, then the fixed part of $|L_k|$ contains a double curve.*

The following definition will be adopted in what follows.

Definition 2.5. A divisor L_r of the blow-up S_r of S at r points in very general position is ***pre-special*** if it is of the form (2.1), it is non-special and L_{r+1} is special on S_{r+1}.

We begin by investigating the fixed part of the linear system defined in Proposition 2.2.

Lemma 2.6. *Let S be a neat surface and let L_r and the E_i's be as in* (2.1). *If $L_r \sim F + M$, where F is the fixed part of $|L_r|$, then $E_i \cdot F \geq 0$ for any i.*

Proof. Suppose that E_r is a component of F, so that $h^0(L_r) = h^0(L_r - E_r)$. If $\pi : S_r \to S_{r-1}$ is the blow-up of E_r and $p := \pi(E_r)$, then the preceding equality is equivalent to $|L_{r-1} - 2p| = |L_{r-1} - 3p|$. Since the point p is in very general position on S_{r-1}, then by [5, Proposition 2.3] we get a contradiction. Since E_r is not a component of F, then $E_r \cdot F \geq 0$ and the same argument applies to E_i for any i. □

Lemma 2.7. *Let S be a neat surface and let L_r be a pre-special divisor of S_r. If $L_r \sim F + nD$, where F is the fixed part of $|L_r|$, then $h^1(D) = 0$. Moreover, $h^1(2D) = 0$ unless or $D = \pi^*B$, where B is an integral curve with $B^2 = 0$ on a $K3$ surface S, in which case $h^1(2D) = 1$.*

Proof. From $2 = E_i \cdot F + n\, E_i \cdot D$ and $E_i \cdot F \geq 0$ by Lemma 2.6, we deduce that $0 \leq D \cdot E_i \leq 1$ because $n > 1$, by Proposition 2.2. Thus

$$D \sim \phi^*B - \sum_{i \in I} E_i,$$

where I is the set of all the i's such that $D \cdot E_i = 1$ and B is an integral curve of S so that $h^1(B) = 0$. Observe that $h^0(D) = h^0(B) - |I|$ because each E_i imposes one independent condition since it corresponds to a simple point of S in very general position. This gives $h^1(D) = 0$.

We now want to determine the possible values of $h^1(2D)$. If $|I| > 0$, then $B^2 > 0$ so that $h^1(2B) = 0$ by Proposition 1.5. Since $|2D|$ is fixed component free, then by Corollary 2.4 we have $h^1(2D) = 0$. If $|I| = 0$, then $D = \phi^*B$ so that $h^1(2D) = h^1(2B)$. By Proposition 1.5 we conclude that $h^1(2B) = 0$ unless S is a $K3$ surface and $B^2 = 0$ in which case $h^1(2B) = 1$. □

The preceding lemma allows one to find the numerical characters of the curve D by means of the Riemann-Roch theorem.

Proposition 2.8. *Let S be a neat surface and let L_r be a pre-special divisor of S_r. If $L_r \sim F + nD$, where F is the fixed part of $|L_r|$, then the general element of $|D|$ is a smooth curve with either*

$$D^2 = \chi(\mathcal{O}_S) - 1, \qquad D \cdot K_{S_r} = 3\chi(\mathcal{O}_S) - 5,$$

or $D = \pi^* B$, where B is an integral curve with $B^2 = 0$ on a K3 surface S.

Proof. We know that $h^0(nD) = n + 1$ for $n = 1, 2$ by Proposition 2.2. Moreover, Lemma 2.7 gives $h^1(D) = 0$. Suppose now that $h^1(2D) = 0$. Then we get

$$\chi(D) = 2, \qquad \chi(2D) = 3.$$

By Riemann-Roch one obtains $D^2 = \chi(\mathcal{O}_S) - 1$ and $D \cdot K_{S_r} = 3\chi(\mathcal{O}_S) - 5$. If $h^1(2D) > 0$, then by Lemma 2.7 we get the remaining case.

To prove that the general element of $|D|$ is smooth, observe that $\chi(\mathcal{O}_S) \leq 2$ by Proposition 1.4. This implies $D^2 \leq 1$ by what was said before, so $|D|$ has at most one base point p. By Bertini's second theorem [2] the general element of $|D|$ is smooth away from p. It has to be smooth also at p, since otherwise two elements of $|D|$ would have a bigger intersection at that point. □

Corollary 2.9. *Let S be a neat surface with $p_g = q = 0$. If L_k, defined as in* (2.1)*, is special, then S is a rational surface.*

Proof. Let r be such that L_r is non-special but L_{r+1} is special and let $L_r \sim F + nD$ be the decomposition given in Proposition 2.2. We know that $|D|$ is a pencil of smooth curves on S_r with $D^2 = 0$ and $D \cdot K_{S_r} = -2$, so that D is rational and $|D|$ has empty base locus. The morphism $\phi_{|D|} : S_r \to \mathbb{P}^1$ is a $\mathbb{P}^1$-fibration. Blowing-down the (-1)-curves which are contained in the fibers of $\phi_{|D|}$ we obtain a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, which is a rational ruled surface (see [1]). This implies that S is also rational. □

Now we wish to investigate the numerical properties of integral curves of the base locus of $|L_r|$ when L_r is pre-special.

Lemma 2.10. *Let S be a neat surface and let L_r be a pre-special divisor of S_r. Let $L_r \sim F + nD$, where F is the fixed part of $|L_r|$, and let C be an integral component of F. Then $\chi(C + sD) \leq s + 1$ for any $s \leq n$.*

Proof. By hypothesis F is the fixed part of $|L_r|$, so we get

$$h^0(C + nD) \leq h^0(F + nD) = h^0(nD),$$

which implies that C is a fixed component of $|C + nD|$. Observe that since $|D|$ does not have fixed components, then also $|kD|$ is a fixed component free for any $k > 0$. This and the equality $C + nD = (C + sD) + (n - s)D$ imply that C is a fixed component of $|C + sD|$. By Serre's duality we have that $h^2(C + sD) = 0$, so from $h^0(C + sD) = h^0(sD) = s + 1$ we get the thesis. □

Proposition 2.11. *Let S be a neat surface with L_r as in* (2.1)*. If C is an integral fixed component of $|L_r|$, then $\chi(C) = 1$. Moreover, if L_r is pre-special and D is defined as in* (2.2)*, then either*

$$C \cdot D \leq \frac{1}{2}(\chi(\mathcal{O}_S) - 1)$$

or S is a K3 surface, $D = \phi^* B$ with $B^2 = 0$ and $C \cdot D \leq 1$.

Proof. Since $Z := \phi(C)$ is integral, $h^i(Z) = 0$ for $i > 0$. Observe that Z is a fixed component of $|H - \sum_i 2p_i|$, for some integral H. By [5, Proposition 2.3] the general element of the last system has multiplicity 2 at each p_i, so that Z has multiplicity at most 2 at each p_i. Thus $h^1(C) = 0$ by Corollary 2.4. By Serre's duality and Proposition 1.4 we have $h^2(C) = 0$, so that $\chi(C) = 1$.

Assume now that L_r is pre-special. By Lemma 2.10 we have $\chi(C+2D) \le 3$. Consider the equality

$$\chi(C+2D) = \chi(C) + \chi(2D) + 2C \cdot D - \chi(\mathcal{O}_S).$$

By Lemma 2.7 and Serre's duality, either $\chi(2D) = h^0(2D) = 3$ or S is a $K3$ surface, $D = \phi^* B$ with $B^2 = 0$ so that $\chi(2D) = 2$. In both the cases we get the thesis. □

3. Applications to some non-rational surfaces

The aim of this section is to apply the results of Section 2 to two classes of smooth projective complex surfaces S. Recall that we denote by $\phi : S_k \to S$ the blow-up map at k very general points of S with exceptional divisors $E_1, \dots, E_s$.

A $K3$ *surface* S is a smooth simply connected compact complex surface with $K_S \sim \mathcal{O}_S$. In what follows we will restrict our attention to the class of projective $K3$ surfaces.

Lemma 3.1. *A projective* $K3$ *surface* S *is neat.*

Proof. Let B be an integral curve on S. If $B^2 > 0$, then B is nef and big so that $h^1(K_S + B) = 0$, and thus $h^1(B) = 0$ since $K_S \sim \mathcal{O}_S$. If $B^2 \le 0$, then by adjunction $B^2 = -2, 0$. Taking cohomology of $0 \to \mathcal{O}_S \to \mathcal{O}_S(B) \to \mathcal{O}_B(B) \to 0$ and using $K_B \sim \mathcal{O}_B(B)$ gives the result. □

Theorem 3.2. *Let* $\phi : S_k \to S$ *be the blow-up of a projective* $K3$ *surface which does not contain elliptic curves at* k *very general points and let* $L_k := \phi^* H - \sum_i 2E_i$ *with* H *integral. Then* L_k *is special if and only if* $k = 2$ *and* $H \sim 2B$ *with* $B^2 = 2$.

Proof. If H is an integral divisor on S with $H^2 > 0$, then H is nef and big because $h^0(H) \ge \chi(H) > 2$ by Serre's duality and the Riemann-Roch theorem. If $H \sim 2B$, then also B is nef and big, so by Kawamata-Viehweg vanishing and $B^2 = 2$ we get $h^0(H) = 6$ and $h^0(B) = 3$. We expect $|L_2|$ to be empty, but $\frac{1}{2} L_2 = \phi^* B - E_1 - E_2$ is effective, so L_2 is special.

Let $r := k - 1$ and suppose now that L_r is pre-special. By Proposition 2.2 we have $L_r \sim F + nD$, where F is the fixed part of $|L_r|$ and $|D|$ is a linear pencil with $D \cdot K_{S_r} = 1$, by Proposition 2.8. The last equality together with $K_{S_r} \sim \sum_i E_i$ imply that $D \sim \phi^* B - E_1$ for some integral curve B of S. If C is an integral component F, then $C \cdot D = 0$ by Proposition 2.11; thus $F \cdot D = 0$. Since $n \ge 2$ and $2 = L_r \cdot E_1 = F \cdot E_1 + nD \cdot E_1$, by Lemma 2.6 we conclude $F \cdot E_1 = 0$. Thus $F \cdot \phi^* B = 0$ so that $\phi(F) \cdot B = 0$, which implies that $\phi(L_r) = H$ is not connected, which is absurd. Hence $F = 0$ and $L_r \sim 2D$, so that $r = 1$. By Proposition 2.8 we have $D^2 = 1$ so that $B^2 = 2$. Since $h^0(L_2) = 1$, by imposing one more general point we get $h^0(L_3) = 0$, so L_3 is non-special. Thus there are no more special divisors. □

Remark 3.3. The hypothesis of Theorem 3.2 is automatically satisfied if $\mathrm{Pic}(S) \cong \mathbb{Z}$. It is still possible to classify special linear systems of type L_k on $K3$ surfaces S with Picard groups of higher rank, but a careful study of the non-reduced fibers of the elliptic fibrations of S has to be performed. Due to the length of this analysis we do not include more results in this direction here.

We recall that an abelian surface is a complex torus admitting a holomorphic line bundle Θ such that $\phi_{|\Theta|}$ is an embedding into a projective space. An abelian surface is *simple* if it does not contain 1-dimensional subgroups.

Lemma 3.4. *A simple abelian surface S is neat.*

Proof. First of all observe that S does not contain integral curves B with $B^2 \leq 0$. We prove the statement by contradiction. If B is such a curve and $p, q \in B$, let $\tau \in \mathrm{Aut}(S)$ be the translation with $\tau(p) = q$. Since $\tau(B) \cdot B = B^2 \leq 0$ and $q \in \tau(B) \cap B$, we deduce that $\tau(B) = B$. This implies that B is isomorphic to a 1-dimensional subgroup of S, which is a contradiction. If B is an integral curve with $B^2 > 0$, then $h^i(B) = h^i(K_S + B) = 0$ for $i > 0$ by Kawamata-Viehweg vanishing. This implies that S is neat. □

Theorem 3.5. *Let S_r be the blow-up of a simple abelian surface S at points in very general position. If L_r is as in* (2.1)*, then it is non-special.*

Proof. If L_r is pre-special, let $L_r \sim F + nD$, with D as in Proposition 2.2. Then $D^2 < 0$ by Proposition 2.8, which is a contradiction. □

4. Applications to some anticanonical rational surfaces

In this section S_n will be the blow-up of $\mathbb{P}^2$ at n points in *very general position*. If $n \leq 9$, then it is known (see [6, Theorem 5.1]) that an effective divisor D on S_n is special if and only if $D \cdot E \leq -2$ for some (-1)-curve E of S_n. If this is the case, then in particular E is a fixed component of $|D|$ so that the general element of $|D|$ is reducible or non-reduced. Thus $h^i(D) = 0$ if D is integral and $i > 0$ so that S_n is neat for $n \leq 9$.

We intend to prove two theorems here:

1. The Harbourne-Hirschowitz conjecture (see [6, Conjecture 4.8]) for linear systems of $\mathbb{P}^2$ through nine points of any multiplicity and through an arbitrary number of additional double points.
2. The classification of the defective secant varieties of S_n for $0 \leq n \leq 9$.

A divisor L of S_r is (-1)***-special*** if $h^0(L) > 0$ and there exists a (-1)-curve E such that $E \cdot L \leq -2$. If L is (-1)-special and $a := -E \cdot L$, then the exact sequence

$$H^1(L - E) \longrightarrow H^1(L) \longrightarrow H^1(\mathcal{O}_{\mathbb{P}^1}(-a)) \longrightarrow 0$$

implies that $h^1(L) > 0$ so that L is special.

Let $\phi : S_{r+9} \to S_9$ be the blow-up map with exceptional divisors $E_1, \ldots, E_r$ and let H be a divisor of S_9. In this section we will adopt the following notation:

$$L_r := \phi^* H - 2E_1 - \cdots - 2E_r. \tag{4.1}$$

Lemma 4.2. *Let L_r be a divisor on S_{r+9} defined as in* (4.1)*. If C_1, C_2 are integral fixed components of $|L_r|$, then $C_1 \cdot C_2 \leq 0$.*

Proof. By Proposition 2.11 we have $\chi(C_i) = 1$. Since $C_1 + C_2$ is contained in the base locus of $|L_r|$, then $h^0(C_1 + C_2) = 1$. This gives $\chi(C_1 + C_2) \leq 1$; thus we get $\chi(C_1 + C_2) = \chi(C_1) + \chi(C_2) + C_1 \cdot C_2 - 1 = 1 + C_1 \cdot C_2$. □

Theorem 4.3. *Let L_r be a divisor on S_{r+9} defined as in* (4.1)*. Then L_r is special if and only if it is* (-1)*-special.*

Proof. One implication has already been proved. Suppose now that L_r is special. If $h^1(H) > 0$, then by [11] there exists a (-1)-curve E of S_9 such that $E \cdot H \leq -2$. Since the points are in very general position, they do not lie on E so that $\phi^* E$ is a (-1)-curve of S_{r+9} and $\phi^* E \cdot L_r = E \cdot H \leq -2$.

If $h^1(H)=0$, then the system $|H-\sum_i 2p_i|$ is special because L_r is special and by Proposition 1.2. Let

$$H \sim B + H',$$

where B is the fixed part of $|H|$. We have that $|H'-\sum_i 2p_i|$ is special because the points are in very general position, so that they can be chosen outside B. Moreover, $h^1(H')=0$ since $|H'|$ does not have fixed components (see the introduction of this section). We deduce that $L'_r=\phi^*H'-\sum_i 2E_i$ is special by Proposition 1.2.

If the general element of $|H'|$ is irreducible, then let $\phi_s : S_{r+9} \to S_{s+9}$ be the blow-up map. Let $0 \leq s < r$ be the biggest integer such that the divisor L'_s of S_{s+9} is non-special. By Proposition 2.2 we have

$$L'_s \sim F + nD,$$

where F is the fixed part of $|L'_s|$ and $|D|$ is a pencil of smooth rational curves with $D^2=0$ by Proposition 2.8. If C is an integral component of F, then $C\cdot D=0$ by Proposition 2.11, so that $D\cdot L'_s=0$. Observe that

$$(\phi_s^*D - E_{s+1})\cdot L'_r = D\cdot L'_s - E_{s+1}\cdot L'_r = -2,$$

where $\phi_s^*D-E_{s+1}$ is a (-1)-curve. This implies that L'_r is (-1)-special.

If the general element of $|H'|$ is reducible, then by [11, Lemma II.6] we deduce that $H'\sim aD$, where $|D|$ is a linear pencil with $D^2=0$ and $-K_{S_9}\cdot D=0,2$. The case $-K_{S_9}\cdot D=0$ can be excluded because of [6, Theorem 5.1], since in this case $h^0(aD)=1$ so that $|L'_r|$ would be empty and thus non-special. If $-K_{S_9}\cdot D=2$, then $p_a(D)=0$ so that the general element of $|D|$ is rational and, by Bertini's second theorem, is smooth. In this case ϕ^*D-E_1 is a (-1)-curve, and from

$$(\phi^*D-E_1)\cdot L'_r = (\phi^*D-E_1)\cdot(\phi^*H' - \sum_{i=1}^{r} 2E_i) = -2$$

we deduce that L'_r is (-1)-special if $r\geq 1$.

We proved that there exists a (-1)-curve E of S_{r+9} such that $E\cdot L'_r \leq -2$. Thus E is a fixed component of $|L'_r|$, and consequently it is a fixed component of $|L_r|$. Thus, by Lemma 4.2 and the fact that ϕ^*B is a fixed curve of $|L_r|$, we get

$$E\cdot L_r = E\cdot(\phi^*B + L'_r) \leq E\cdot L'_r \leq -2,$$

so that L_r is (-1)-special. □

As an application of Theorem 4.3, we find the dimension of the secant variety of any projective embedding of S_r with $r\leq 9$.

Lemma 4.4. *If H is an ample and integral divisor of S_n, with $2\leq n\leq 9$, then $p_a(H)>0$.*

Proof. We prove the statement by contradiction. Assume that H is ample and $p_a(H)=0$. If $n\geq 3$, since H is ample, then, by [10, Theorem 1.1], we have that H is linearly equivalent to a non-negative sum of the classes E_0, E_0-E_1, $2E_0-E_1-E_2$, $-K_i := 3E_0-E_1-\cdots-E_i$, where E_0 is the pull-back of a line and the E_i, with $0<i\leq n$, are the exceptional divisors. Since H is ample, then $H\cdot E_n>0$, so that $H+K_{S_n}$ is effective. Thus we get $H^2 = H\cdot(H+K_{S_n}-K_{S_n}) \geq -H\cdot K_{S_n}$. Since $p_a(H)=0$, we have $H^2=-H\cdot K_{S_n}-2$; hence $H^2<-H\cdot K_{S_n}$, which is a contradiction.

If $n=2$, let $H=dE_0-m_1E_1-m_2E_2$. Then $-2=2p_a(H)-2=d^2-3d-m_1^2+m_1-m_2^2+m_2$. On the other hand we have $d>m_1+m_2$ because $H\cdot(E_0-E_1-E_2)>0$.

By substituting $d = m_1 + m_2 + 1$ in the right hand side of the equation we obtain the non-negative number $2m_1m_2 - 2$, which is a contradiction. $\square$

Theorem 4.5. *Let H be a very ample divisor of S_n, with $0 \le n \le 9$. The r-secant variety of $\phi_{|H|}(S_n)$ is defective if and only if (H, n, r) is one of the following:*

$$(\mathcal{O}_{\mathbb{P}^2}(2), 0, 1), \quad (\mathcal{O}_{\mathbb{P}^2}(4), 0, 4), \quad (\phi^*\mathcal{O}_{\mathbb{P}^2}(2a) - (2a-2)E_1, 1, 2a-1).$$

Proof. By Terracini's lemma, the r-secant variety of $\phi_{|H|}(S_n)$ is defective if and only if $L_{r+1} := \phi^*H - 2E_1 - \cdots - 2E_{r+1}$ is special; see [6, Lemma 7.4].

If L_r is non-special and L_{r+1} is special, then we are in the hypothesis of Proposition 2.2, so we get $L_r \sim F + mD$, where F is the fixed part of $|L_r|$, $m > 1$ and $|D|$ is a linear pencil. By Proposition 2.8 the general element of $|D|$ is a smooth rational curve with $D^2 = 0$. Moreover, $D \cdot F = 0$ by Proposition 2.11, so that $D \cdot L_r = 0$.

If $D \cdot E_i = 0$ for any i, then $D = \phi^*D'$, where $D' = \phi(D)$, so that $0 = D \cdot L_r = D' \cdot H$, which is a contradiction since H is ample. Thus we deduce that $D \cdot E_i > 0$ for some i, and this gives $2 = L_r \cdot E_i = (F + mD) \cdot E_i \ge m(D \cdot E_i)$, where the last inequality is due to Lemma 2.6. Since $m > 1$ we deduce that $D \cdot E_i = 1$, $m = 2$ and $F \cdot E_i = 0$.

Suppose now that $D \cdot E_k = 0$ and let $D_k := D + E_i - E_k$. The general element of $|D_k|$ is irreducible and $D_k^2 = 0$, because the same is true for $|D|$ and we are just exchanging the role of the points p_i and p_k, which are in very general position. In particular D_k is a nef divisor. Observe that $D_k \cdot L_r = D \cdot L_r = 0$, so $D_k \cdot (F + mD) = 0$. Since D_k is nef, we get $D_k \cdot D = 0$, which is a contradiction.

We proved that $D \cdot E_i = 1$ for all i so that $D = \phi^*D' - \sum_i E_i$ and $F = \phi^*F'$, because $F \cdot E_i = 0$ for any i. This implies that $D' \cdot F' = D \cdot F = 0$. Since $H \sim 2D' + F'$ is very ample, it is connected, so that $F' \sim \mathcal{O}_{S_n}$ and consequently $L_r \sim 2D$. Since D' is ample and $p_a(D') = 0$, because $p_a(D) = 0$, we get $n = 0, 1$ by Lemma 4.4. In the first case D' is linearly equivalent to either $\mathcal{O}_{\mathbb{P}^2}(1)$ or $\mathcal{O}_{\mathbb{P}^2}(2)$, while in the second it is linearly equivalent to $\phi^*\mathcal{O}_{\mathbb{P}^2}(a) - (a-1)E_1$ for some $a \ge 2$.

Since $D^2 = 0$, then $r = D'^2$. This allows us to determine L_r. In any such case we get that $h^0(L_{r+1}) = 1$ so that L_{r+2} is non-special because $h^0(L_{r+2}) = 0$. $\square$

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