

SPECIAL SYSTEMS THROUGH DOUBLE POINTS ON AN ALGEBRAIC SURFACE

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ABSTRACT. Let S be a smooth projective algebraic surface satisfying the following property: $H^i(S, B) = 0$ for $i > 0$, for any irreducible and reduced curve B of S . The aim of this paper is to provide a characterization of special linear systems on S which are singular along a set of double points in very general position. As an application, the dimension of such systems is evaluated in case S is a simple Abelian surface, a $K3$ surface which does not contain elliptic curves or an anticanonical rational surface.

INTRODUCTION

In what follows S will be a smooth projective algebraic surface defined over the complex numbers.

Let H be an integral divisor of S . The problem of determining the dimension of the non-complete linear subsystem of $|H|$ made by curves through $r + 1$ double points, i.e. singular at those points, in very general position on S is strictly connected with the problem of evaluating the dimension of the r -secant variety of S by Terracini's Lemma [14, Lemma 3.4.28]. The subject and its generalizations have been studied by many authors (see for example [4, 5, 7, 16]), and the main results are about classifying the *defective surfaces*, i.e. surfaces whose r -secant variety does not have the expected dimension. In this case H is assumed to be very ample, and even under this hypothesis it is not easy to determine the numerical characters of the special pairs (S, H) . Trying to fill this gap, this paper is mainly devoted to the study of linear systems through double points on those surfaces S which have the following property:

$$(0.1) \quad H^i(S, B) = 0 \quad \text{for} \quad i > 0$$

for any integral curve B of S . As an application a complete characterization of special linear systems of this type on simple abelian surfaces, $K3$ surfaces which do not contain elliptic curves and anticanonical rational surfaces is given.

The paper is organized as follows: in Section 1 we introduce some preliminary material about linear systems and in Proposition 1.4 give a partial classification of surfaces satisfying (0.1). Section 2 deals with the main part of the paper, where the characterization of these special systems is stated and proved. As an application, in

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Section 3, special linear systems on simple abelian surfaces and $K3$ surfaces which do not contain elliptic curves are completely classified. As a consequence none of these surfaces is defective. Finally, Section 4 focuses on the proof of the Gimigliano-Harbourne-Hirschowitz-Segre conjecture [9, 11, 13, 15] for linear systems of \mathbb{P}^2 with nine points of any multiplicity and r double points. The complete list of defective blow-ups of \mathbb{P}^2 at most nine very general points is given.

1. NOTATION AND PRELIMINARIES

In what follows S will be a smooth algebraic surface defined over \mathbb{C} with canonical bundle K_S . A divisor L and its associated line bundle will be denoted by the same letter. We adopt the notation $h^i(L) := \dim H^i(S, L)$ for the dimension of the cohomology groups. A compact notation like the one used in the formula $\chi = p_g - q + 1$ will be adopted in what follows for denoting the main invariants of a surface. The arithmetic genus of a curve B of S will be denoted by $p_a(B) := \frac{1}{2}(B^2 + B \cdot K_S) + 1$. We recall that by Riemann-Roch, the Euler characteristic $\chi(B) := h^0(B) - h^1(B) + h^2(B)$ of the line bundle $\mathcal{O}_S(B)$ is equal to

$$(1.1) \quad \chi(B) = \frac{1}{2}(B^2 - B \cdot K_S) + \chi(\mathcal{O}_S).$$

See [1, 3] for the main properties of these invariants. The base locus of a linear system $|L|$ is denoted by $\text{Bs}|L|$. A divisor L is *special* if

$$h^0(L) \cdot h^1(L) > 0.$$

Let p_1, \dots, p_r be points in very general position on S and let $|H - \sum_i 2p_i|$ be the linear systems of divisors of $|H|$ which are singular at all the p_i 's. We say that the linear system $|H - \sum_i 2p_i|$ is *special* if

$$\dim |H - \sum_i 2p_i| > \max\{-1, \dim |H| - 3r\}.$$

Proposition 1.2. *Let $\phi : S_r \rightarrow S$ be the blow-up at all the p_i 's with exceptional divisors E_i . If $h^l(H) = 0$ for $l > 0$, then $L_r := \phi^*H - \sum_i 2E_i$ is special if and only if $|H - \sum_i 2p_i|$ is special.*

Proof. Since ϕ has connected fibers, then $\phi_*\mathcal{O}_{S_r} = \mathcal{O}_S$ by the Zariski connectedness theorem. This, the projection formula [12, II, Exercise 5.1 (d)] and $R^l\phi_*\phi^*H = 0$ for $l > 0$ imply the equalities $h^l(\phi^*H) = h^l(H)$ for any l . Since $\phi^*H \cdot E_i = 0$ for all i and $\chi(2E_i) = -3$, the Riemann-Roch theorem and what was proved before give

$$\chi(L_r) = \chi(\phi^*H) - 3r = \chi(H) - 3r = h^0(H) - 3r,$$

where the last equality is by hypothesis. Let $E := \sum_i 2E_i$ and consider the exact sequence of sheaves:

$$0 \longrightarrow \mathcal{O}_{S_r}(D - E_i) \longrightarrow \mathcal{O}_{S_r}(D) \longrightarrow \mathcal{O}_{E_i}(D) \longrightarrow 0.$$

If $D \cdot E_i \geq 0$ and $h^2(D) = 0$, then taking cohomology of the exact sequence and using $h^1(D|_{E_i}) = 0$, we deduce that $h^2(D - E_i) = 0$. Taking D to be ϕ^*H , $\phi^*H - E_1, \phi^*H - 2E_1, \dots, \phi^*H - \sum_i 2E_i = L_r$ we deduce that $h^2(L_r) = 0$. Thus, by what was proved before, L_r is special if and only if $h^0(L_r) > \max\{0, h^0(H) - 3r\}$. We conclude by observing that an element of $|L_r|$ is the strict transform of an element of $|H - \sum_i 2p_i|$ so that the dimensions of the two linear systems are equal. \square

We recall that an abelian surface S is *simple* if it does not contain 1-dimensional subgroups. In particular S is simple if it does not contain elliptic curves.

Definition 1.3. In what follows a *neat surface* is a smooth algebraic projective surface which satisfies property (0.1).

Proposition 1.4. *If S is a neat surface, then it is one of the following:*

p_g	q	χ	Type of surface
0	0	1	
1	0	2	$K3$
1	2	0	Simple abelian

Proof. Assume that K_S is effective. Then either $K_S \sim \mathcal{O}_S$ or $K_S - C$ is effective for some integral curve C of S . In the second case by Serre duality $h^2(C) = h^0(K_S - C) > 0$, which is a contradiction. This implies that $K_S \sim \mathcal{O}_S$, so by [1, Theorem VIII.2] S is either a $K3$ or an abelian surface. If S is abelian and C is an elliptic curve on it, then $C^2 = 0$ by adjunction, so $h^0(C) - h^1(C) = \chi(C) = 0$ gives $h^1(C) > 0$, which is a contradiction. This implies that S is simple.

Now assume K_S is not effective, hence $p_g = 0$. If $q(S) > 0$, then by [1, Proposition V.15] the Albanese morphism $\alpha : S \rightarrow \text{Alb}(S)$ has connected 1-dimensional fibers. By [3, Theorem 20.1] the general fiber F of α is smooth. The Riemann-Roch theorem and $F^2 = 0$ give

$$p_a(F) + q - 1 = \frac{1}{2}(F \cdot K_S) + q = 1 - \chi(F) \leq 0,$$

where the last inequality is due to $h^i(F) = 0$ for $i > 0$ and $h^0(F) > 0$. This leaves us with the case $p_a(F) = 0$ and $q = 1$. By [1, Chapter VI] the minimal model of S is a ruled surface S_{\min} whose basis $B = \alpha(S)$ is a smooth elliptic curve. Let $\alpha = \phi \circ \alpha_{\min}$, where $\alpha_{\min} : S_{\min} \rightarrow B$ is obtained by blowing down all the (-1) -curves contained in the fibers of α . By [12, Proposition 2.9], α_{\min} has a section C with $C^2 \leq 0$. Let $\tilde{C} \subset S$ be the strict transform of C through ϕ . Thus $\tilde{C}^2 \leq C^2 \leq 0$. Taking the exact sequence of \tilde{C} ,

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(\tilde{C}) \longrightarrow \mathcal{O}_{\tilde{C}}(\tilde{C}) \longrightarrow 0,$$

one obtains that $h^1(\tilde{C}) > 0$, which is a contradiction. So if $p_g(S) = 0$, then $q(S) = 0$. □

Proposition 1.5. *Let S be a neat surface and let B be an integral curve such that $h^0(B) \geq 2$. Then either $h^1(2B) = 0$ or S is a $K3$ surface, $h^1(2B) = 1$ and $B^2 = 0$.*

Proof. If $p_g = 0$, consider the exact sequence

$$0 \longrightarrow \mathcal{O}_S((n - 1)B) \longrightarrow \mathcal{O}_S(nB) \longrightarrow \mathcal{O}_B(nB) \longrightarrow 0.$$

When $n = 1$ the equalities $h^1(B) = h^2(\mathcal{O}_S) = 0$ imply that $h^1(B|_B) = 0$, so that $h^1(nB|_B) = 0$ for any positive n . Taking $n = 2$ we deduce that $h^1(2B) = 0$.

If $p_g > 0$, then $K_S \sim \mathcal{O}_S$ by Proposition 1.4. If $B^2 > 0$, then $2B$ is a nef and big divisor so that $h^1(2B) = 0$ by Kawamata-Viehweg vanishing. If $B^2 = 0$, then $\chi(\mathcal{O}_S) = \chi(B) = h^0(B) \geq 2$, where the first equality is due to the Riemann-Roch theorem and the second to the hypothesis. Thus S is a $K3$ surface and $\chi(\mathcal{O}_S) = 2$, by Proposition 1.4. Since $\chi(2B) = 2$ and $h^0(2B) = 3$, by Riemann-Roch we deduce $h^1(2B) = 1$. □

2. LINEAR SYSTEMS THROUGH DOUBLE POINTS ON SURFACES

Let S be a neat surface and let $\phi : S_r \rightarrow S$ be the blow-up map at r points in very general position. Let H be an integral curve of S and

$$(2.1) \quad L_r := \phi^*H - 2E_1 - \cdots - 2E_r,$$

where $E_i = \phi^{-1}(p_i)$ are the exceptional divisors.

Proposition 2.2. *Let S be a neat surface and let L_r be as in (2.1). If L_r is non-special and L_{r+1} is special, then*

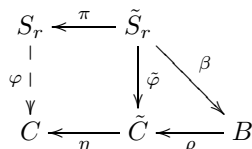
$$L_r \sim F + nD,$$

where F is the fixed part of $|L_r|$ and $H^0(nD) = \text{Sym}^n H^0(D)$, where $n > 1$.

Proof. By hypothesis $h^1(L_r) = 0$, so by Proposition 1.2 and the fact that L_{r+1} is special, we have $\dim |L_r - 2p| > \max\{-1, \dim |L_r| - 3\}$ for a point $p \in S_r$ in very general position. Let $L_r \sim F + M$, where F is the fixed part of $|L_r|$. Since p can be chosen to lie outside F , then

$$(2.3) \quad \dim |M - 2p| > \max\{-1, \dim |M| - 3\}.$$

Let $\varphi : S \dashrightarrow \mathbb{P}^N$ be the rational map defined by the linear system $|M|$, let $C := \varphi(S)$ and let $q := \varphi(p)$. Observe that a hyperplane H of \mathbb{P}^N contains the tangent space $T_q C$ if and only if $\varphi^{-1}(H) \in |M|$ is singular at p . Thus the elements of $|M - 2p|$ are in one to one correspondence with hyperplanes H such that $H \supset T_q C$. From (2.3) we deduce that $T_q C$ imposes less than 3 conditions on the hyperplanes containing it, and this implies that $\dim T_q C < 2$. If $\dim T_q C = 0$, i.e. C is a point, then $\dim |M| = 1$ so that $\dim |M - 2p| = -1$, a contradiction. Thus $\dim T_q C = 1$ and C is a curve. Consider the following diagram of maps:



where π is a blow-up map, η is a normalization map, $\tilde{\varphi}$ is the lifting of the resolution of indeterminacy of φ to \tilde{C} and $\rho \circ \beta$ is the Stein factorization of $\tilde{\varphi}$, i.e. β has connected fibers and ρ is a finite map. Observe that on the bottom line of the diagram we have curves and on the top line we have surfaces.

Assume that B is non-rational and let E be a (-1) -curve of \tilde{S}_r . Since E is rational, $\beta(E)$ is a point. Thus β descends to a morphism $\beta_S : S \rightarrow B$ which pulls back all the non-trivial holomorphic 1-forms of B to corresponding 1-forms on S so that $q(S) > 0$. Then S is an abelian surface by Proposition 1.4. If $C_q := \beta_S^{-1}(q)$ for some $q \in B$, then $C_q^2 = 0$ so that $h^0(C_q) - h^1(C_q) = \chi(C_q) = 0$ by Riemann-Roch, Serre's duality and $K_S \sim \mathcal{O}_S$. Thus $h^1(C_q) > 0$, which is a contradiction. We proved that B is rational so that if \tilde{D} is a fiber of β , then

$$H^0(a\tilde{D}) \cong \text{Sym}^a H^0(\tilde{D}).$$

A general element Z of $|M|$ is the closure of $\varphi^{-1}(H \cap C)$, where H is a hyperplane of \mathbb{P}^N which avoids the singularities of C . Thus $Z = \pi(n\tilde{D})$, where $n = \deg(\rho) \deg(C)$. This implies that $H^0(M) = \text{Sym}^n H^0(D)$, where $D := \pi(\tilde{D})$. \square

Following the lines of the last proof, it is easy to observe that even if S does not satisfy property (0.1), the fixed part of system $|L_r - 2p|$ contains a double curve through p . That is why we have the following well-known result (see [8, Theorem 4.1] or [16]).

Corollary 2.4. *Let S be a smooth projective algebraic surface and let L_k be defined as in (2.1). If L_k is special, then the fixed part of $|L_k|$ contains a double curve.*

The following definition will be adopted in what follows.

Definition 2.5. A divisor L_r of the blow-up S_r of S at r points in very general position is **pre-special** if it is of the form (2.1), it is non-special and L_{r+1} is special on S_{r+1} .

We begin by investigating the fixed part of the linear system defined in Proposition 2.2.

Lemma 2.6. *Let S be a neat surface and let L_r and the E_i 's be as in (2.1). If $L_r \sim F + M$, where F is the fixed part of $|L_r|$, then $E_i \cdot F \geq 0$ for any i .*

Proof. Suppose that E_r is a component of F , so that $h^0(L_r) = h^0(L_r - E_r)$. If $\pi : S_r \rightarrow S_{r-1}$ is the blow-up of E_r and $p := \pi(E_r)$, then the preceding equality is equivalent to $|L_{r-1} - 2p| = |L_{r-1} - 3p|$. Since the point p is in very general position on S_{r-1} , then by [5, Proposition 2.3] we get a contradiction. Since E_r is not a component of F , then $E_r \cdot F \geq 0$ and the same argument applies to E_i for any i . □

Lemma 2.7. *Let S be a neat surface and let L_r be a pre-special divisor of S_r . If $L_r \sim F + nD$, where F is the fixed part of $|L_r|$, then $h^1(D) = 0$. Moreover, $h^1(2D) = 0$ unless or $D = \pi^*B$, where B is an integral curve with $B^2 = 0$ on a K3 surface S , in which case $h^1(2D) = 1$.*

Proof. From $2 = E_i \cdot F + n E_i \cdot D$ and $E_i \cdot F \geq 0$ by Lemma 2.6, we deduce that $0 \leq D \cdot E_i \leq 1$ because $n > 1$, by Proposition 2.2. Thus

$$D \sim \phi^*B - \sum_{i \in I} E_i,$$

where I is the set of all the i 's such that $D \cdot E_i = 1$ and B is an integral curve of S so that $h^1(B) = 0$. Observe that $h^0(D) = h^0(B) - |I|$ because each E_i imposes one independent condition since it corresponds to a simple point of S in very general position. This gives $h^1(D) = 0$.

We now want to determine the possible values of $h^1(2D)$. If $|I| > 0$, then $B^2 > 0$ so that $h^1(2B) = 0$ by Proposition 1.5. Since $|2D|$ is fixed component free, then by Corollary 2.4 we have $h^1(2D) = 0$. If $|I| = 0$, then $D = \phi^*B$ so that $h^1(2D) = h^1(2B)$. By Proposition 1.5 we conclude that $h^1(2B) = 0$ unless S is a K3 surface and $B^2 = 0$ in which case $h^1(2B) = 1$. □

The preceding lemma allows one to find the numerical characters of the curve D by means of the Riemann-Roch theorem.

Proposition 2.8. *Let S be a neat surface and let L_r be a pre-special divisor of S_r . If $L_r \sim F + nD$, where F is the fixed part of $|L_r|$, then the general element of $|D|$ is a smooth curve with either*

$$D^2 = \chi(\mathcal{O}_S) - 1, \quad D \cdot K_{S_r} = 3\chi(\mathcal{O}_S) - 5,$$

or $D = \pi^*B$, where B is an integral curve with $B^2 = 0$ on a K3 surface S .

Proof. We know that $h^0(nD) = n + 1$ for $n = 1, 2$ by Proposition 2.2. Moreover, Lemma 2.7 gives $h^1(D) = 0$. Suppose now that $h^1(2D) = 0$. Then we get

$$\chi(D) = 2, \quad \chi(2D) = 3.$$

By Riemann-Roch one obtains $D^2 = \chi(\mathcal{O}_S) - 1$ and $D \cdot K_{S_r} = 3\chi(\mathcal{O}_S) - 5$. If $h^1(2D) > 0$, then by Lemma 2.7 we get the remaining case.

To prove that the general element of $|D|$ is smooth, observe that $\chi(\mathcal{O}_S) \leq 2$ by Proposition 1.4. This implies $D^2 \leq 1$ by what was said before, so $|D|$ has at most one base point p . By Bertini's second theorem [2] the general element of $|D|$ is smooth away from p . It has to be smooth also at p , since otherwise two elements of $|D|$ would have a bigger intersection at that point. \square

Corollary 2.9. *Let S be a neat surface with $p_g = q = 0$. If L_k , defined as in (2.1), is special, then S is a rational surface.*

Proof. Let r be such that L_r is non-special but L_{r+1} is special and let $L_r \sim F + nD$ be the decomposition given in Proposition 2.2. We know that $|D|$ is a pencil of smooth curves on S_r with $D^2 = 0$ and $D \cdot K_{S_r} = -2$, so that D is rational and $|D|$ has empty base locus. The morphism $\phi_{|D|} : S_r \rightarrow \mathbb{P}^1$ is a \mathbb{P}^1 -fibration. Blowing-down the (-1) -curves which are contained in the fibers of $\phi_{|D|}$ we obtain a \mathbb{P}^1 -bundle over \mathbb{P}^1 , which is a rational ruled surface (see [1]). This implies that S is also rational. \square

Now we wish to investigate the numerical properties of integral curves of the base locus of $|L_r|$ when L_r is pre-special.

Lemma 2.10. *Let S be a neat surface and let L_r be a pre-special divisor of S_r . Let $L_r \sim F + nD$, where F is the fixed part of $|L_r|$, and let C be an integral component of F . Then $\chi(C + sD) \leq s + 1$ for any $s \leq n$.*

Proof. By hypothesis F is the fixed part of $|L_r|$, so we get

$$h^0(C + nD) \leq h^0(F + nD) = h^0(nD),$$

which implies that C is a fixed component of $|C + nD|$. Observe that since $|D|$ does not have fixed components, then also $|kD|$ is a fixed component free for any $k > 0$. This and the equality $C + nD = (C + sD) + (n - s)D$ imply that C is a fixed component of $|C + sD|$. By Serre's duality we have that $h^2(C + sD) = 0$, so from $h^0(C + sD) = h^0(sD) = s + 1$ we get the thesis. \square

Proposition 2.11. *Let S be a neat surface with L_r as in (2.1). If C is an integral fixed component of $|L_r|$, then $\chi(C) = 1$. Moreover, if L_r is pre-special and D is defined as in (2.2), then either*

$$C \cdot D \leq \frac{1}{2}(\chi(\mathcal{O}_S) - 1)$$

or S is a K3 surface, $D = \phi^*B$ with $B^2 = 0$ and $C \cdot D \leq 1$.

Proof. Since $Z := \phi(C)$ is integral, $h^i(Z) = 0$ for $i > 0$. Observe that Z is a fixed component of $|H - \sum_i 2p_i|$, for some integral H . By [5, Proposition 2.3] the general element of the last system has multiplicity 2 at each p_i , so that Z has multiplicity at most 2 at each p_i . Thus $h^1(C) = 0$ by Corollary 2.4. By Serre's duality and Proposition 1.4 we have $h^2(C) = 0$, so that $\chi(C) = 1$.

Assume now that L_r is pre-special. By Lemma 2.10 we have $\chi(C + 2D) \leq 3$. Consider the equality

$$\chi(C + 2D) = \chi(C) + \chi(2D) + 2C \cdot D - \chi(\mathcal{O}_S).$$

By Lemma 2.7 and Serre's duality, either $\chi(2D) = h^0(2D) = 3$ or S is a $K3$ surface, $D = \phi^*B$ with $B^2 = 0$ so that $\chi(2D) = 2$. In both the cases we get the thesis. \square

3. APPLICATIONS TO SOME NON-RATIONAL SURFACES

The aim of this section is to apply the results of Section 2 to two classes of smooth projective complex surfaces S . Recall that we denote by $\phi : S_k \rightarrow S$ the blow-up map at k very general points of S with exceptional divisors E_1, \dots, E_s .

A $K3$ surface S is a smooth simply connected compact complex surface with $K_S \sim \mathcal{O}_S$. In what follows we will restrict our attention to the class of projective $K3$ surfaces.

Lemma 3.1. *A projective $K3$ surface S is neat.*

Proof. Let B be an integral curve on S . If $B^2 > 0$, then B is nef and big so that $h^1(K_S + B) = 0$, and thus $h^1(B) = 0$ since $K_S \sim \mathcal{O}_S$. If $B^2 \leq 0$, then by adjunction $B^2 = -2, 0$. Taking cohomology of $0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_S(B) \rightarrow \mathcal{O}_B(B) \rightarrow 0$ and using $K_B \sim \mathcal{O}_B(B)$ gives the result. \square

Theorem 3.2. *Let $\phi : S_k \rightarrow S$ be the blow-up of a projective $K3$ surface which does not contain elliptic curves at k very general points and let $L_k := \phi^*H - \sum_i 2E_i$ with H integral. Then L_k is special if and only if $k = 2$ and $H \sim 2B$ with $B^2 = 2$.*

Proof. If H is an integral divisor on S with $H^2 > 0$, then H is nef and big because $h^0(H) \geq \chi(H) > 2$ by Serre's duality and the Riemann-Roch theorem. If $H \sim 2B$, then also B is nef and big, so by Kawamata-Viehweg vanishing and $B^2 = 2$ we get $h^0(H) = 6$ and $h^0(B) = 3$. We expect $|L_2|$ to be empty, but $\frac{1}{2}L_2 = \phi^*B - E_1 - E_2$ is effective, so L_2 is special.

Let $r := k - 1$ and suppose now that L_r is pre-special. By Proposition 2.2 we have $L_r \sim F + nD$, where F is the fixed part of $|L_r|$ and $|D|$ is a linear pencil with $D \cdot K_{S_r} = 1$, by Proposition 2.8. The last equality together with $K_{S_r} \sim \sum_i E_i$ imply that $D \sim \phi^*B - E_1$ for some integral curve B of S . If C is an integral component F , then $C \cdot D = 0$ by Proposition 2.11; thus $F \cdot D = 0$. Since $n \geq 2$ and $2 = L_r \cdot E_1 = F \cdot E_1 + nD \cdot E_1$, by Lemma 2.6 we conclude $F \cdot E_1 = 0$. Thus $F \cdot \phi^*B = 0$ so that $\phi(F) \cdot B = 0$, which implies that $\phi(L_r) = H$ is not connected, which is absurd. Hence $F = 0$ and $L_r \sim 2D$, so that $r = 1$. By Proposition 2.8 we have $D^2 = 1$ so that $B^2 = 2$. Since $h^0(L_2) = 1$, by imposing one more general point we get $h^0(L_3) = 0$, so L_3 is non-special. Thus there are no more special divisors. \square

Remark 3.3. The hypothesis of Theorem 3.2 is automatically satisfied if $\text{Pic}(S) \cong \mathbb{Z}$. It is still possible to classify special linear systems of type L_k on $K3$ surfaces S with Picard groups of higher rank, but a careful study of the non-reduced fibers of the elliptic fibrations of S has to be performed. Due to the length of this analysis we do not include more results in this direction here.

We recall that an abelian surface is a complex torus admitting a holomorphic line bundle Θ such that $\phi_{|\Theta|}$ is an embedding into a projective space. An abelian surface is *simple* if it does not contain 1-dimensional subgroups.

Lemma 3.4. *A simple abelian surface S is neat.*

Proof. First of all observe that S does not contain integral curves B with $B^2 \leq 0$. We prove the statement by contradiction. If B is such a curve and $p, q \in B$, let $\tau \in \text{Aut}(S)$ be the translation with $\tau(p) = q$. Since $\tau(B) \cdot B = B^2 \leq 0$ and $q \in \tau(B) \cap B$, we deduce that $\tau(B) = B$. This implies that B is isomorphic to a 1-dimensional subgroup of S , which is a contradiction. If B is an integral curve with $B^2 > 0$, then $h^i(B) = h^i(K_S + B) = 0$ for $i > 0$ by Kawamata-Viehweg vanishing. This implies that S is neat. \square

Theorem 3.5. *Let S_r be the blow-up of a simple abelian surface S at points in very general position. If L_r is as in (2.1), then it is non-special.*

Proof. If L_r is pre-special, let $L_r \sim F + nD$, with D as in Proposition 2.2. Then $D^2 < 0$ by Proposition 2.8, which is a contradiction. \square

4. APPLICATIONS TO SOME ANTICANONICAL RATIONAL SURFACES

In this section S_n will be the blow-up of \mathbb{P}^2 at n points in *very general position*. If $n \leq 9$, then it is known (see [6, Theorem 5.1]) that an effective divisor D on S_n is special if and only if $D \cdot E \leq -2$ for some (-1) -curve E of S_n . If this is the case, then in particular E is a fixed component of $|D|$ so that the general element of $|D|$ is reducible or non-reduced. Thus $h^i(D) = 0$ if D is integral and $i > 0$ so that S_n is neat for $n \leq 9$.

We intend to prove two theorems here:

1. The Harbourne-Hirschowitz conjecture (see [6, Conjecture 4.8]) for linear systems of \mathbb{P}^2 through nine points of any multiplicity and through an arbitrary number of additional double points.
2. The classification of the defective secant varieties of S_n for $0 \leq n \leq 9$.

A divisor L of S_r is (-1) -**special** if $h^0(L) > 0$ and there exists a (-1) -curve E such that $E \cdot L \leq -2$. If L is (-1) -special and $a := -E \cdot L$, then the exact sequence

$$H^1(L - E) \longrightarrow H^1(L) \longrightarrow H^1(\mathcal{O}_{\mathbb{P}^1}(-a)) \longrightarrow 0$$

implies that $h^1(L) > 0$ so that L is special.

Let $\phi : S_{r+9} \rightarrow S_9$ be the blow-up map with exceptional divisors E_1, \dots, E_r and let H be a divisor of S_9 . In this section we will adopt the following notation:

$$(4.1) \quad L_r := \phi^* H - 2E_1 - \dots - 2E_r.$$

Lemma 4.2. *Let L_r be a divisor on S_{r+9} defined as in (4.1). If C_1, C_2 are integral fixed components of $|L_r|$, then $C_1 \cdot C_2 \leq 0$.*

Proof. By Proposition 2.11 we have $\chi(C_i) = 1$. Since $C_1 + C_2$ is contained in the base locus of $|L_r|$, then $h^0(C_1 + C_2) = 1$. This gives $\chi(C_1 + C_2) \leq 1$; thus we get $\chi(C_1 + C_2) = \chi(C_1) + \chi(C_2) + C_1 \cdot C_2 - 1 = 1 + C_1 \cdot C_2$. \square

Theorem 4.3. *Let L_r be a divisor on S_{r+9} defined as in (4.1). Then L_r is special if and only if it is (-1) -special.*

Proof. One implication has already been proved. Suppose now that L_r is special. If $h^1(H) > 0$, then by [11] there exists a (-1) -curve E of S_9 such that $E \cdot H \leq -2$. Since the points are in very general position, they do not lie on E so that $\phi^* E$ is a (-1) -curve of S_{r+9} and $\phi^* E \cdot L_r = E \cdot H \leq -2$.

If $h^1(H) = 0$, then the system $|H - \sum_i 2p_i|$ is special because L_r is special and by Proposition 1.2. Let

$$H \sim B + H',$$

where B is the fixed part of $|H|$. We have that $|H' - \sum_i 2p_i|$ is special because the points are in very general position, so that they can be chosen outside B . Moreover, $h^1(H') = 0$ since $|H'|$ does not have fixed components (see the introduction of this section). We deduce that $L'_r = \phi^*H' - \sum_i 2E_i$ is special by Proposition 1.2.

If the general element of $|H'|$ is irreducible, then let $\phi_s : S_{r+9} \rightarrow S_{s+9}$ be the blow-up map. Let $0 \leq s < r$ be the biggest integer such that the divisor L'_s of S_{s+9} is non-special. By Proposition 2.2 we have

$$L'_s \sim F + nD,$$

where F is the fixed part of $|L'_s|$ and $|D|$ is a pencil of smooth rational curves with $D^2 = 0$ by Proposition 2.8. If C is an integral component of F , then $C \cdot D = 0$ by Proposition 2.11, so that $D \cdot L'_s = 0$. Observe that

$$(\phi_s^*D - E_{s+1}) \cdot L'_r = D \cdot L'_s - E_{s+1} \cdot L'_r = -2,$$

where $\phi_s^*D - E_{s+1}$ is a (-1) -curve. This implies that L'_r is (-1) -special.

If the general element of $|H'|$ is reducible, then by [11, Lemma II.6] we deduce that $H' \sim aD$, where $|D|$ is a linear pencil with $D^2 = 0$ and $-K_{S_9} \cdot D = 0, 2$. The case $-K_{S_9} \cdot D = 0$ can be excluded because of [6, Theorem 5.1], since in this case $h^0(aD) = 1$ so that $|L'_r|$ would be empty and thus non-special. If $-K_{S_9} \cdot D = 2$, then $p_a(D) = 0$ so that the general element of $|D|$ is rational and, by Bertini's second theorem, is smooth. In this case $\phi^*D - E_1$ is a (-1) -curve, and from

$$(\phi^*D - E_1) \cdot L'_r = (\phi^*D - E_1) \cdot (\phi^*H' - \sum_{i=1}^r 2E_i) = -2$$

we deduce that L'_r is (-1) -special if $r \geq 1$.

We proved that there exists a (-1) -curve E of S_{r+9} such that $E \cdot L'_r \leq -2$. Thus E is a fixed component of $|L'_r|$, and consequently it is a fixed component of $|L_r|$. Thus, by Lemma 4.2 and the fact that ϕ^*B is a fixed curve of $|L_r|$, we get

$$E \cdot L_r = E \cdot (\phi^*B + L'_r) \leq E \cdot L'_r \leq -2,$$

so that L_r is (-1) -special. □

As an application of Theorem 4.3, we find the dimension of the secant variety of any projective embedding of S_r with $r \leq 9$.

Lemma 4.4. *If H is an ample and integral divisor of S_n , with $2 \leq n \leq 9$, then $p_a(H) > 0$.*

Proof. We prove the statement by contradiction. Assume that H is ample and $p_a(H) = 0$. If $n \geq 3$, since H is ample, then, by [10, Theorem 1.1], we have that H is linearly equivalent to a non-negative sum of the classes $E_0, E_0 - E_1, 2E_0 - E_1 - E_2, -K_i := 3E_0 - E_1 - \dots - E_i$, where E_0 is the pull-back of a line and the E_i , with $0 < i \leq n$, are the exceptional divisors. Since H is ample, then $H \cdot E_n > 0$, so that $H + K_{S_n}$ is effective. Thus we get $H^2 = H \cdot (H + K_{S_n} - K_{S_n}) \geq -H \cdot K_{S_n}$. Since $p_a(H) = 0$, we have $H^2 = -H \cdot K_{S_n} - 2$; hence $H^2 < -H \cdot K_{S_n}$, which is a contradiction.

If $n = 2$, let $H = dE_0 - m_1E_1 - m_2E_2$. Then $-2 = 2p_a(H) - 2 = d^2 - 3d - m_1^2 + m_1 - m_2^2 + m_2$. On the other hand we have $d > m_1 + m_2$ because $H \cdot (E_0 - E_1 - E_2) > 0$.

By substituting $d = m_1 + m_2 + 1$ in the right hand side of the equation we obtain the non-negative number $2m_1m_2 - 2$, which is a contradiction. \square

Theorem 4.5. *Let H be a very ample divisor of S_n , with $0 \leq n \leq 9$. The r -secant variety of $\phi_{|H|}(S_n)$ is defective if and only if (H, n, r) is one of the following:*

$$(\mathcal{O}_{\mathbb{P}^2}(2), 0, 1), \quad (\mathcal{O}_{\mathbb{P}^2}(4), 0, 4), \quad (\phi^* \mathcal{O}_{\mathbb{P}^2}(2a) - (2a - 2)E_1, 1, 2a - 1).$$

Proof. By Terracini's lemma, the r -secant variety of $\phi_{|H|}(S_n)$ is defective if and only if $L_{r+1} := \phi^*H - 2E_1 - \cdots - 2E_{r+1}$ is special; see [6, Lemma 7.4].

If L_r is non-special and L_{r+1} is special, then we are in the hypothesis of Proposition 2.2, so we get $L_r \sim F + mD$, where F is the fixed part of $|L_r|$, $m > 1$ and $|D|$ is a linear pencil. By Proposition 2.8 the general element of $|D|$ is a smooth rational curve with $D^2 = 0$. Moreover, $D \cdot F = 0$ by Proposition 2.11, so that $D \cdot L_r = 0$.

If $D \cdot E_i = 0$ for any i , then $D = \phi^*D'$, where $D' = \phi(D)$, so that $0 = D \cdot L_r = D' \cdot H$, which is a contradiction since H is ample. Thus we deduce that $D \cdot E_i > 0$ for some i , and this gives $2 = L_r \cdot E_i = (F + mD) \cdot E_i \geq m(D \cdot E_i)$, where the last inequality is due to Lemma 2.6. Since $m > 1$ we deduce that $D \cdot E_i = 1$, $m = 2$ and $F \cdot E_i = 0$.

Suppose now that $D \cdot E_k = 0$ and let $D_k := D + E_i - E_k$. The general element of $|D_k|$ is irreducible and $D_k^2 = 0$, because the same is true for $|D|$ and we are just exchanging the role of the points p_i and p_k , which are in very general position. In particular D_k is a nef divisor. Observe that $D_k \cdot L_r = D \cdot L_r = 0$, so $D_k \cdot (F + mD) = 0$. Since D_k is nef, we get $D_k \cdot D = 0$, which is a contradiction.

We proved that $D \cdot E_i = 1$ for all i so that $D = \phi^*D' - \sum_i E_i$ and $F = \phi^*F'$, because $F \cdot E_i = 0$ for any i . This implies that $D' \cdot F' = D \cdot F = 0$. Since $H \sim 2D' + F'$ is very ample, it is connected, so that $F' \sim \mathcal{O}_{S_n}$ and consequently $L_r \sim 2D$. Since D' is ample and $p_a(D') = 0$, because $p_a(D) = 0$, we get $n = 0, 1$ by Lemma 4.4. In the first case D' is linearly equivalent to either $\mathcal{O}_{\mathbb{P}^2}(1)$ or $\mathcal{O}_{\mathbb{P}^2}(2)$, while in the second it is linearly equivalent to $\phi^* \mathcal{O}_{\mathbb{P}^2}(a) - (a - 1)E_1$ for some $a \geq 2$.

Since $D^2 = 0$, then $r = D'^2$. This allows us to determine L_r . In any such case we get that $h^0(L_{r+1}) = 1$ so that L_{r+2} is non-special because $h^0(L_{r+2}) = 0$. \square

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