THE LINEAR DUAL OF THE DERIVED CATEGORY OF A SCHEME

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ABSTRACT. Let $X \to S$ be a projective morphism of schemes. We study the category $D(X/S)^*$ of S-linear exact functors $D(X) \to D(S)$, and we study the Fourier transform $D(X) \to D(X/S)^*$.

INTRODUCTION

Let $p\colon X \to S$ be an S-scheme. Let $D(X) = D_{qc}(X)$ be the derived category of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology. Given an object $K \in D(X)$, we define the functor

$$\omega_K\colon D(X) \to D(S)$$
$$M \mapsto p_*(K \otimes M).$$

(Since we are dealing with derived categories all functors are assumed to be derived and we use the abbreviated notation: $p_* = \mathbf{R}p_*$, $\otimes = \overset{\mathbf{L}}{\otimes}$, etc.). This functor satisfies the following properties:

1) It is an additive, graded (i.e., commutes with translations) and is an exact functor (i.e., takes exact triangles into exact triangles).

2) It is S-linear: One has a natural isomorphism $\omega_K(M \otimes p^*N) \simeq \omega_K(M) \otimes N$, for any $M \in D(X)$, $N \in D(S)$. This follows by the projection formula.

An S-linear form on $D(X)$ is a functor $\omega\colon D(X) \to D(S)$ satisfying 1) and endowed with an isomorphism such as 2). An S-linear morphism between S-linear forms is defined in the obvious way. We shall denote by $D(X/S)^*$ the category of S-linear forms and S-linear morphisms and call it the S-linear dual category of $D(X)$. An S-linear form of the type ω_K, $K \in D(X)$, is called an integral linear form (of kernel K). We have a functor (which we call a Fourier transform)

$$\text{Fourier}\colon D(X) \to D(X/S)^*$$
$$K \mapsto \omega_K.$$

An S-linear form $\omega\colon D(X) \to D(S)$ is said to be *bounded and coherent* (resp. *perfect*) if it maps $D_c^b(X)$ into $D_c^b(S)$ (resp. $D_{\text{perf}}(X)$ into $D_{\text{perf}}(S)$). We shall denote by $D(X/S)_{bc}^*$ (resp. $D(X/S)_{\text{perf}}^*$) the faithful subcategory of $D(X/S)^*$ whose

objects are the bounded and coherent S-linear forms (resp. the perfect S-linear forms).

Assume now for simplicity that S is regular and $X \to S$ is projective. The main result of the paper is to prove that the Fourier transform induces equivalences

$$D_c^b(X) \xrightarrow{\sim} D(X/S)_{\mathrm{perf}}^*, \qquad D_{\mathrm{perf}}(X) \xrightarrow{\sim} D(X/S)_{bc}^*.$$

In other words, every perfect S-linear form (resp. bounded and coherent S-linear form) is integral and its kernel is a unique bounded and coherent (resp. perfect) object of $D(X)$. This is obtained in Theorem 1.14 and Corollary 1.16, where S is not assumed to be regular. Indeed, the only hypothesis in Theorem 1.14 (resp. Corollary 1.16) is the projectivity of the morphism $X \to S$ (resp. projectivity and finite Tor-dimension). Even flatness is not required. Hence these results apply for very general situations. We have also proved these results for the S-linear dual categories of $D_c^b(X)$ and $D_{\mathrm{perf}}(X)$. See Theorems 1.23 and 1.26.

These results are connected with Orlov's result about the integrality of exact functors. Orlov proved in [6] that if X and Y are smooth projective varieties over a field k and $F \colon D_c^b(X) \to D_c^b(Y)$ is exact and fully faithful, then F is an integral functor; i.e., there exists an object $K \in D_c^b(X \times Y)$ such that $F \simeq \Phi_K$, where Φ_K is the integral functor with kernel K:

$$\Phi_K(\_) = p_*(K \otimes q^*(\_)), \qquad
\begin{array}{ccc}
X \times Y & \xrightarrow{\ p\ } & Y \\
\downarrow{\scriptstyle q} & & \\
X. & &
\end{array}$$

This has been generalized to singular schemes in [7] and [2]. However, there are many integral functors that are not fully faithful. It is widely believed that this result should hold also without the faithfulness hypothesis: is any exact functor integral? One could go further and state the integrality conjecture in a relative setting: let $X \to S$ and $Y \to S$ be two projective S-schemes (and assume for simplicity that all the schemes are regular) and let K be an object in $D_c^b(X \times_S Y)$. One has a (relative) integral functor

$$\begin{array}{l}
\Phi_K \colon D_c^b(X) \to D_c^b(Y) \\
\qquad M \mapsto p_*(K \otimes q^*M),
\end{array}
\qquad
\begin{array}{ccc}
X \times_S Y & \xrightarrow{\ p\ } & Y \\
\downarrow{\scriptstyle q} & & \\
X. & &
\end{array}$$

This functor is exact and S-linear. One could ask if any exact S-linear functor from $D_c^b(X)$ to $D_c^b(Y)$ is of this type. It is not difficult to find a counterexample if we do not assume $Y \to S$ to be flat. Our result proves the conjecture in the case $Y = S$. But in this case our result is not only an integrality result because it is also shown that the morphisms between the kernels are the same as the S-linear morphisms between the associated functors, which is known to fail in general (i.e. for $Y \neq S$).

Regarding the general integrality conjecture, under the flatness assumption on $Y \to S$, we give a linear criterion for the integrality of a functor in Proposition 1.30.

Moreover, our results give new evidence of the validity of the integrality conjecture. If $F \colon D_c^b(X) \to D_c^b(Y)$ is a (relative) integral functor with kernel $K \in D_c^b(X \times_S Y)$, then this same kernel defines a functor in the opposite direction. That

is, a kernel K has two associated functors $F\colon D_c^b(X) \to D_c^b(Y)$ and $F^*\colon D_c^b(Y) \to D_c^b(X)$, with the same kernel. Let us say that these are *partners*. Our results prove that any exact S-linear functor $F\colon D_c^b(X) \to D_c^b(Y)$ has a partner $F^*\colon D_c^b(Y) \to D_c^b(X)$, which is nothing but the "dual" functor.

1. LINEAR FUNCTORS

We shall denote $D(X) = D_{qc}(X)$ to be the derived category of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology. A functor $F\colon D(X) \to D(Y)$ means an additive, graded and exact functor, i.e., an additive functor that commutes with the shift functor and takes exact triangles into exact triangles. For simplicity all schemes are assumed to be quasi-compact, quasi-separated and of finite type over a field k. Whenever one has a morphism of schemes $f\colon X \to Y$, we still denote by f the morphism $X \times_Y Y' \to Y'$ induced after a base change $Y' \to Y$.

Definition 1.1. Let $p\colon X \to S$ and $q\colon Y \to S$ be two S-schemes. An S-*linear functor* $F\colon D(X) \to D(Y)$ is a covariant functor endowed with a bi-additive and bi-graded bi-functorial isomorphism $\theta_F(M,E)\colon F(M \otimes p^*E) \simeq F(M) \otimes q^*E$. That is, an S-linear functor is a pair (F, θ_F), though we shall usually denote it by F.

An S-linear morphism $\phi\colon F \to F'$ of S-linear functors is a morphism of functors which is compatible with the θ's, i.e., such that the diagram

$$
\begin{array}{ccc}
F(M \otimes p^*E) & \xrightarrow{\;\sim\;} & F(M) \otimes q^*E \\
\phi(M \otimes p^*E) \downarrow & & \downarrow \phi(M) \otimes 1 \\
F'(M \otimes p^*E) & \xrightarrow{\;\sim\;} & F'(M) \otimes q^*E
\end{array}
$$

is commutative.

Definition 1.2. Let X be an S-scheme. An S-*linear form on* $D(X)$ is an S-linear functor $\omega\colon D(X) \to D(S)$. A morphism of S-linear forms is an S-linear morphism of functors.

We shall denote by $D(X/S)^*$ the category of S-linear forms on $D(X)$ and S-linear morphisms.

Example. Let $K \in D(X \times_S Y)$. Let us denote by $p\colon X \times_S Y \to Y$ and $q\colon X \times_S Y \to X$ the natural projections. The functor $\Phi_K\colon D(X) \to D(Y)$ defined by

$$\Phi_K(M) = p_*(q^*M \otimes K)$$

is S-linear, with the θ provided by the projection formula. We say that Φ_K is a relative integral functor of kernel K. In particular, for each $K \in D(X)$, we have an S-linear form on $D(X)$:

$$\omega_K\colon D(X) \to D(S), \qquad M \mapsto p_*(M \otimes K).$$

We say that ω_K is an integral S-linear form on $D(X)$ of kernel K.

Definition 1.3. We say that a functor $F\colon D(X) \to D(Y)$ is *bounded and coherent* if it takes $D_c^b(X)$ into $D_c^b(Y)$. We say that F is *perfect* if it takes $D_{\mathrm{perf}}(X)$ into $D_{\mathrm{perf}}(Y)$. We say that F is *quasi-perfect* if it takes $D_{\mathrm{perf}}(X)$ into $D_c^b(Y)$.

Definition 1.4. A morphism $f\colon X \to Y$ is of *finite Tor-dimension* if $f^*\colon D(Y) \to D(X)$ is bounded and coherent.

1.1. Complexes of finite homological dimension over S.

Definition 1.5. Let $p\colon X \to S$ be an S-scheme. An object $M \in D(X)$ is said to be *of finite homological dimension over S* (fhd over S for short) if the functor

$$D(S) \to D(X)$$
$$E \mapsto p^*E \otimes M$$

is bounded and coherent. We shall denote by $D_{\mathrm{fhd}/S}(X)$ the faithful subcategory of $D(X)$ whose objects are the complexes of finite homological dimension over S.

Remark 1.6.   (1) If S is a regular scheme, then $D_{\mathrm{fhd}/S}(X) = D_c^b(X)$.
  (2) If f has finite Tor-dimension (for example f flat or S regular), then $D_{\mathrm{perf}}(X)$
     $\subset D_{\mathrm{fhd}/S}(X)$.
  (3) If f has finite Tor-dimension and X is regular, then $D_{\mathrm{perf}}(X) = D_{\mathrm{fhd}/S}(X)$
     $= D_c^b(X)$.
  (4) If $X = S$, then $D_{\mathrm{fhd}/S}(S) = D_{\mathrm{perf}}(S)$.

We shall mention some properties of complexes of finite homological dimension over S. They can be found in [4].

Proposition 1.7. *Assume that $p\colon X \to S$ is projective and let $\mathcal{O}_X(1)$ be a relatively ample invertible sheaf on X. Let $M \in D(X)$. The following conditions are equivalent:*

  (1) *M is fhd over S.*
  (2) *$p_*(M \otimes \mathcal{O}_X(n))$ is perfect for any n.*
  (3) *The functor $\mathbf{R}\mathcal{H}om_X^\bullet(M, p^!(\underline{}))\colon D(S) \to D(X)$ is bounded coherent.*

Two important properties of complexes of finite homological dimension over S are the following:

Proposition 1.8. *Assume that $p\colon X \to S$ is locally projective. Let us denote by $D_{X/S} = p^!\mathcal{O}_S$ the relative dualizing complex and for any $M \in D(X)$ let us denote by $M^\vee = \mathbf{R}\mathcal{H}om_X^\bullet(M, D_{X/S})$. If $K \in D_{fhd/S}(X)$, then*
(1) For any $N \in D(S)$ one has an isomorphism

$$K^\vee \otimes p^*N \overset{\sim}{\to} \mathbf{R}\mathcal{H}om_X^\bullet(K, p^!N).$$

(2) $K^\vee$ is of finite homological dimension over S and the natural map $K \to K^{\vee\vee}$ is an isomorphism.

Definition 1.9. We say that a functor $F\colon D(X) \to D(Y)$ is *of finite homological dimension over S* (fhd over S for short) if it takes $D_{\mathrm{fhd}/S}(X)$ into $D_{\mathrm{fhd}/S}(Y)$. We say that F is *quasi-fhd over S* if it takes $D_{\mathrm{fhd}/S}(X)$ into $D_c^b(Y)$.

Proposition 1.10. *Assume that $p\colon X \to S$ is projective and let $\omega_K\colon D(X) \to D(S)$ be an integral S-linear form. Then*

  (1) *ω_K is bounded and coherent $\Leftrightarrow$ K is perfect.*
  (2) *ω_K is perfect $\Leftrightarrow$ K is fhd over S.*
  (3) *ω_K is quasi-perfect $\Leftrightarrow$ K is bounded and coherent.*

Proof. (1) Assume that ω_K is bounded and coherent. Let us see that K is perfect. It suffices to see that $K \otimes N$ is bounded and coherent for any bounded and coherent N. For this, it suffices to see that $p_*(K \otimes N \otimes \mathcal{O}_X(n))$ is bounded and coherent for

any n (see [4, Lemma 2.5] and its following paragraph). Since $p_*(K \otimes N \otimes \mathcal{O}_X(n)) = \omega_K(N \otimes \mathcal{O}_X(n))$, one concludes the proof of part (1). The converse is clear.

(2) Assume that ω_K is perfect. Then $\omega_K(\mathcal{O}_X(n))$ is perfect for any n; i.e., $p_*(K \otimes \mathcal{O}_X(n))$ is perfect and then K is fhd over S by Proposition 1.7. Conversely, assume that K is fhd over S and let $M \in D_{\mathrm{perf}}(X)$. Then, for any $N \in D_c^b(S)$, $\omega_K(M) \otimes N = \omega_K(M \otimes p^*N) = p_*(K \otimes M \otimes p^*N)$ is bounded and coherent. Hence $\omega_K(M)$ is perfect.

(3) Assume that ω_K is quasi-perfect. Then $p_*(K \otimes \mathcal{O}_X(n)) = \omega_K(\mathcal{O}_X(n))$ is bounded and coherent; hence K is bounded and coherent. The converse is clear. $\square$

Proposition 1.11. *Let $F\colon D(X) \to D(Y)$ be an S-linear functor. If F is bounded and coherent, then it is fhd over S.*

Proof. Let $P \in D_{\mathrm{fhd}/S}(X)$. For any $N \in D_c^b(S)$, $P \otimes p^*N$ is bounded and coherent and then $F(P \otimes p^*N)$ is bounded and coherent. Since F is S-linear, it follows that $F(P) \otimes q^*N$ is bounded and coherent. Hence $F(P)$ is fhd over S. $\square$

Proposition 1.12. *Assume that $p\colon X \to S$ is of finite Tor-dimension and let $\omega\colon D(X) \to D(S)$ be an S-linear form on $D(X)$. Then*

ω *is fhd over S (resp. quasi-fhd over S) $\Rightarrow \omega$ is perfect (resp. quasi-perfect) .*

Proof. Since p is of finite Tor-dimension, $D_{\mathrm{perf}}(X) \subset D_{\mathrm{fhd}/S}(X)$. One concludes the proof because $D_{\mathrm{fhd}/S}(S) = D_{\mathrm{perf}}(S)$. $\square$

Notation. Let $p\colon X \to S$ be an S-scheme. We shall denote by $D(X/S)^*_{\mathrm{perf}}$ (resp. $D(X/S)^*_{\mathrm{bc}}$, $D(X/S)^*_{\mathrm{fhd}/S}$, $D(X/S)^*_{\mathrm{quasi\text{-}perf}}$, $D(X/S)^*_{\mathrm{quasi\text{-}fhd}/S}$) the category of perfect (resp. bounded and coherent, fhd/S, quasi-perfect, quasi fhd/S) S-linear forms.

Proposition 1.13. *Let $p\colon X \to S$ be a projective morphism. If $\omega\colon D(X) \to D(S)$ is an S-linear form, then it has a right adjoint.*

Proof. Brown representability (see [5]) says that ω has a right adjoint if we assume that ω commutes with infinite direct sums. But one can copy the proof, replacing this condition by the S-linearity one. $\square$

Theorem 1.14. *Let $p\colon X \to S$ be a projective morphism. The functor*

$$D_{fhd/S}(X) \to D(X/S)^*_{\mathrm{perf}}$$
$$K \mapsto \omega_K$$

is an equivalence.

Proof. Sketch: We shall construct a quasi-inverse $D(X/S)^*_{\mathrm{perf}} \to D_{\mathrm{fhd}/S}(X)$ in the following way: let $\omega^\#\colon D(S) \to D(X)$ be the right adjoint of ω, and define $K_\omega = \mathbf{R}\mathcal{H}om_X^\bullet(\omega^\#(\mathcal{O}_S), D_{X/S})$, where $D_{X/S}$ is the relative dualizing complex of X over S. We shall see that the correspondences $K \mapsto \omega_K$ and $\omega \mapsto K_\omega$ give the desired equivalence.

Now we give more details. By Proposition 1.13, ω has a right adjoint $\omega^\#\colon D(S) \to D(X)$. So one has

$$\mathrm{Hom}_{D(S)}(\omega(M), N) \simeq \mathrm{Hom}_{D(X)}(M, \omega^\#(N)).$$

Let us see that this yields an isomorphism

$$\mathbf{R}\mathcal{H}om_S^\bullet(\omega(M), N) \xrightarrow{\sim} p_*\mathbf{R}\mathcal{H}om_X^\bullet(M, \omega^\#(N)).$$

In fact, for any $E \in D(S)$ one has

$$\operatorname{Hom}_{D(S)}(E, \mathbf{R}\mathcal{H}om^\bullet_S(\omega(M), N)) \simeq \operatorname{Hom}_{D(S)}(E \otimes \omega(M), N)$$
$$\simeq \operatorname{Hom}_{D(S)}(\omega(p^*E \otimes M), N)$$
$$\simeq \operatorname{Hom}_{D(X)}(p^*E \otimes M, \omega^\#(N))$$
$$\simeq \operatorname{Hom}_{D(S)}(E, p_*\mathbf{R}\mathcal{H}om^\bullet_X(M, \omega^\#(N))).$$

If one takes $M = \mathcal{O}_X(-n)$ one obtains

$$(1.1) \qquad \mathbf{R}\mathcal{H}om^\bullet_S(\omega(\mathcal{O}_X(-n)), N) \simeq p_*(\omega^\#(N) \otimes \mathcal{O}_X(n)).$$

Assume that ω is perfect. Then, if N is perfect, $\omega^\#(N)$ is of finite homological dimension over S: in fact, if N is perfect, then $\mathbf{R}\mathcal{H}om^\bullet_S(\omega(\mathcal{O}_X(-n)), N)$ is perfect, because $\omega(\mathcal{O}_X(-n))$ is perfect. Then $p_*(\omega^\#(N) \otimes \mathcal{O}_X(n))$ is perfect for any n by (1.1). By Proposition 1.7, $\omega^\#(N)$ is fhd over S.

Let us denote by $D_{X/S} = f^!\mathcal{O}_S$ the relative dualizing complex and let $M^\vee = \mathbf{R}\mathcal{H}om^\bullet_X(M, D_{X/S})$. Let us denote

$$K_\omega = \omega^\#(\mathcal{O}_S)^\vee.$$

By Proposition 1.8, K_ω has finite homological dimension over S and $\omega^\#(\mathcal{O}_S) \xrightarrow{\sim} K_\omega^\vee$. Let us prove that

$$\mathbf{R}\mathcal{H}om^\bullet_X(K_\omega, p^!N) \simeq \omega^\#(N).$$

Let us first define a morphism $\mathbf{R}\mathcal{H}om^\bullet_X(K_\omega, p^!N) \to \omega^\#(N)$. Since K_ω has finite homological dimension over S one has $\omega^\#(\mathcal{O}_S) \otimes p^*N \simeq \mathbf{R}\mathcal{H}om^\bullet_X(K_\omega, p^!N)$. So one has to define a morphism $\omega^\#(\mathcal{O}_S) \otimes p^*N \to \omega^\#(N)$. The unit morphism $(\omega \circ \omega^\#)(\mathcal{O}_S) \to \mathcal{O}_S$ induces a morphism $(\omega \circ \omega^\#)(\mathcal{O}_S) \otimes N \to N$. By the S-linearity of ω one has a morphism $\omega(\omega^\#(\mathcal{O}_S) \otimes p^*N) \to N$ and then a morphism $\omega^\#(\mathcal{O}_S) \otimes p^*N \to \omega^\#(N)$. Now let us see that it is an isomorphism. It suffices to see that it is an isomorphism after tensoring by $\mathcal{O}_X(n)$ and taking the direct image p_*. But

$$p_*(\omega^\#(\mathcal{O}_S) \otimes p^*N \otimes \mathcal{O}_X(n)) \simeq p_*(\omega^\#(\mathcal{O}_S) \otimes \mathcal{O}_X(n)) \otimes N$$
$$\overset{(1.1)}{\simeq} \mathbf{R}\mathcal{H}om^\bullet_S(\omega(\mathcal{O}_X(-n)), \mathcal{O}_X) \otimes N$$
$$\simeq \mathbf{R}\mathcal{H}om^\bullet_S(\omega(\mathcal{O}_X(-n)), N)$$
$$\overset{(1.1)}{\simeq} p_*(\omega^\#(N) \otimes \mathcal{O}_X(n)).$$

We have then proved that $\omega^\# \simeq \mathbf{R}\mathcal{H}om^\bullet_X(K_\omega, p^!(\underline{}))$. Taking left adjoints one concludes that $\omega \simeq \omega_{K_\omega}$. Moreover K_ω is fhd over S. The uniqueness of K_ω follows from the construction.

Conversely, if K is fhd over S, then ω_K is perfect by Proposition 1.10 and $K_{\omega_K} \simeq K$. Indeed, the right adjoint of ω_K is

$$(1.2) \qquad \omega_K^\#(\underline{}) = \mathbf{R}\mathcal{H}om^\bullet_S(K, p^!(\underline{})) \simeq K^\vee \otimes p^*(\underline{})$$

by Proposition 1.8 and then $K_{\omega_K} = \omega_K^\#(\mathcal{O}_S)^\vee \simeq K^{\vee\vee} \simeq K$ by (2) of Proposition 1.8.

The correspondences $\omega \mapsto K_\omega$ and $K \mapsto \omega_K$ are clearly functorial. To conclude, we have to see that a morphism $K \to L$ in $D_{\mathrm{fhd}/S}(X)$ is equivalent to an S-linear morphism of functors $\omega_K \to \omega_L$. Giving an S-linear morphism $\omega_K \to \omega_L$ is

equivalent to giving an S-linear morphism $\omega_L^{\#} \to \omega_K^{\#}$. By (1.2), this is equivalent to giving a morphism $L^{\vee} \to K^{\vee}$. One concludes by (2) of Proposition 1.8. $\qquad\square$

Remark 1.15.   (1) In the proof of the theorem it has been shown that one can relax the hypothesis of ω being perfect and replace it by the weaker hypothesis: $\omega(\mathcal{O}_X(i))$ is perfect for any i.
    (2) The kernel K_ω associated to an S-linear form ω does not depend on the S-linear structure of ω, since it was defined as $\omega^{\#}(\mathcal{O}_S)^{\vee}$, with $\omega^{\#}$ the right adjoint of ω. The S-linear structure of ω is used to construct the isomorphism $\omega \simeq \omega_{K_\omega}$. One deduces that the S-linear structure of ω is essentially unique: assume that (ω, θ) and (ω, θ') are two S-linear structures on ω. Since $K_{(\omega,\theta)} = K_{(\omega,\theta')}$, one obtains an S-linear isomorphism $(\omega, \theta) \simeq (\omega, \theta')$.

Corollary 1.16. *Let $p\colon X \to S$ be a projective morphism of finite Tor-dimension. The functor*

$$D_{\mathrm{perf}}(X) \to D(X/S)_{bc}^*$$
$$K \mapsto \omega_K$$

is an equivalence.

Proof. Let ω be a bounded and coherent S-linear form on $D(X)$. By Proposition 1.11, ω is fhd over S, and hence it is perfect by Proposition 1.12. By Theorem 1.14, ω is isomorphic to ω_K, with $K \in D_{\mathrm{fhd}/S}(X)$. Finally K is perfect by Proposition 1.10. $\qquad\square$

Corollary 1.17. *Let $p\colon X \to S$ be a projective morphism and assume that S is regular. Then*

$$D(X/S)_{\mathrm{quasi\text{-}perf}}^* = D(X/S)_{\mathrm{perf}}^* \simeq D_c^b(X) = D_{\mathrm{fhd}/S}(X)$$

and

$$D(X/S)_{\mathrm{fhd}/S}^* = D(X/S)_{\mathrm{bc}}^* \simeq D_{\mathrm{perf}}(X).$$

Proof. Since S is regular, $D_{\mathrm{perf}}(S) = D_c^b(S)$ and then $D(X/S)_{\mathrm{quasi\text{-}perf}}^* = D(X/S)_{\mathrm{perf}}^*$. By Theorem 1.14, $D(X/S)_{\mathrm{perf}}^* \simeq D_{\mathrm{fhd}/S}(X)$. Finally $D_c^b(X) = D_{\mathrm{fhd}/S}(X)$ because S is regular.

For the second part, $D(X/S)_{\mathrm{fhd}/S}^* = D(X/S)_{\mathrm{bc}}^*$ because $D_c^b(X) = D_{\mathrm{fhd}/S}(X)$ and $D_c^b(S) = D_{\mathrm{fhd}/S}(S)$. Finally, $D(X/S)_{\mathrm{bc}}^* \simeq D_{\mathrm{perf}}(X)$ by Corollary 1.16. $\qquad\square$

Corollary 1.18. *Let $p\colon X \to S$ be a projective morphism of finite Tor-dimension and assume that X is regular. Then*

$$D_c^b(X) = D_{\mathrm{perf}}(X) = D_{\mathrm{fhd}/S}(X)$$

and

$$D(X/S)_{\mathrm{quasi\text{-}perf}}^* = D(X/S)_{\mathrm{perf}}^* = D(X/S)_{\mathrm{fhd}/S}^* = D(X/S)_{\mathrm{bc}}^*$$
$$= D(X/S)_{\mathrm{quasi\text{-}fhd}/S}^* \simeq D_c^b(X).$$

Proof. $D_c^b(X) = D_{\mathrm{perf}}(X)$ because X is regular, and $D_c^b(X) = D_{\mathrm{fhd}/S}(X)$ because f has finite Tor-dimension and X is regular. It follows that $D(X/S)_{\mathrm{fhd}/S}^* = D(X/S)_{\mathrm{perf}}^*$ and $D(X/S)_{\mathrm{bc}}^* = D(X/S)_{\mathrm{quasi\text{-}fhd}/S}^* = D(X/S)_{\mathrm{quasi\text{-}perf}}^*$. Now

$D(X/S)^*_{\mathrm{perf}} \simeq D_{\mathrm{fhd}/S}(X)$ by Theorem 1.14, and $D(X/S)^*_{\mathrm{bc}} \simeq D_{\mathrm{perf}}(X)$ by Corollary 1.16. □

As an immediate consequence of Theorem 1.14 and Corollary 1.16 one obtains the "partner" of a functor:

Corollary 1.19. *Let X and Y be two projective S-schemes and let $F\colon D(X) \to D(Y)$ be a perfect S-linear functor. Then it induces a functor:*

$$F^*\colon D_{\mathrm{fhd}/S}(Y) \to D_{\mathrm{fhd}/S}(X).$$

Corollary 1.20. *Let X and Y be two projective S-schemes of finite Tor-dimension over S and let $F\colon D(X) \to D(Y)$ be a bounded and coherent S-linear functor. Then it induces a functor:*

$$F^*\colon D_{\mathrm{perf}}(Y) \to D_{\mathrm{perf}}(X).$$

Of course, if F is both perfect and bounded and coherent, the partners F^* of Corollaries 1.19 and 1.20 coincide.

Remark 1.21. If $F\colon D(X) \to D(Y)$ is a (relative) integral functor of kernel $K \in D(X \times_S Y)$, then F^* is also an integral functor with the *same* kernel.

1.2. **Dual categories of $D_{\mathbf{perf}}(X)$ and $D^b_c(X)$.** In this subsection we shall reproduce the results of the previous one for the categories $D_{\mathrm{perf}}(X)$ and $D^b_c(X)$. That is, we want to study the categories of S-linear functors $D_{\mathrm{perf}}(X) \to D_{\mathrm{perf}}(S)$ and $D^b_c(X) \to D^b_c(S)$.

We shall study first the linear dual of $D_{\mathrm{perf}}(X)$.

Definition 1.22. Let $p\colon X \to S$ be an S-scheme. An S-linear form on $D_{\mathrm{perf}}(X)$ is a functor $\omega\colon D_{\mathrm{perf}}(X) \to D_{\mathrm{perf}}(S)$ endowed with a bi-additive, bi-graded and bi-functorial isomorphism $\omega(M \otimes p^*E) \simeq \omega(M) \otimes E$, with $M \in D_{\mathrm{perf}}(X)$, $E \in D_{\mathrm{perf}}(S)$. An S-linear morphism between S-linear forms is defined as in §1.1.

We shall denote by $D_{\mathrm{perf}}(X)^*$ the category of S-linear forms on $D_{\mathrm{perf}}(X)$ and S-linear morphisms. One has a natural restriction functor $D(X/S)^*_{\mathrm{perf}} \to D_{\mathrm{perf}}(X)^*$.

By Proposition 1.10, if K is an object of $D(X)$ of finite homological dimension over S, then $\omega_K\colon D(X) \to D(S)$ takes perfect complexes into perfect complexes. Then one has a functor

$$D_{\mathrm{fhd}/S}(X) \to D_{\mathrm{perf}}(X)^*.$$

Theorem 1.23. *Assume that X and S are projective schemes over a field k. The functor*

$$D_{\mathrm{fhd}/S}(X) \to D_{\mathrm{perf}}(X)^*$$
$$K \mapsto \omega_K$$

is an equivalence.

Proof. The proof is similar to that of Theorem 1.14. Instead of Brown representability we need the following result: It is proved in [3] that any contravariant cohomological functor of finite type over $D_{\mathrm{perf}}(X)$ (X a projective scheme over k) is representable by a bounded complex with coherent homology. Moreover, one has an equivalence between $D^b_c(X)$ and the category of contravariant cohomological functors of finite type over $D_{\mathrm{perf}}(X)$ (see [1]). We shall refer to this result as the *locally finite duality on $D_{\mathrm{perf}}(X)$*.

It follows that if $\omega\colon D_{\mathrm{perf}}(X) \to D_{\mathrm{perf}}(S)$ is an S-linear form, it has a "right adjoint" $\omega^{\#}\colon D_c^b(S) \to D_c^b(X)$; that is, one has

$$\mathrm{Hom}_{D(S)}(\omega(P), N) \simeq \mathrm{Hom}_{D(X)}(P, \omega^{\#}(N))$$

for any $P \in D_{\mathrm{perf}}(X)$, $N \in D_c^b(S)$. Now, the S-linearity of ω yields that $\omega^{\#}(E) \simeq \omega^{\#}(\mathcal{O}_S) \otimes p^* E$ for any $E \in D_{\mathrm{perf}}(S)$. Indeed, for any $P \in D_{\mathrm{perf}}(X)$, $E \in D_{\mathrm{perf}}(S)$ one has

$$\mathrm{Hom}_{D(S)}(P, \omega^{\#}(E)) \simeq \mathrm{Hom}_{D(X)}(\omega(P), E) \simeq \mathrm{Hom}_{D(X)}(\omega(P) \otimes E^*, \mathcal{O}_S)$$

$$\simeq \mathrm{Hom}_{D(X)}(\omega(P \otimes p^* E^*), \mathcal{O}_S) \simeq \mathrm{Hom}_{D(S)}(P \otimes p^* E^*, \omega^{\#}(\mathcal{O}_S))$$

$$\simeq \mathrm{Hom}_{D(S)}(P, \omega^{\#}(\mathcal{O}_S) \otimes p^* E),$$

where $E^* = \mathbf{R}\mathcal{H}om_S^{\bullet}(E, \mathcal{O}_S)$. One concludes that $\omega^{\#}(E) \simeq \omega^{\#}(\mathcal{O}_S) \otimes p^* E$ by the locally finite duality on $D_{\mathrm{perf}}(X)$.

Let us now see that $\omega^{\#}(\mathcal{O}_S)$ has finite homological dimension over S. Let us denote $L = \omega^{\#}(\mathcal{O}_S)$. For any $P \in D_{\mathrm{perf}}(X)$, $E \in D_{\mathrm{perf}}(S)$ one has

$$\mathrm{Hom}_{D(S)}(E, \omega(P)^*) \simeq \mathrm{Hom}_{D(S)}(E \otimes \omega(P), \mathcal{O}_S)$$

$$\simeq \mathrm{Hom}_{D(S)}(\omega(p^* E \otimes P), \mathcal{O}_S)$$

$$\simeq \mathrm{Hom}_{D(X)}(p^* E \otimes P, L)$$

$$\simeq \mathrm{Hom}_{D(S)}(E, p_* \mathbf{R}\mathcal{H}om_X^{\bullet}(P, L))$$

$$\simeq \mathrm{Hom}_{D(S)}(E, \omega_L(P^*)).$$

By the locally finite duality on $D_{\mathrm{perf}}(X)$ one obtains $\omega(P)^* \simeq \omega_L(P^*)$. By Proposition 1.10, L has finite homological dimension over S.

Let us denote $K = L^{\vee}$ and let us prove that $\omega \simeq \omega_K$. For any $P \in D_{\mathrm{perf}}(X)$, $E \in D_{\mathrm{perf}}(S)$ one has

$$\mathrm{Hom}_{D(S)}(\omega(P), E) \simeq \mathrm{Hom}_{D(X)}(P, \omega^{\#}(E)) \simeq \mathrm{Hom}_{D(X)}(P, L \otimes p^* E)$$

$$\simeq \mathrm{Hom}_{D(X)}(P, K^{\vee} \otimes p^* E) \simeq \mathrm{Hom}_{D(X)}(P, \mathbf{R}\mathcal{H}om_X^{\bullet}(K, p^! E))$$

$$\simeq \mathrm{Hom}_{D(S)}(p_*(P \otimes K), E)$$

and then $\omega \simeq \omega_K$. The rest of the proof is as in Theorem 1.14. $\square$

Corollary 1.24. *Under the same hypothesis, the natural restriction functor*

$$D(X/S)_{\mathrm{perf}}^* \to D_{\mathrm{perf}}(X)^*$$

is an equivalence.

Now we shall study the linear dual of $D_c^b(X)$.

Definition 1.25. An S-linear form on $D_c^b(X)$ is a functor $\omega\colon D_c^b(X) \to D_c^b(S)$ endowed with a bi-additive, bi-graded and bi-functorial isomorphism

$$\omega(M \otimes p^* E) \simeq \omega(M) \otimes E$$

for any $M \in D_c^b(X)$, $E \in D_{\mathrm{perf}}(S)$.

Theorem 1.26. *Assume that X and S are projective schemes over a field k. Then the functor*

$$D_{\mathrm{perf}}(X) \to D_c^b(X)^*$$

$$K \mapsto \omega_K$$

is an equivalence.

Proof. Let $\omega\colon D_c^b(X) \to D_c^b(S)$ be an S-linear form. It is proved in [8] that any locally-finite homological functor on $D_c^b(X)$ (X a projective scheme over a field k) is representable by an object of $D_{\mathrm{perf}}(X)$. It follows (see [2, Prop. 2.2]) that the category of locally-finite homological functors on $D_c^b(X)$ is equivalent to $D_{\mathrm{perf}}(X)$. Hence ω has a "left adjoint" $\widetilde{\omega}\colon D_{\mathrm{perf}}(S) \to D_{\mathrm{perf}}(X)$; that is, one has

$$\mathrm{Hom}_{D(S)}(E, \omega(M)) \simeq \mathrm{Hom}_{D(X)}(\widetilde{\omega}(E), M)$$

for any $E \in D_{\mathrm{perf}}(S)$, $M \in D_c^b(X)$. Now the S-linearity of ω yields that $\widetilde{\omega}(E) \simeq \widetilde{\omega}(\mathcal{O}_S) \otimes p^*E$. Indeed,

$$\begin{aligned}
\mathrm{Hom}_{D(X)}(\widetilde{\omega}(E), M) &\simeq \mathrm{Hom}_{D(S)}(E, \omega(M)) \simeq \mathrm{Hom}_{D(S)}(\mathcal{O}_S, E^* \otimes \omega(M)) \\
&\simeq \mathrm{Hom}_{D(S)}(\mathcal{O}_S, \omega(p^*E^* \otimes M)) \simeq \mathrm{Hom}_{D(X)}(\widetilde{\omega}(\mathcal{O}_S), p^*E^* \otimes M) \\
&\simeq \mathrm{Hom}_{D(X)}(\widetilde{\omega}(\mathcal{O}_S) \otimes p^*E, M).
\end{aligned}$$

Let us denote $K = \widetilde{\omega}(\mathcal{O}_S)^*$. Since $\widetilde{\omega}(\_) \simeq \widetilde{\omega}(\mathcal{O}_S) \otimes p^*(\_)$, one deduces, by "adjointness", that $\omega(\_) \simeq p_*(K \otimes \_)$. That is, $\omega \simeq \omega_K$. $\square$

Corollary 1.27. *Assume also that $p\colon X \to S$ has finite Tor-dimension. The natural restriction functor*

$$D(X/S)_{bc}^* \to D_c^b(X)^*$$

is an equivalence.

1.3. **A linear criterion for integrality.** Let $p\colon X \to S$ and $q\colon Y \to S$ be two S-schemes. For any base change $S' \to S$ we denote $X_{S'} = X \times_S S'$ and we still denote by $p\colon X_{S'} \to S'$ the morphism induced by $p\colon X \to S$ under base change.

Definition 1.28. Let $F\colon D(X) \to D(Y)$ be an S-linear functor and $f\colon S' \to S$ a base change. We say that F *extends to* S' if there exists an S'-linear functor $F_{S'}\colon D(X_{S'}) \to D(Y_{S'})$ such that the diagram

$$\begin{array}{ccc}
D(X_{S'}) & \xrightarrow{\ F_{S'}\ } & D(Y_{S'}) \\
{\scriptstyle f_*}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\
D(X) & \xrightarrow{\ \ F\ \ } & D(Y)
\end{array}$$

is commutative, i.e., $f_* \circ F_{S'} \simeq F \circ f_*$.

Definition 1.29. An S-linear functor $F\colon D(X) \to D(Y)$ is said to be *geometric* if it extends to any base change $S' \to S$.

Proposition 1.30. *Assume that $q\colon Y \to S$ is flat. Let $F\colon D(X) \to D(Y)$ be an S-linear functor. Then F is integral if and only if F is geometric.*

Proof. Assume that $F = \Phi_K$ is an integral functor of kernel $K \in D(X \times_S Y)$. Let $f\colon S' \to S$ be a base change and $K_{S'} = f^*K \in D(X_{S'} \times_{S'} Y_{S'})$. By the projection formula and flat base change it follows easily that

$$\Phi_K(f_*M) \simeq f_*\Phi_{K_{S'}}(M), \qquad M \in D(X_{S'}).$$

This means that Φ_K extends to S', an extension being $\Phi_{K_{S'}}$.

Assume now that F is geometric. Let us take $S' = X$, $f = p$. By hypothesis there exists an S'-linear functor $F_{S'}\colon D(X \times_S S') \to D(Y \times_S S')$ such that $F(f_*M) \simeq f_*F_{S'}(M)$, for any $M \in D(X \times_S S')$. Let us denote by $\mathcal{O}_\Delta \in D(X \times_S S')$ the

structure sheaf of the diagonal subscheme. For any $N \in D(X)$ one has $N \simeq f_*(\mathcal{O}_\Delta \otimes p^*N)$. Then, by the S'-linearity of $F_{S'}$,

$$F(N) \simeq F(f_*(\mathcal{O}_\Delta \otimes p^*N)) \simeq f_*F_{S'}(\mathcal{O}_\Delta \otimes p^*N) \simeq f_*(F_{S'}(\mathcal{O}_\Delta) \otimes q^*N).$$

This proves that F is an integral functor of kernel $F_{S'}(\mathcal{O}_\Delta)$.   □

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