BUCHSBAUM VARIETIES WITH NEXT TO SHARP BOUNDS ON CASTELNUOVO-MUMFORD REGULARITY

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ABSTRACT. This paper is devoted to the study of the next extremal case for a Castelnuovo-type bound $\operatorname{reg} V \leq \lceil (\deg V - 1)/\operatorname{codim} V \rceil + 1$ of the Castelnuovo-Mumford regularity for a nondegenerate Buchsbaum variety V. A Buchsbaum variety with the maximal regularity is known to be a divisor on a variety of minimal degree if the degree of the variety is large enough. We show that a Buchsbaum variety satisfying $\operatorname{reg} V = \lceil (\deg V - 1)/\operatorname{codim} V \rceil$ is a divisor on a Del Pezzo variety if $\deg V \gg 0$.

1. INTRODUCTION

The Castelnuovo-Mumford regularity is one of the most important invariants measuring a complexity of the defining equations of a projective variety. There have been various studies on bounding the regularity of a variety.

Let k be an algebraically closed field. Let $S = k[X_0, \cdots, X_N]$ be the polynomial ring over k. A variety $V \subset \mathbb{P}_k^N = \operatorname{Proj} S$ means a nondegenerate irreducible reduced projective scheme over k. For a coherent sheaf $\mathcal{F}$ on $\mathbb{P}_k^N$ and an integer $m \in \mathbb{Z}$, $\mathcal{F}$ is said to be m-regular if $\operatorname{H}^i(\mathbb{P}_k^N, \mathcal{F}(m-i)) = 0$ for all $i \geq 1$. For a projective scheme $X \subseteq \mathbb{P}_k^N$, X is said to be m-regular if the ideal sheaf $\mathcal{I}_X$ is m-regular. The Castelnuovo-Mumford regularity of $X \subseteq \mathbb{P}_k^N$ is the least such integer m and is denoted by $\operatorname{reg} X$. A projective scheme X is m-regular if and only if for every $p \geq 0$ the minimal generators of the pth syzygy module of the defining ideal $I(= \Gamma_* \mathcal{I}_X \subset S)$ of X occur in degree $\leq m + p$; see [3, 4].

For a rational number $m \in \mathbb{Q}$, we write $\lceil m \rceil$ for the minimal integer which is greater than or equal to m and $\lfloor m \rfloor$ for the maximal integer which is less than or equal to m.

The Eisenbud-Goto conjecture $\operatorname{reg} V \leq \deg V - \operatorname{codim} V + 1$ for a nondegenerate projective variety V is one of the most important problems in this field, and it is widely open to get the bound and the classification of boundary examples in the case of higher dimensional projective varieties. However, if V is an ACM variety, that

is, the coordinate ring of V is Cohen-Macaulay, then a regularity bound $\operatorname{reg} V \leq \lceil (\deg V - 1)/\operatorname{codim} V \rceil + 1$ easily follows from the uniform position principle for a generic hyperplane section of a projective curve. Arising from this viewpoint, one has extended the bounding the regularity with invariants coming from the deficiency module of the variety. If $V \subset \mathbb{P}_k^N$ is a nondegenerate projective curve, then $\operatorname{reg} V \leq \lceil (\deg V - 1)/\operatorname{codim} V \rceil + \max\{k(V), 1\}$, where $k(V) = \min\{v \geq 0 | \mathfrak{m}^v \mathrm{M}(V) = 0\}$ and $\mathrm{M}(V) = \mathrm{H}_*^1 \mathcal{I}_V$. Further, the boundary curve and the next boundary curve are classified for $\deg V \gg 0$; see [10, 12].

This paper considers a Buchsbaum variety, which is an extended class of an ACM variety. A projective variety $V \subset \mathbb{P}_k^N$ is called a Buchsbaum variety if the coordinate of V is a Buchsbaum ring. A result of Stückrad and Vogel [18] states that $\operatorname{reg} V \leq \lceil (\deg V - 1)/\operatorname{codim} V \rceil + 1$ for a nondegenerate Buchsbaum variety $V \subset \mathbb{P}_k^N$. The extremal case is known to be a divisor on a variety of minimal degree if $\deg V \gg 0$ by Yanagawa for curves [19] and by Nagel [15] in general. In this paper, we classify the next extremal case of the Castelnuovo-type bound of the Castelnuovo-Mumford regularity for a Buchsbaum variety. We show that a Buchsbaum variety $V \subset \mathbb{P}_k^N$ satisfying $\operatorname{reg} V = \lceil (\deg V - 1)/\operatorname{codim} V \rceil$ with $\deg V \gg 0$ is a divisor on a Del Pezzo variety; see Theorem 2.7. More interestingly, our result reminds us of an analogy for the next extremal case of the relationship between the secant lines and the Eisenbud-Goto bound; see [8, §0].

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2. Castelnuovo-Mumford regularity of Buchsbaum varieties

First we recall the definition of a Buchsbaum variety. For background information concerning Buchsbaum rings we refer to the book [17].

Let $S = k[x_0, \cdots, x_N]$ be the polynomial ring over k with the homogeneous maximal ideal $\mathfrak{m}$. Let $V(\subset \mathbb{P}_k^N = \operatorname{Proj} S)$ be a projective scheme. Let $I = \Gamma_* \mathcal{I}_{V/\mathbb{P}^N}$ be the defining ideal of V. Let $R = S/I$ be the coordinate ring of V and put $\mathfrak{n} = \mathfrak{m}/I$. The deficiency module of V is defined as $\mathrm{M}^i(V) = \mathrm{H}_*^i \mathcal{I}_{V/\mathbb{P}^N} = \bigoplus_{\ell \in \mathbb{Z}} \mathrm{H}^i(\mathcal{I}_{V/\mathbb{P}^N}(\ell))$ for $i = 1, \cdots, \dim V$; see, e.g., [9].

Definition 2.1. The projective scheme V is called a quasi-Buchsbaum scheme if $\mathfrak{m} \mathrm{M}^i(V) = 0$ holds for $i = 1, \cdots, \dim V$. The projective scheme V is called a Buchsbaum scheme if $R_\mathfrak{n}$ is a Buchsbaum ring. In case V is irreducible and reduced, we call V a Buchsbaum variety.

Proposition 2.2. Let $V(\subset \mathbb{P}_k^N = \operatorname{Proj} S)$ be a projective scheme with $\dim V = n$. The scheme V is a Buchsbaum scheme if and only if for any hyperplane $H_1, \cdots, H_{n-1}$ satisfying that $\dim V_j = \dim V - j$, where $V_{n-j} = V \cap H_1 \cap \cdots \cap H_j$ for $j = 0, \cdots, n-1$, the scheme V_j is quasi-Buchsbaum.

Remark 2.3. We simply call it a Buchsbaum variety in this paper, while it is called an arithmetically Buchsbaum variety in [15, 19]. If V is a Buchsbaum variety, then a generic hyperplane section of V is also a Buchsbaum variety.

In this section we investigate a Castelnuovo-type bound for the Castelnuovo-Mumford regularity for Buchsbaum varieties. Let us start by stating a result of Stückrad-Vogel [18].

Proposition 2.4. *Let V be a Buchsbaum variety of $\mathbb{P}_k^N$ with $n = \dim V \geq 1$ over an algebraically closed field. Let $W = V \cap H$ be a generic hyperplane section. Then we have $\operatorname{reg} W = \operatorname{reg} V$. Consequently, $\operatorname{reg} V \leq \lceil (\deg V - 1)/\operatorname{codim} V \rceil + 1$.*

In general, a nondegenerate projective variety $V \subset \mathbb{P}_k^N$ satisfies $\deg V \geq \operatorname{codim} V + 1$. The projective variety V is called a variety of minimal degree if $\deg V = \operatorname{codim} V + 1$. In this case, the variety V is classified to be a hyperquadric, a (cone over the) Veronese surface, or a rational normal scroll; see [6, (3.10)] and [5, (5.10)]. A nondegenerate projective variety V is a variety of almost minimal degree if $\deg V = \operatorname{codim} V + 2$, which is classified to be either a normal Del Pezzo variety or the image of a variety of minimal degree via a projection; see [2, 5, 16].

The following result describes Buchsbaum varieties with the maximal regularity of Castelnuovo-type.

Proposition 2.5 ([15, 19]). *Let $V \subseteq \mathbb{P}_k^N$ be a nondegenerate Buchsbaum variety over an algebraically closed field k with $\operatorname{char} k = 0$ or $\operatorname{codim} V \geq 5$. If $\deg V \geq (\operatorname{codim} V)^2 + 2 \operatorname{codim} V + 2$ and $\operatorname{reg} V = \lceil (\deg V - 1)/\operatorname{codim} V \rceil + 1$, then V is a divisor on a variety of minimal degree.*

The converse of (2.5) is also obtained by Nagel [15, (4.2)].

Lemma 2.6. *If a nondegenerate Buchsbaum variety $V \subseteq \mathbb{P}_k^N$ is a divisor on a variety of minimal degree, then $\operatorname{reg} V = \lceil (\deg V - 1)/\operatorname{codim} V \rceil + 1$.*

Let $S = k[X_0, \cdots, X_N]$ be the polynomial ring over a field k with the homogeneous maximal ideal $\mathfrak{m} = S_+$. Let M be a finitely generated graded S-module. Then we define $a_-(M) = \min\{\ell | [M]_\ell \neq 0\}$. Also, we define the socle of M as $\operatorname{Soc}(M) = \{x \in M | \mathfrak{m}x = 0\}$, which is a graded S-module; see, e.g., [7].

We will study Buchsbaum varieties with next to sharp bounds of Castelnuovo-type on the Castelnuovo-Mumford regularity.

Theorem 2.7. *Let $V \subset \mathbb{P}_k^N$ be a nondegenerate Buchsbaum variety over an algebraically closed field k with $\operatorname{char} k = 0$. Assume $\deg V \geq (\operatorname{codim} V)^2 + 4 \operatorname{codim} V + 2$. If $\operatorname{reg} V = \lceil (\deg V - 1)/\operatorname{codim} V \rceil$, then V is a divisor on a Del Pezzo variety.*

Proof. Let $n = \dim V$. Let us take generic hyperplanes $H_1, \cdots, H_n$. Let us define $V_{n-j} = V \cap H_1 \cap \cdots \cap H_j$ and $L_{n-j} = H_1 \cap \cdots \cap H_j$ for $j = 0, \cdots, n$. Then $\dim V_i = i$ and $L_i \cong \mathbb{P}_k^{N-n+i}$ for $i = 0, \cdots, n$. From (2.4), we have $\operatorname{reg} V_0 = \operatorname{reg} V_1 = \cdots = \operatorname{reg} V_n$. So, $\operatorname{reg} V_0 = \lceil (\deg V_0 - 1)/\operatorname{codim} V_0 \rceil$. By [12, (2.3)] and Lemma 2.6, V_0 lies on an elliptic normal curve Y_0. By an elliptic normal curve we mean an irreducible nondegenerate ACM curve of arithmetic genus one. The defining equations of an elliptic normal curve consist of quadric equations except for the case Y_0 of a plane cubic curve.

Let $c = \operatorname{codim} V = \operatorname{codim} V_0$ and $d = \deg V = \deg V_0$. Then we see $\deg Y_0 = \operatorname{codim} Y_0 + 2 = c + 1$.

Now we will show that if $c \geq 3$, then $\Gamma(\mathcal{I}_{Y_0}(2)) \cong \Gamma(\mathcal{I}_{V_0}(2))$. Indeed, if there exists a hyperquadric Q such that $V_0 \subseteq Q$ and $Y_0 \not\subseteq Q$, then $V_0 \subseteq Y_0 \cap Q$ and $d \leq 2(c+1)$ by the Bezout theorem, which contradicts the assumption $d \geq c^2 + 4c + 2$. On the other hand, if $c = 2$, that is, Y_0 is a plane cubic curve, then $\Gamma(\mathcal{I}_{Y_0}(3)) \cong \Gamma(\mathcal{I}_{V_0}(3))$. Indeed, if not, we similarly obtain an inequality $d \leq 3(c+1) = 9$, which contradicts the assumption $d \geq 14$.

In order to prove that $\Gamma(\mathcal{I}_{V_1}(2)) \to \Gamma(\mathcal{I}_{V_0}(2))$ is surjective for $c \geq 3$ and that $\Gamma(\mathcal{I}_{V_1}(3)) \to \Gamma(\mathcal{I}_{V_0}(3))$ is surjective for $c = 2$, we have only to show $\mathrm{H}^1(\mathcal{I}_{V_1}(1)) = 0$ and $\mathrm{H}^1(\mathcal{I}_{V_1}(2)) = 0$ respectively.

The exact sequence $\mathrm{H}^1_*(\mathcal{I}_{V_1}(-1)) \overset{\cdot h}{\to} \mathrm{H}^1_*(\mathcal{I}_{V_1}) \to \mathrm{H}^1_*(\mathcal{I}_{V_0})$ leads to an injective map $\mathrm{H}^1_*(\mathcal{I}_{V_1}) \to \mathrm{Soc}(\mathrm{H}^1_*(\mathcal{I}_{V_0}))$ because V_1 is a Buchsbaum variety and $\mathfrak{m}\mathrm{H}^1_*(\mathcal{I}_{V_1}) = 0$. So, let us study the structure of $\mathrm{Soc}(\mathrm{H}^1_*(\mathcal{I}_{V_0}))$ in the positive graded part as an S-graded module. Since Y_0 is ACM, we have the exact sequence $\mathrm{H}^1_*(\mathcal{I}_{Y_0}) = 0 \to \mathrm{H}^1_*(\mathcal{I}_{V_0}) \to \mathrm{H}^1_*(\mathcal{I}_{V_0/Y_0}) \to \mathrm{H}^2_*(\mathcal{I}_{Y_0})$. Note that $\mathrm{H}^2(\mathcal{I}_{Y_0}(\ell)) \cong \mathrm{H}^1(\mathcal{O}_{Y_0}(\ell)) \cong (\mathrm{H}^0(\mathcal{O}_{Y_0}(-\ell)))' = 0$ for $\ell > 0$. Thus we have $\mathrm{Soc}(\mathrm{H}^1_*(\mathcal{I}_{V_0})) = \mathrm{Soc}(\mathrm{H}^1_*(\mathcal{I}_{V_0/Y_0}))$ in the positive graded part.

Note that V_0 may contain a singular point of Y_0, which happens only in the case where Y_0 is a rational curve with $p_a(Y_0) = 1$. If not, we see that $\mathcal{I}_{V_0/Y_0} \cong \mathcal{O}_{Y_0}(-V_0)$. In this case, by Serre duality, the graded S-module $\mathrm{Soc}(\mathrm{H}^1_*(\mathcal{I}_{V_0/Y_0}))$ is isomorphic to the dual of $\Gamma_*(\mathcal{O}_{Y_0}(V_0))/\mathfrak{m}\Gamma_*(\mathcal{O}_{Y_0}(V_0))$. Let $\mathcal{F} = \mathcal{O}_{Y_0}(V_0)$. Then we have $\mathrm{H}^1(\mathcal{F} \otimes \mathcal{O}_{Y_0}(m-1)) = 0$ for $d + (c+1)(m-1) \geq 1$. In other words, $\mathcal{F}$ is m-regular for $m \geq (c - d + 2)/(c+1)$. Let us put $m = \lceil (c - d + 2)/(c+1) \rceil$. Then $\Gamma(\mathcal{F} \otimes \mathcal{O}_{Y_0}(\ell)) \otimes \Gamma(\mathcal{O}_{Y_0}(1)) \to \Gamma(\mathcal{F}(\ell+1))$ is surjective for $\ell \geq m$ by [13]. Hence we obtain $a_-(\mathrm{Soc}(\mathrm{H}^1_*\mathcal{I}_{V_0/Y_0})) \geq -m$.

Next assume that Y_0 is singular and V_0 contains a singular point of Y_0. The normalization $\pi : \tilde{Y}_0 \to Y_0$ is indeed a one-point linear projection of a rational normal curve $\tilde{Y}_0$. Let $\tilde{V}_0 = \pi^*(V_0)$. Then we have $\mathrm{H}^1_*(\mathcal{I}_{V_0/Y_0}) \cong \mathrm{H}^1_*(f_*f^*\mathcal{I}_{V_0/Y_0}) \cong \mathrm{H}^1_*(\mathcal{I}_{\tilde{V}_0/\tilde{Y}_0})$. Let $\mathcal{F} = \mathcal{O}_{\tilde{Y}_0}(\tilde{V}_0)$. Note that $\tilde{Y}_0$ is a rational normal curve. Thus we have $\mathrm{H}^1(\mathcal{F} \otimes \mathcal{O}_{\tilde{Y}_0}(m-1)) = 0$ for $d + 1 + (c+1)(m-1) \geq -1$. From the previous argument, we have $a_-(\mathrm{Soc}(\mathrm{H}^1_*\mathcal{I}_{V_0/Y_0})) \geq -m$, where $m = \lceil (c - d - 1)/(c+1) \rceil$.

Conclusively, we see that $a_-(\mathrm{Soc}(\mathrm{H}^1_*(\mathcal{I}_{V_0}))) \geq 2$ if $d \geq 3c + 4$ and that $a_-(\mathrm{Soc}(\mathrm{H}^1_*(\mathcal{I}_{V_0}))) \geq 3$ if $d \geq 4c+5$. Since $d \geq c^2 + 4c + 2$, we obtain $\mathrm{H}^1(\mathcal{I}_{V_1}(1)) = 0$ for $c \geq 3$ and $\mathrm{H}^1(\mathcal{I}_{V_1}(2)) = 0$ for $c = 2$.

For $c \geq 3$, we have a surjective map $\Gamma(\mathcal{I}_{V_1}(2)) \to \Gamma(\mathcal{I}_{V_0}(2)) \cong \Gamma(\mathcal{I}_{Y_0}(2))$. Note that Y_0 is the intersection of the hyperquadrics of $L_0(\cong \mathbb{P}_k^{N-n})$ containing V_0. Let Y_1' be the intersection of the hyperquadrics of $L_1(\cong \mathbb{P}_k^{N-n+1})$ containing V_1. Since $Y_1' \cap H_1 = Y_0$, there is an irreducible component Y_1 of Y_1' such that $Y_1 \cap H_1 = Y_0$ and Y_1 is nondegenerate. For $c = 2$, we are similarly done as in the case $c \geq 3$. Hence there exists a surface Y_1 containing V_1 such that $Y_1 \cap H_0 = Y_0$, $\mathrm{codim}\, Y_1 = \mathrm{codim}\, Y_0$ and $\deg Y_1 = \deg Y_0$.

Let $1 \leq i \leq n - 1$. Assume that V_i is a divisor on a variety Y_i of almost minimal degree. We will proceed to construct inductively a variety Y_{i+1} of almost minimal degree containing V_{i+1}. As in the case of $i = 0$, we need to show that $\Gamma(\mathcal{I}_{V_{i+1}}(2)) \to \Gamma(\mathcal{I}_{V_i}(2))$ is surjective for $c \geq 3$. By the inductive process, $\mathrm{H}^1(\mathcal{I}_{V_i}(1)) = 0$ implies $\mathrm{H}^1(\mathcal{I}_{V_{i+1}}(1)) = 0$, which gives the surjectivity of $\Gamma(\mathcal{I}_{V_{i+1}}(2)) \to \Gamma(\mathcal{I}_{V_i}(2))$. For $c = 2$, we can similarly show that $\Gamma(\mathcal{I}_{V_{i+1}}(3)) \to \Gamma(\mathcal{I}_{V_i}(3))$ is surjective. By the same method as for $i = 0$, we have a nondegenerate projective variety $Y_{i+1} \subset L_{i+1}(\cong \mathbb{P}_k^{N-n+i+1})$ with $Y_{i+1} \cap H_i = Y_i$ and $\deg Y_{i+1} = \deg Y_i$ such that V_{i+1} is a divisor on Y_{i+1} for $i = 1, \cdots, n - 1$. The variety Y_i can be easily shown to be a Del Pezzo variety by induction on i from [5, (1.6.4.5)]. Hence the assertion is proved. $\qquad\square$

Remark 2.8. For a nondegenerate Buchsbaum variety $V \subseteq \mathbb{P}_k^N$ with $\mathrm{char}\, k = 0$ which is not a divisor on any variety of minimal degree, if $\deg V \geq 2(\mathrm{codim}\, V + 1)^2$,

then $\operatorname{reg} V < \lceil (\deg V + 1)/(\operatorname{codim} V + 1) \rceil$ by [15, (4.3)]. As a result, there are no Buchsbaum varieties $V \subseteq \mathbb{P}_k^N$ with $(\operatorname{codim} V)^2 + 4\operatorname{codim} V + 2 \leq \deg V \leq 2(\operatorname{codim} V + 1)^2 - 1$ such that $\operatorname{reg} V = \lceil (\deg V + 1)/(\operatorname{codim} V + 1) \rceil$.

Remark 2.9. The assumption $\operatorname{char} k = 0$ is necessary for the use of [12, (2.3)] in this proof. However, if V_0 is in uniform position, then [12, (2.3)] works for $\operatorname{char} k > 0$; see, e.g. [14, (4.1)]. Although a generic hyperplane section V_0 of a projective curve may not be in uniform position for $\operatorname{char} k > 0$, the Castelnuovo's method as in [1, 11] gives $\operatorname{reg} V_0 \leq \lceil (\deg V_0 - 1)/\operatorname{codim} V_0 \rceil - 1$ if $\deg V_0 \gg 0$. Thus the corresponding statement of (2.7) works without describing the degree condition explicitly. "*For a nondegenerate Buchsbaum variety $V \subset \mathbb{P}_k^N$ with $\deg V \gg 0$ and $\operatorname{codim} V \geq 3$, if $\operatorname{reg} V = \lceil (\deg V - 1)/\operatorname{codim} V \rceil$, then V is a divisor on a Del Pezzo variety.*"

Finally we describe examples of nondegenerate non-ACM Buchsbaum curves V with $\operatorname{reg} V = \lceil (\deg V - 1)/\operatorname{codim} V \rceil$ on a classical Del Pezzo surface.

Example 2.10 ([12, (4.2), (4.3)])**.** Let $Y = \mathbb{P}_k^1 \times \mathbb{P}_k^1$. We write $\mathcal{O}_Y(a,b)$ for $\pi_1^* \mathcal{O}_Y(a) \otimes \pi_2^* \mathcal{O}_Y(b)$, where π_1 and π_2 are the first and second projections respectively. Let Z_1 and Z_2 be divisors corresponding to $\mathcal{O}_Y(1,0)$ and $\mathcal{O}_Y(0,1)$ respectively. Through a 2-uple embedding of Y by $H = 2Z_1 + 2Z_2$, Y is a Del Pezzo surface of degree 8 in $\mathbb{P}_k^8$. Let V be a divisor on Y linearly equivalent to $aZ_1 + bZ_2$, where $a \leq b$. Then V is a non-ACM Buchsbaum variety with $\operatorname{reg} V = \lceil (\deg V - 1)/\operatorname{codim} V \rceil$ if and only if $(a,b) = (2m, 2m+2)$ for $5 \leq m \leq 11$, $(a,b) = (2m, 2m+3)$ for $3 \leq m \leq 9$, $(a,b) = (2m+1, 2m+4)$ for $6 \leq m \leq 12$ or $(a,b) = (2m+1, 2m+5)$ for $4 \leq m \leq 10$.

Let $\pi : Y = \mathbb{P}(\mathcal{E}) \to \mathbb{P}_k^1$ be a projective bundle, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}_k^1} \oplus \mathcal{O}_{\mathbb{P}_k^1}(-1)$. Let Z be a minimal section and F be a fiber of π. Through an embedding of Y by $H = 2Z + 3F$, Y is a Del Pezzo surface of degree 8 in $\mathbb{P}_k^8$. Let V be a divisor on Y linearly equivalent to $aZ + bF$. Then V is a non-ACM Buchsbaum variety with $\operatorname{reg} V = \lceil (\deg V - 1)/\operatorname{codim} V \rceil$ if and only if $(a,b) = (2m, 3m+2)$ for $5 \leq m \leq 11$, $(a,b) = (2m+1, 3m+3)$ for $2 \leq m \leq 8$ or $(a,b) = (2m+1, 3m+4)$ for $7 \leq m \leq 13$.

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