GENERATORS OF A PICARD MODULAR GROUP
IN TWO COMPLEX DIMENSIONS

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ABSTRACT. The goal of the article is to prove that four explicitly given trans-
formations, two Heisenberg translations, a rotation and an involution generate
the Picard modular group with Gaussian integers acting on the two dimen-
sional complex hyperbolic space. The result answers positively a question
raised by A. Kleinschmidt and D. Persson.

1. INTRODUCTION

Our main goal in this article is to give a simple, self-contained proof that four
explicitly given transformations, two Heisenberg translations, a rotation and an
involution generate the two dimensional Picard modular group with Gaussian inte-
gers acting on the two dimensional complex hyperbolic space. The result answers
positively a question raised by A. Kleinschmidt and D. Persson [KP], and it is used
in the work [BKNPP] on string compactifications. The method used in the paper
gives a simple algorithm to decompose any transformation in the Picard group as
a product of the generators.

The complex hyperbolic space $\mathbf{CH}^2$ is the rank one Hermitian symmetric space
of noncompact type, $SU(2,1)/S(U(2) \times U(1))$. A standard model of the com-
plex hyperbolic space is the complex unit ball of $\mathbf{C}^2$, $B^2 = \{z \in \mathbf{C}^2; \ |z| < 1\}$
equipped with the Bergman metric $g = \sum_{j,k=1}^2 g_{j,k}(z)dz_j \otimes d\bar{z}_k$, where $g_{j,k} =$
const $\cdot \partial_j \overline{\partial}_k \log(1 - |z|^2)$. This model is the bounded realization of the Hermitian
symmetric space $\mathbf{CH}^2$. We shall use the unbounded hyperquadric model of the
complex hyperbolic space, that is, $D^2 = \{z \in \mathbf{C}^2; \ \Re e z_2 > \frac{1}{2}|z_1|^2\}$.

The holomorphic automorphism group of $\mathbf{CH}^2$, $\mathbf{Aut}(\mathbf{CH}^2)$, consists of rational
functions $G = (g_1, g_2) : D^2 \mapsto D^2$,

$$(1.1) \qquad g_j(z) = \frac{g_{j+1,1} + g_{j+1,2}z_1 + g_{j+1,3}z_2}{g_{1,1} + g_{1,2}z_1 + g_{1,3}z_2},$$

$j = 1, 2$. These automorphisms act linearly in homogeneous coordinates $(\zeta_0, \zeta_1, \zeta_2)$,
$z_j = \frac{\zeta_j}{\zeta_0}$, $j = 1, 2$. The corresponding matrix $G = (g_{jk})_{j,k=1}^3$ satisfies the condition

$$(1.2) \qquad\qquad G^*CG = C,$$

where

$$C \equiv \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The determinant of the matrix G is normalized to be equal to 1. The matrix C is the matrix of the quadratic form of a defining function of D^2 written in homogeneous coordinates. More precisely,

$$D^2 = \{[\zeta] \in \mathbf{CP}^2;\ r(\zeta) \equiv \langle C\zeta, \zeta \rangle = -\zeta_2\bar\zeta_0 - \zeta_0\bar\zeta_2 + |\zeta_1|^2 < 0\}.$$

The Picard modular groups are

$$SU(2, 1; \mathcal{O}_d),$$

where $\mathcal{O}_d$ is the ring of algebraic integers of the imaginary quadratic extension $\mathbf{Q}(i\sqrt{d})$ for any positive squarefree integer d (see Holzapfel [H1]). The elements of the ring $\mathcal{O}_d$ can be described easily (Hardy-Wright [HW]):

$$\mathcal{O}_d = \begin{cases} \mathbf{Z}[i\sqrt{d}] & \text{if } d \equiv 1, 2 \pmod 4, \\ \mathbf{Z}[\frac{1+i\sqrt{d}}{2}] & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

It is well known that the ring $\mathcal{O}_d$ is Euclidean for positive square free integer d if and only if $d = 1, 2, 3, 7, 11$; see [ST], page 92. The Picard modular groups $SU(2, 1; \mathcal{O}_d)$ are discrete holomorphic automorphism subgroups of $\mathbf{CH}^2$.

Geometric and spectral properties of discrete lattices acting on complex hyperbolic spaces has recently attracted a lot of attention. See, for example, the work of Goldman and Parker [GP], Francsics and Lax [FL1], [FL2], [FL3], Falbel and Parker [FP], Deraux, Falbel and Paupert [DFP], Schwartz [Sch], or the survey paper of Parker [P] on the geometric properties of complex hyperbolic lattices. Spectral properties of the automorphic complex hyperbolic Laplace-Beltrami operator were investigated by Epstein, Melrose and Mendoza [EMM], Reznikov [R], and Lindenstrauss and Venkatesh [LV]. Despite the remarkable progress, several important algebraic, geometric and analytic problems are still open. To name a few, we mention the existence of nonarithmetic lattices or the existence of embedded eigenvalues in the continuous spectrum of the automorphic Laplace-Beltrami operator. A general construction of a fundamental domain was obtained in [GR] for Lie groups. However, the exact algebraic and geometric structure is known explicitly only for very few lattices and fundamental domains in higher dimensions. This is in sharp contrast to the case of the real hyperbolic spaces $\mathbf{H}^2$ and $\mathbf{H}^3$. Since the influential work of Mostow [M], it is well known that discrete holomorphic automorphism subgroups acting on the complex hyperbolic space $\mathbf{CH}^n$ are particularly hard to analyze.

2. Preliminaries

Three important classes of holomorphic automorphisms are Heisenberg translations, dilations, and rotations.

The Heisenberg translation by $a \in \partial D^2$, $N_a \in \mathbf{Aut}(\mathbf{CH}^2)$ is defined as

$$N_a(z_1, z_2) = (z_1 + a_1, z_2 + a_2 + z_1\bar a_1).$$

If we write $a = (a_1, a_2) = (\gamma, \frac{1}{2}|\gamma|^2 + ir)$ with $\gamma \in \mathbf{C}$, $r \in \mathbf{R}$, then the Heisenberg translation is given by $N_{(\gamma, \frac{1}{2}|\gamma|^2 + ir)}(z_1, z_2) = (z_1 + \gamma, z_2 + \frac{1}{2}|\gamma|^2 + ir + z_1\bar\gamma)$. The

corresponding matrix representation is

$$N_a \equiv \begin{pmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_2 & \bar{a}_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ \frac{1}{2}|\gamma|^2 + ir & \bar{\gamma} & 1 \end{pmatrix}.$$

The product of the Heisenberg translations N_a, N_b is the Heisenberg translation

(2.1) $$N_a \circ N_b = N_{(a_1+b_1, a_2+b_2+\bar{a}_1 b_1)}$$

with the parameter $(a_1 + b_1, a_2 + b_2 + \bar{a}_1 b_1) \in \partial D^2$. Moreover, the inverse of N_a is the Heisenberg translation

(2.2) $$N_a^{-1} = N_{(-a_1, -a_2 + |a_1|^2)}.$$

The holomorphic automorphism of D^2,

$$A_\delta(z) = (\delta z_1, \delta^2 z_2),$$

is called a dilation with parameter $\delta > 0$. Its matrix representation is

$$A \equiv \begin{pmatrix} \frac{1}{\delta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \delta \end{pmatrix}.$$

Rotation in the first variable by $e^{i\varphi}$,

$$M_{e^{i\varphi}}(z_1, z_2) = (e^{i\varphi}z_1, z_2),$$

is a holomorphic automorphism of D^2 with $\varphi \in \mathbf{R}$. There are three matrices,

$$M \equiv \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta^{-2} & 0 \\ 0 & 0 & \beta \end{pmatrix},$$

$\beta = e^{-i\varphi/3 + 2\pi i k/3}$, $k = 0, 1, 2$, corresponding to the same rotation.

The holomorphic involution

$$J(z_1, z_2) = (z_1/z_2, 1/z_2)$$

will also play a significant role. A matrix representation of J is

$$J \equiv \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Notice that $J^2 = I$ and J maps ∞ into $(0, 0)$.

Let z be a boundary point of D^2, i.e. $z \in \partial D^2 \cup \{\infty\}$. The stabilizer subgroup (isotropy subgroup) Γ_z of z contains all the holomorphic automorphisms that leave z fixed, that is $\Gamma_z \equiv \{g \in \mathrm{SU}(2,1); g(z) = z\}$. The stabilizer subgroup of ∞ consists of lower triangular matrices, that is

$$\mathcal{P} \equiv \Gamma_\infty = \{P \in \mathrm{SL}(3, \mathbf{C}); P^*CP = C, \ p_{12} = p_{13} = p_{23} = 0\}.$$

The Langlands decomposition of the stabilizer subgroup of ∞, $\mathcal{P} \equiv \Gamma_\infty$, will play an important role in our method. Any element of the stabilizer subgroup $P \in \mathcal{P}$ can be decomposed as a product of a Heisenberg translation, dilation, and a rotation:

$$P = \begin{pmatrix} p_{11} & 0 & 0 \\ p_{21} & p_{22} & 0 \\ p_{31} & p_{32} & p_{33} \end{pmatrix} = NAM = \begin{pmatrix} \frac{\beta}{\delta} & 0 & 0 \\ \frac{\beta\gamma}{\delta} & \beta^{-2} & 0 \\ \frac{\beta}{\delta}(\frac{1}{2}|\gamma|^2 + ir) & \bar{\gamma}\beta^{-2} & \beta\delta \end{pmatrix}.$$

The parameters $\beta, \gamma \in \mathbf{C}$ and $\delta, r \in \mathbf{R}$ satisfy the restrictions $|\beta| = 1$ and $\delta > 0$.

We recall from [FL1], [FL2] that the Langlands decomposition can also be used to parametrize a holomorphic automorphism $G = (g_{jk})_{j,k=1}^3$ not in the stabilizer subgroup of the ideal point ∞. Let $N_{G(\infty)}$ be the Heisenberg translation mapping $(0,0)$ into $G(\infty)$. Then the transformation $P \equiv JN_{G(\infty)}^{-1}G$ belongs to the stabilizer subgroup of ∞, so

$$(2.3) \qquad G = N_{G(\infty)}JP = N_{G(\infty)}JNAM.$$

The transformations N and P in the decomposition of G are not necessarily in the Picard modular group $\Gamma \equiv SU(2,1;\mathcal{O}_d)$, even if $G \in \Gamma$. The entries of N, P are not necessarily integers in the ring $\mathcal{O}_d$. However,

$$(2.4) \qquad g_{1j} = -p_{3j}, \;\; j = 1,2,3,$$

and so p_{3j}, $j = 1, 2, 3$, are integers in the ring $\mathcal{O}_d$.

3. Statement of the results

It is well known that the modular group $PSL(2, \mathbf{Z}) \equiv SL(2, \mathbf{Z})/\{\pm I\}$ is generated by the transformations $z \mapsto z + 1$ and $z \mapsto -\frac{1}{z}$. A. Kleinschmidt and D. Persson [KP] raised the question of an analogous statement for the Picard modular group $SU(2,1;\mathbf{Z}[i])$, namely if there is a simple description of $SU(2,1;\mathbf{Z}[i])$ in terms of generators. Our main result is to give an elementary proof that the four transformations, $N_{(0,1)}$, $N_{(1+i,1)}$, M_i and J, are sufficient to generate $SU(2,1;\mathbf{Z}[i])$. An application of this description to instanton corrections in string theory can be found in [BKNPP]. From now on, we restrict ourselves to the case $d = 1$.

Theorem 3.1. *The Picard modular group $SU(2,1;\mathbf{Z}[i])$ is generated by the Heisenberg translations*

$$(3.1) \qquad N_{(0,i)} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ i & 0 & 1 \end{pmatrix}, \;\; N_{(1+i,1)} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 1+i & 1 & 0 \\ 1 & 1-i & 1 \end{pmatrix},$$

the rotation by i in the first coordinate

$$(3.2) \qquad M_i \equiv \begin{pmatrix} i & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{pmatrix},$$

and the involution

$$(3.3) \qquad J \equiv \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Remark 3.2. The method used in the paper is constructive. It gives an algorithm to decompose any transformation in $SU(2,1;\mathbf{Z}[i])$ as a product of the four generators in (3.1), (3.2), and (3.3). The main ingredients are the decomposition (2.3) and the Euclidean algorithm.

Remark 3.3. It would be interesting to know if our method can be extended to the other Euclidean rings $\mathcal{O}_d$, more precisely, to the Picard modular groups $SU(2,1;\mathcal{O}_d)$, $d = 2,3,7,11$. Little is known about the geometric and algebraic properties, e.g., explicit fundamental domains, generators, and presentations of

these Picard modular groups, except in the case $d = 3$ (see Falbel-Parker [FP]). We also mention a related result on quaternionic hyperbolic space: Woodward [W] obtained generators for the Picard modular group $PU(2, 1; \mathcal{H})$, where $\mathcal{H}$ is the ring of Hurwitz integral quaternions.

In [FL1], [FL2] an explicit fundamental domain was constructed for the Picard modular group $SU(2, 1; \mathbf{Z}[i])$. The symmetry of this fundamental domain under a nonholomorphic isometry was used to obtain information about the embedded eigenvalues (Maass cusp forms) of the automorphic Laplace-Beltrami operator of the group $SU(2, 1; \mathbf{Z}[i])$ in [FL3]. In [FFP] the authors construct a different fundamental domain; determine the geometric, combinatorial structure of this fundamental domain; and obtain a presentation of the group. See also the work [Y].

Remark 3.4. We mention that very little is known about the fundamental domains and combinatorial structure of the higher dimensional Picard modular groups $SU(n, 1; \mathcal{O}_d)$. More generally, it is a major challenge in complex hyperbolic geometry to understand the geometric structure of discrete lattices in higher dimensions.

4. PROOF

We start by characterizing the stabilizer subgroup of infinity, $\mathcal{P}$, and describing the arithmetic properties of the entries of a transformation in $\mathcal{P}$.

Lemma 4.1. *Let* $G = (g_{jk}) \in SU(2, 1)$. *Then* $G \in \mathcal{P}$ *if and only if* $g_{13} = 0$. *Moreover, if* $d = 1$, *then* $P \in \mathcal{P} \equiv \Gamma_\infty(2, 1; \mathbf{Z}[i])$ *if and only if the parameters in the Langlands decomposition of* P *satisfy the conditions*

$$(4.1) \qquad \delta = 1, \quad \beta = 1, i, -1, -i, \quad r \in \mathbf{Z}, \quad \gamma \in \mathbf{Z}[i], \quad |\gamma|^2 \in 2\mathbf{Z}.$$

Proof of Lemma 4.1. It is well known that if $G \in \mathcal{P}$, then G is lower triangular, so it is enough to prove the converse. Computing and comparing the entries in the lower right corner of (1.2) we obtain

$$-\bar{g}_{33}g_{13} + |g_{23}|^2 - \bar{g}_{13}g_{33} = 0.$$

So $g_{13} = 0$ implies that $g_{23} = 0$. Similarly, comparing the entries in the third row, second column gives the equation

$$-\bar{g}_{33}g_{12} + \bar{g}_{23}g_{22} - \bar{g}_{13}g_{32} = 0.$$

Thus $\bar{g}_{33}g_{12} = 0$. However, $g_{13} = g_{23} = 0$ and $\det G = 1$ exclude the fact that $g_{33} = 0$. Therefore $g_{12} = 0$ and G is lower triangular. Let $P \in \Gamma_\infty(2, 1; \mathbf{Z}[i])$. Since $p_{11} = \beta/\delta$ and $p_{33} = \beta\delta$ are nonzero Gaussian integers with $|\beta| = 1$ and $\delta > 0$, it follows immediately that $\delta = 1$ and $\beta = \pm 1, \pm i$. Moreover, $p_{31}/\beta = \frac{1}{2}|\gamma|^2 + ir$ and $p_{21}/\beta = \gamma$ are also Gaussian integers. This proves the second part of Lemma 4.1. $\square$

Proposition 4.2. *The stabilizer subgroup of infinity,* $\mathcal{P}$, *in the Picard modular group* $SU(2, 1; \mathbf{Z}[i])$ *is generated by the Heisenberg translations* $N_{(0,i)}$ *and* $N_{(1+i,1)}$ *and by the rotation* M_i.

Proof of Proposition 4.2. Let P be an element in the stabilizer subgroup $\mathcal{P}$. We know that P is lower triangular. According to Lemma 4.1 there is no dilation

component in its Langlands decomposition; that is,

$$P = NM = \begin{pmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ \frac{1}{2}|\gamma|^2 + ir & \bar{\gamma} & 1 \end{pmatrix} \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta^{-2} & 0 \\ 0 & 0 & \beta \end{pmatrix}.$$

Since $\beta^4 = 1$, the rotation in P is M_i, $M_{-1} = M_i^2$, $M_{-i} = M_i^3$, or $I = M_i^4$. Therefore the rotation component of P in the Langlands decomposition is generated by M_i.

According to (2.1) the Heisenberg translation part of P splits as

$$(4.2) \qquad N_{(\gamma, \frac{1}{2}|\gamma|^2 + ir)} = N_{(0, ri)} \circ N_{(\gamma, \frac{1}{2}|\gamma|^2)}.$$

Here $N_{(0, ri)}$ can be written as

$$(4.3) \qquad N_{(0, ri)} = N_{(0, i)}^r, \quad r \in \mathbf{Z},$$

observing that the inverse of $N_{(0, i)}$ is $N_{(0, -i)}$.

The next step is to decompose $N_{(\gamma, \frac{1}{2}|\gamma|^2)}$ as a product of Heisenberg translations in the directions $(1 + i, 1)$, $(-1 + i, 1)$, and $(0, i)$. Let $\gamma = m + in \in \mathbf{Z}[i]$. Then $|\gamma|^2 = m^2 + n^2 \in 2\mathbf{Z}$ according to (4.1). This means that m and n have the same parity. Therefore we can write γ as

$$\gamma = k(1 + i) + l(-1 + i)$$

with $k \equiv \frac{m+n}{2} \in \mathbf{Z}$, $l \equiv \frac{n-m}{2} \in \mathbf{Z}$ and $|\gamma|^2 = 2(k^2 + l^2)$. Since $(0, -2kli) \in \partial D^2$, $(k(1+i), k^2) \in \partial D^2$, and $(l(-1+i), l^2) \in \partial D^2$, it follows from (2.1) and (4.3) that

$$\begin{aligned} N_{(\gamma, \frac{1}{2}|\gamma|^2)} &= N_{(0, -2kli)} \circ N_{(\gamma, \frac{1}{2}|\gamma|^2 + 2kli)} \\ &= N_{(0, -2kli)} \circ N_{(k(1+i)+l(-1+i), k^2+l^2+2kli)} \\ &= N_{(0, -2kli)} \circ N_{(k(1+i), k^2)} \circ N_{(l(-1+i), l^2)} \\ (4.4) \qquad &= N_{(0, i)}^{-2kl} \circ N_{(k(1+i), k^2)} \circ N_{(l(-1+i), l^2)}. \end{aligned}$$

Here the inverse of $N_{(1+i, 1)}$ is $N_{(-1-i, 1)}$, and an easy induction argument shows that

$$(4.5) \qquad N_{(k(1+i), k^2)} = N_{(1+i, 1)}^k$$

for any $k \in \mathbf{Z}$. Moreover the third factor in (4.4) can be written as

$$(4.6) \qquad N_{(l(-1+i), l^2)} = M_i \circ N_{(l(1+i), l^2)} \circ M_i^{-1} = M_i \circ N_{(1+i, 1)}^l \circ M_i^{-1}$$

using (4.5). Combining (4.2), (4.3), (4.4), (4.5), and (4.6) we obtain that

$$\begin{aligned} N_{(\gamma, \frac{1}{2}|\gamma|^2 + ir)} &= N_{(0, i)}^r \circ N_{(0, i)}^{-2kl} \circ N_{(1+i, 1)}^k \circ M_i \circ N_{(1+i, 1)}^l \circ M_i^{-1} \\ &= N_{(0, i)}^{r - 2kl} \circ N_{(1+i, 1)}^k \circ M_i \circ N_{(1+i, 1)}^l \circ M_i^{-1}. \end{aligned}$$

This completes the proof of Proposition 4.2. $\qquad \square$

Proof of Theorem 3.1. Let $G = (g_{jk})_{j,k=1}^3$ be an element of the group $SU(2, 1; \mathbf{Z}[i])$. We may assume that G does not belong to the stabilizer subgroup of infinity, $\mathcal{P}$. Then $g_{13} \neq 0$ and G maps infinity to $(g_{23}/g_{13}, g_{33}/g_{13})$. Since $G(\infty)$ is in ∂D^2,

$$(4.7) \qquad \Re e \frac{g_{33}}{g_{13}} = \frac{1}{2} \left| \frac{g_{23}}{g_{13}} \right|^2.$$

Consider the Heisenberg translation $N_{G(\infty)}$ that maps $(0,0)$ to $G(\infty)$. Note that the translation $N_{G(\infty)}$ is not necessarily in the Picard modular group $SU(2,1;\mathbf{Z}[i])$ except if $|g_{13}| = 1$. However, we know from (2.3) that

$$JN_{G(\infty)}^{-1}G = P.$$

We will successively approximate $N_{G(\infty)}^{-1}$ by Heisenberg translations in the Picard group to decrease the value $|g_{13}|^2 \in \mathbf{Z}$ until it becomes 0. Then G belongs to the stabilizer subgroup $\mathcal{P}$ according to Lemma 4.1 and can be expressed as a product of the generators (3.1) and (3.2) according to Proposition 4.2. This approximation process has finitely many steps and uses the fact that the ring $\mathcal{O}_1 \equiv \mathbf{Z}[i]$ is Euclidean.

Write

$$(4.8) \qquad -\frac{g_{23}}{g_{13}} = x(1+i) + y(-1+i) = (x-y) + (x+y)i$$

with real numbers $x, y \in \mathbf{R}$. Select integers $m, n \in \mathbf{Z}$ such that

$$(4.9) \qquad |x-m| \le 1/2, \quad |y-n| \le 1/2,$$

i.e., the nearest integers in $\mathbf{R}$. Let

$$(4.10) \qquad \gamma = m - n + i(m+n),$$

and select an integer $k \in \mathbf{Z}$ such that

$$(4.11) \qquad \left| k + \Im m\left(\bar{\gamma}\frac{g_{23}}{g_{13}}\right) + \Im m\frac{g_{33}}{g_{13}} \right| \le \frac{1}{2}.$$

Then we approximate $N_{G(\infty)}^{-1}$ by the translation

$$N_{(\gamma,\frac{1}{2}|\gamma|^2+ik)} \equiv N_{(m-n+i(m+n),m^2+n^2+ik)}.$$

Notice that $N_{(\gamma,\frac{1}{2}|\gamma|^2+ik)}$ is in the Picard modular group because

$$(4.12) \qquad |\gamma|^2 = (m-n)^2 + (m+n)^2 = 2m^2 + 2n^2 \in 2\mathbf{Z}.$$

Next we calculate the entry in the upper right corner of the product:

$$(4.13) \qquad \begin{aligned} G_1 &\equiv JN_{(\gamma,\frac{1}{2}|\gamma|^2+ik)}G \\ &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ \frac{1}{2}|\gamma|^2 + ik & \bar{\gamma} & 1 \end{pmatrix} G \\ &= \begin{pmatrix} -\frac{1}{2}|\gamma|^2 - ik & -\bar{\gamma} & -1 \\ -\gamma & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} G. \end{aligned}$$

So $g_{13}^{(1)}$, the entry in the upper right corner of $G_1 = (g_{jk}^{(1)})$, is equal to

$$
\begin{aligned}
g_{13}^{(1)} &\equiv -(\tfrac{1}{2}|\gamma|^2 + ik)g_{13} - \bar{\gamma}g_{23} - g_{33} \\
&= -g_{13}\big(\tfrac{1}{2}|\gamma|^2 + ik + \bar{\gamma}\frac{g_{23}}{g_{13}} + \frac{g_{33}}{g_{13}}\big) \\
&= -g_{13}\big[\tfrac{1}{2}|\gamma|^2 + \Re e(\bar{\gamma}\frac{g_{23}}{g_{13}}) + \Re e\frac{g_{33}}{g_{13}}\big] \\
&\quad -ig_{13}\big[k + \Im m(\bar{\gamma}\frac{g_{23}}{g_{13}}) + \Im m\frac{g_{33}}{g_{13}}\big] \\
&\equiv -g_{13}(I_1 + iI_2).
\end{aligned}
$$
(4.14)

Using (4.7), (4.8), the definition of γ in (4.10), and (4.12) we can simplify I_1:

$$
\begin{aligned}
I_1 &= \frac{1}{2}|\gamma|^2 + \Re\left(\overline{\gamma}\frac{g_{23}}{g_{13}}\right) + \Re\frac{g_{33}}{g_{13}} \\
&= \frac{1}{2}|\gamma|^2 + \Re\left(\overline{\gamma}\frac{g_{23}}{g_{13}}\right) + \frac{1}{2}\left|\frac{g_{23}}{g_{13}}\right|^2 \\
&= \frac{1}{2}\left|\gamma + \frac{g_{23}}{g_{13}}\right|^2 \\
&= \frac{1}{2}\Big|(m-n) + (m+n)i - (x-y) - (x+y)i\Big|^2 \\
&= (x-m)^2 + (y-n)^2.
\end{aligned}
$$

Then (4.9) gives the upper bound

$$
|I_1| \le \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}.
$$
(4.15)

The selection of k in (4.11) gives the inequality $|I_2| \le 1/2$ for the second term in (4.14). Therefore we can estimate $g_{13}^{(1)}$ by combining (4.14) with (4.11) and (4.15):

$$
|g_{13}^{(1)}|^2 = |g_{13}|^2|I_1 + iI_2|^2 = |g_{13}|^2(I_1^2 + I_2^2) \le |g_{13}|^2\left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right] = \frac{1}{2}|g_{13}|^2.
$$

Repeating this approximation procedure finitely many times we reduce the matrix of the transformation G to the matrix of a transformation G_n with $g_{13}^{(n)} = 0$. However, according to Lemma 4.1, this condition implies that the G_n belongs to the stabilizer subgroup of infinity $\mathcal{P}$. Proposition 4.2 guarantees that G_n is generated by (3.1) and (3.2). Since the approximation procedure (4.13) uses the transformation J and transformations in $\mathcal{P}$, Proposition 4.2 implies that G is generated by (3.1), (3.2), and (3.3). This completes the proof of Theorem 3.1. □

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