SOME REMARKS ON SYMPLECTIC INJECTIVE STABILITY

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ABSTRACT. It is shown that if A is an affine algebra of odd dimension d over an infinite field of cohomological dimension at most one, with $(d+1)!A = A$, and with $4|(d-1)$, then $\mathrm{Um}_{d+1}(A) = e_1\mathrm{Sp}_{d+1}(A)$. As a consequence it is shown that if A is a non-singular affine algebra of dimension d over an infinite field of cohomological dimension at most one, and $d!A = A$, and $4|d$, then $\mathrm{Sp}_d(A) \cap \mathrm{ESp}_{d+2}(A) = \mathrm{ESp}_d(A)$. This result is a partial analogue for even-dimensional algebras of the one obtained by Basu and Rao for odd-dimensional algebras earlier.

1. INTRODUCTION

The rings R we work with are always commutative with 1. In [2] Bass-Milnor-Serre began the study of the stabilization for the linear groups $\mathrm{GL}_n(R)/\mathrm{E}_n(R)$, for $n \geq 3$. They showed that the natural map $\mathrm{GL}_n(R)/\mathrm{E}_n(R) \longrightarrow \mathrm{GL}_{n+1}(R)/\mathrm{E}_{n+1}(R)$ is an isomorphism for $n \geq d+3$, where d is the Jacobson-Krull dimension of R (i.e. the dimension of its maximum spectrum). In [3], §11, Bass conjectured that $K_1(R) = \mathrm{GL}_{d+2}(R)/\mathrm{E}_{d+2}(R)$, where d is the Jacobson-Krull dimension. In [23], L.N. Vaserstein proved this conjecture for an algebra A, which is finite as a module over a commutative ring R and whose spectrum of maximal ideals is a noetherian space of dimension d.

In [14] R.A. Rao and W. van der Kallen showed that if A is a non-singular affine algebra of dimension $d > 1$ over a field of cohomological dimension at most one, then the natural map $\mathrm{SL}_n(A)/\mathrm{E}_n(A) \longrightarrow \mathrm{SL}_{n+1}(A)/\mathrm{E}_{n+1}(A)$ is injective for $n \geq d+1$, if $(d+1)!A = A$. The expectation that the injective stability would improve over these special rings was motivated by the famous theorem of A. Suslin (in [22]) who showed that stably free projective modules of rank d over such rings were free, improving the earlier estimate of $(d+1)$ of H. Bass over general commutative rings.

L.N. Vaserstein in [24] pursued the question of injective stability for the other classical groups: the symplectic groups $\mathrm{Sp}_{2n}(R)/\mathrm{ESp}_{2n}(R)$, and the orthogonal groups $\mathrm{O}_{2n}(R)/\mathrm{EO}_{2n}(R)$. He showed that the natural maps $\mathrm{Sp}_{2n}(R)/\mathrm{ESp}_{2n}(R) \to \mathrm{Sp}_{2n+2}(R)/\mathrm{ESp}_{2n+2}(R)$ and $\mathrm{O}_{2n}(R)/\mathrm{EO}_{2n}(R) \to \mathrm{O}_{2n+2}(R)/\mathrm{EO}_{2n+2}(R)$ are bijective for $n \geq (2d+4)$, where d is the stable dimension of R.

In [4] it was shown that if A is a non-singular affine algebra of dimension d over a field of cohomological dimension at most one, then in the symplectic situation the

above map is injective for $n \geq d+1$. In ([6], Corollary 6.8) the above result of [4] is re-proved via a relative version of a lemma of L.N. Vaserstein in ([17], Lemma 5.5). Moreover, it is shown that if A is a non-singular affine algebra over a finite field or its algebraic closure, of even dimension $d \geq 4$, of characteristic $\neq 2$, then the above map is injective for $n \geq d$.

In this paper we extend the results in the symplectic case, and our main result is the following:

Theorem 1. *Let A be an affine algebra of even dimension d over a field k of cohomological dimension at most one. If $(d+1)!A = A$ and $4|d$, then a $\sigma \in Sp_d(A) \cap ESp_{d+2}(A)$ is symplectic homotopic to the identity. In particular, if A is non-singular, then the natural map*

$$\frac{\mathrm{Sp}_{2n}(R)}{\mathrm{ESp}_{2n}(R)} \longrightarrow \frac{\mathrm{Sp}_{2n+2}(R)}{\mathrm{ESp}_{2n+2}(R)}$$

is injective for $2n \geq \max\{4, d\}$.

The improvement in the bound is mainly due to the use of Suslin matrices, as done in [15], the emphasis here being to use the sizes when the Suslin matrices are symplectic. This explains the need for the condition on the dimension as the Suslin matrices exhibit four different characteristics depending on the size of the row. Here are a few more comments on the proof.

For an affine algebra of dimension d over a field of cohomological dimension at most one, A. Suslin in [22] has shown that $\mathrm{Um}_{d+1}(A) = e_1 \mathrm{SL}_{d+1}(A)$. As a consequence of our argument one can improve this result in some cases and show the following.

Theorem 2. *Let A be an affine algebra of odd dimension $d \geq 5$ over a field of cohomological dimension at most one. Assume that $d!A = A$. If $4|(d-1)$, then $Um_{d+1}(A) = e_1 Sp_{d+1}(A)$.*

Theorem 1 is a consequence of Theorem 2.

The study of injective stabilization is deeply connected to the question of whether stably free modules are free. We refer the reader to ([15], §2), where this connection has been detailed.

Suslin's expectation in [21] is that a stably free module of rank $d-1$ over an affine algebra over an algebraically closed field is free. J. Fasel has recently proved a remarkable result in [8] that this is true over smooth threefolds. In view of the connection mentioned above between freeness of projective modules defined by a unimodular row and stably elementary matrices, we expect that the correct estimate will be $2n \geq \max\{4, d-1\}$ for non-singular algebras A (with $d!A = A$) over a field of cohomological dimension one. (If this is shown, then one can show that Suslin's conjecture holds for affine algebras of dimension $d \equiv 1$ modulo (4).)

J. Fasel in ([8], Corollary 3.3) has shown that if A is a smooth affine algebra of dimension three over a field of cohomological dimension at most one and $3!A = A$, then $\mathrm{Um}_4(A)/\mathrm{E}_4(A)\mathrm{Sp}_4(A) = 0$, provided also that -1 is a square in A. This raises the question whether $\mathrm{Um}_{d+1}(R) = e_1\mathrm{Sp}_{d+1}(R)$ when $d!A = A$ and $2|(d-1)$ over an affine algebra of dimension d over a field of cohomological dimension at most one. Our method does not apply unless $4|(d-1)$ and $(d+1)!A = A$.

This is the first occasion that the *form* of the Suslin matrices has been used to get a positive feedback on the question of injective stability in the case of the

symplectic groups. On an earlier occasion in [16] the *theory of Suslin matrices* expounded by Jose-Rao in [9, 10] has been used to show that the injective stability bounds do not improve in general for the orthogonal groups; see ([5], Corollary 2.7), where an example of an affine algebra of dimension 3 over a field of cohomological dimension one is given for which the map between orthogonal quotients is bijective for $2n \geq 8$ but is not injective for $2n = 6$.

2. PRELIMINARIES

2.1. **A relative version of a lemma of L.N. Vaserstein.** We begin with some preliminary definitions.

A row $v = (v_1, \ldots, v_n) \in R^n$ is said to be *unimodular* if there are elements $w_1, \ldots, w_n$ in R such that $v_1 w_1 + \cdots + v_n w_n = 1$. $\mathrm{Um}_n(R)$ will denote the set of all unimodular rows $v \in R^n$. Let I be an ideal in R. We denote by $\mathrm{Um}_n(R, I)$ the set of all unimodular rows of length n which are congruent to $e_1 = (1, 0, \ldots, 0)$ modulo I. (If $I = R$, then $\mathrm{Um}_n(R, I)$ is $\mathrm{Um}_n(R)$.)

The group $\mathrm{GL}_n(R)$ of invertible matrices acts on R^n in a natural way: $v \mapsto v\sigma$, if $v \in R^n$, $\sigma \in \mathrm{GL}_n(R)$. This map preserves $\mathrm{Um}_n(R)$, so $\mathrm{GL}_n(R)$ acts on $\mathrm{Um}_n(R)$. Note that any subgroup G of $\mathrm{GL}_n(R)$ also acts on $\mathrm{Um}_n(R)$. If $v, w \in \mathrm{Um}_n(R)$, we denote $v \sim_G w$ or $v \in wG$ if there is a $g \in G$ such that $v = wg$.

Let $\mathrm{E}_n(R)$ denote the subgroup of $\mathrm{SL}_n(R)$ consisting of all *elementary* matrices, i.e. those matrices which are a finite product of the *elementary generators* $\mathrm{E}_{ij}(\lambda) = \mathrm{I}_n + \lambda e_{ij}$, $1 \leq i \neq j \leq n$, $\lambda \in R$, where $e_{ij} \in \mathrm{M}_n(R)$ has only one non-zero entry 1 in its (i, j)-th position.

Definition 2.1 (The Relative Groups $\mathrm{E}_n(I)$, $\mathrm{E}_n(R, I)$). Let I be an ideal of R. The relative elementary group $\mathrm{E}_n(I)$ is the subgroup of $\mathrm{E}_n(R)$ generated as a group by the elements $\mathrm{E}_{ij}(x)$, $x \in I$, $1 \leq i \neq j \leq n$.

The relative elementary group $\mathrm{E}_n(R, I)$ is the normal closure of $\mathrm{E}_n(I)$ in $\mathrm{E}_n(R)$.

An important theorem of A.A. Suslin in ([19], Corollary 1.4) is that $\mathrm{E}_n(R, I)$ is a normal subgroup of $\mathrm{GL}_n(R)$, for $n \geq 3$.

The following is a special case of Bass' stabilization results in ([1], Chapter 4, Theorem 3.4, pg. 183):

Proposition 2.2. *Let A be a commutative ring whose maximum spectrum has dimension $\leq d$, and let I be an ideal of A. Then $\mathrm{Um}_r(A, I) = e_1 E_r(A, I)$, for $r \geq \max\{3, d + 2\}$.*

Definition 2.3 (Symplectic Group $\mathrm{Sp}_{2n}(R)$). This is the isotropy group of the standard symplectic form $\psi_n = \sum_{i=1}^n e_{2i-1, 2i} - \sum_{i=1}^n e_{2i, 2i-1}$. In other words, $\mathrm{Sp}_{2n}(R) = \{\alpha \in \mathrm{GL}_{2n}(R) \mid \alpha^t \psi_n \alpha = \psi_n\}$.

Let σ denote the permutation of the natural numbers given by $\sigma(2i) = 2i - 1$ and $\sigma(2i - 1) = 2i$.

Definition 2.4 (Elementary Symplectic Group $\mathrm{ESp}_{2n}(R)$). We define for $z \in R$, $1 \leq i \neq j \leq 2n$,

$$se_{ij}(z) = \begin{cases} 1_{2n} + ze_{ij} & \text{if } i = \sigma(j), \\ 1_{2n} + ze_{ij} - (-1)^{i+j} z e_{\sigma(j)\sigma(i)} & \text{if } i \neq \sigma(j) \text{ and } i < j. \end{cases}$$

(It is easy to check that all these generators belong to $\mathrm{Sp}_{2n}(R)$.) We call them the elementary symplectic matrices over R, and the subgroup of $\mathrm{Sp}_{2n}(R)$ generated by them is called the elementary symplectic group $\mathrm{ESp}_{2n}(R)$.

Definition 2.5 (The Relative Group $\mathrm{ESp}_{2n}(I)$, $\mathrm{ESp}_{2n}(R, I)$)**.** Let I be an ideal of R. The relative elementary group $\mathrm{ESp}_{2n}(I)$ is the subgroup of $\mathrm{ESp}_{2n}(R)$ generated as a group by the elements $se_{ij}(x)$, $x \in I$ and $1 \leq i \neq j \leq 2n$.

The relative elementary group $\mathrm{ESp}_{2n}(R, I)$ is the normal closure of $\mathrm{ESp}_{2n}(I)$ in $\mathrm{ESp}_{2n}(R)$.

We now recall a well-known lemma of L.N. Vaserstein:

Lemma 2.6 ([17, Lemma 5.5])**.** *For any natural number $n \geq 2$ and any alternating matrix φ from $\mathrm{GL}_{2n}(R)$, we have*

$$e_1(\mathrm{E}_{2n}(R)) \;\; = \;\; e_1(\mathrm{E}_{2n}(R) \cap \mathrm{Sp}_{\varphi}(R)),$$

where

$$\mathrm{Sp}_{\varphi}(R)) \;\; = \;\; \{\alpha \in \mathrm{SL}_{2n}(R) \mid \alpha^t \varphi \alpha = \varphi\}.$$

In this special case when $\varphi = \psi_r$ the proof is much easier to establish. In this case his proof shows the following:

Lemma 2.7. *For any natural number $n \geq 2$, $e_1 \mathrm{E}_{2n}(R) = e_1 \mathrm{ESp}_{2n}(R)$.*

The relative version of Lemma 2.7 was established in [6]:

Theorem 2.8 ([6, Theorem 5.5])**.** *Let I be an ideal of R and $2R = R$. Let $v \in \mathrm{Um}_{2n}(R, I)$. Then $v\mathrm{E}_{2n}(R, I) = v\mathrm{ESp}_{2n}(R, I)$, for $n \geq 2$.*

Remark 2.9. One can similarly define the groups $\mathrm{ESp}_{\varphi}(R)$, $\mathrm{ESp}_{\varphi}(R, I)$, w.r.t. an ideal I of R, for any alternating matrix $\varphi \in \mathrm{SL}_{2n}(\mathbb{Z})$, which is given by a (signed) permutation matrix (i.e. which is a permutation matrix if you ignore the signs). We have all the above results (Lemma 2.6, Lemma 2.7, Theorem 2.8) even in this situation. In the sequel we will be using these results with $\varphi = J_r$, a special alternating matrix defined by A. Suslin. (See Subsection 2.2 below for the definition of J_r.)

We next recall the role of the excision algebra (introduced by W. van der Kallen in ([14], Proposition 3.3) to convert questions of actions of relative groups to actions of absolute groups. We abstract the proof of ([14], Proposition 3.3) as:

Lemma 2.10. *Let $v \in \mathrm{Um}_r(A)$, $r \geq 3$, with $v \equiv e_1$ modulo (t), for some $t \in A$. Let $v = e_1 + tw$, for $w \in A^r$. Let $B = A[Z]/(Z^2 - tZ)$. Let $v^* = e_1 + Zw$. Then its image in B^r is unimodular. Moreover, if it is completable (to a (symplectic) matrix of determinant one) over B, then v is completable (to a (symplectic) matrix of determinant one) over A which is congruent to the identity modulo (t).*

B is called the excision algebra w.r.t. (t).

We now recall a few more results which will be needed in the sequel.

2.2. The Suslin matrix. Given two vectors $v, w \in R^{r+1}$, A.A. Suslin in ([18], §5) gave an inductive process to construct the (Suslin) matrix $S_r(v, w)$. We recall this process: Let $v = (a_0, v_1)$, $w = (b_0, w_1)$, where $a_0, b_0 \in R$ and $v_1, w_1 \in \mathrm{M}_{1r}(R)$. Set $S_0(v, w) = a_0$, and set

$$S_r(v, w) \;\; = \;\; \begin{pmatrix} a_0 I_{2^{r-1}} & S_{r-1}(v_1, w_1) \\ -S_{r-1}(w_1, v_1)^T & b_0 I_{2^{r-1}} \end{pmatrix}.$$

In ([18], Lemma 5.1) it is noted that $S_r(v,w)S_r(w,v)^T = (v \cdot w^T)I_{2^r} = S_r(w,v)^T S_r(v,w)$, and that $\det S_r(v,w) = (v \cdot w^T)^{2^{r-1}}$, for $r \geq 1$.

A. Suslin introduced these matrices and showed that a unimodular row of the form $(a_0, a_1, a_2^2, \ldots, a_r^r)$ (which is the same as $\chi_{r!}([v])$, for $v = (a_0, a_1, \ldots, a_r)$ in the notation of L. Vaserstein in [25]) can be completed to an invertible matrix. In fact, in ([20], Proposition 2.2, Corollary 2.5) it is shown that there is a $\beta_r(v,w)$ with $[\beta_r(v,w)] = [S_r(v,w)]$ in $K_1(A)$ whose first row is $\chi_{r!}([v])$. In ([15], Corollary 4.3 (iii)) it is shown that if $v, w \in \mathrm{Um}_{r+1}(R, I)$, then we may assume that $\beta_r(v,w) \equiv I_{r+1}$ modulo I.

A.A. Suslin then describes a sequence of forms $J_r \in \mathrm{M}_{2^r}(R)$ by the recurrence formulae:

$$J_r = \begin{cases} 1 & \text{for } r = 0, \\ J_{r-1} \perp -J_{r-1}, & \text{for } r \text{ even}, \\ J_{r-1}\top - J_{r-1}, & \text{for } r \text{ odd}. \end{cases}$$

$$\left(\text{Here } \alpha \perp \beta = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \text{ while } \alpha\top\beta = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}. \right)$$

It is easy to see that $\det(J_r) = 1$, for all r, and that $J_r^T = J_r^{-1} = (-1)^{\frac{r(r+1)}{2}} J_r$. Moreover, J_r is antisymmetric if $r = 4k+1$ and $r = 4k+2$, whereas J_r is symmetric for $r = 4k$ and $r = 4k + 3$. In ([18], Lemma 5.3), it is noted that the following formulae are valid: The *Suslin identities* are

$$\begin{aligned} \text{for } r = 4k : (S_r(v,w)J_r)^T &= S_r(v,w)J_r; \\ \text{for } r = 4k+1 : S_r(v,w)J_rS_r(v,w)^T &= (v \cdot w^T)J_r; \\ \text{for } r = 4k+2 : (S_r(v,w)J_r)^T &= -S_r(v,w)J_r; \\ \text{for } r = 4k+3 : S_r(v,w)J_rS_r(v,w)^T &= (v \cdot w^T)J_r. \end{aligned}$$

Remark 2.11. Consider the Suslin matrix $S_r(v,w)$, with $\langle v, w \rangle = 1$. By the Suslin identities, this is a symplectic matrix w.r.t. J_r when $r \equiv 1$ modulo (4) and an orthogonal matrix w.r.t. J_r when $r \equiv 3$ modulo (4).

Lemma 2.13 is the relative symplectic analogue of ([15], Corollary 4.3). Before this we recall the excision ring $(\mathbb{Z} \oplus I)$ considered by W. van der Kallen in [12].

The excision ring $(\mathbb{Z} \oplus I)$. If I is an ideal in R, one can then construct the ring $\mathbb{Z} \oplus I$ with multiplication defined by

$$(n \oplus i)(m \oplus j) = (nm \oplus nj + mi + ij),$$

for $m, n \in \mathbb{Z}$, $i, j \in I$.

The maximal spectrum of the ring $\mathbb{Z} \oplus I$ is noetherian, being the union of finitely many subspaces of dimension $\leq \dim(R)$.

There is a natural homomorphism $\varphi : \mathbb{Z} \oplus I \longrightarrow R$ given by $(m \oplus i) \mapsto m + i \in R$.

Next recall the lemma which enables one to reduce to the relative case in [15]:

Lemma 2.12 ([15, Lemma 4.2]). *Let R be a ring, and let J be an ideal of R. Let $\alpha \in \mathrm{SL}_n(R)$ with $\alpha \equiv I_n$ modulo J. Then there is a unique matrix $S \in \mathrm{SL}_n(\mathbb{Z} \oplus J)$ such that $S \equiv I_n$ modulo $(0 \oplus J)$, and with $\varphi(S) = \alpha$, where $\varphi : \mathbb{Z} \oplus J \longrightarrow R$ is the natural homomorphism.*

Moreover, if $n = 2^m$, for some m, and α is a Suslin matrix of determinant one, then S is also a Suslin matrix of determinant one.

We shall follow a process of "cutting down of size" of the symplectic form J_r below; i.e. we shall look at the forms which are obtained successively from J_r by deleting their i-th and $\pi(i)$-th rows and columns, where π is the underlying permutation of J_r. For simplicity, we shall just write J_r for these forms, without referring to their size, as we are really considering the symplectic group of these forms sitting inside the symplectic group of J_r.

Lemma 2.13. *Let R be a commutative ring of dimension d, and let I be an ideal of R. Let $S_r(v, w) \in \mathrm{SL}_{2^r}(R, I)$, for $v, w \in \mathrm{Um}_{r+1}(R, I)$, and with $\langle v, w \rangle = 1$. Assume that $r \geq d$. Further assume that $r \equiv 1 \ (mod \ 4)$.*

Then there exists $\varepsilon_{J_r} \in \mathrm{ESp}(J_r)$ such that $S_r(v, w)\varepsilon_{J_r} = I_k \perp \gamma$, for some $\gamma \in \mathrm{Sp}_J(R, I)$ and for suitable k. Moreover, γ and $\beta_r(v, w)$ are stably elementarily equivalent.

Proof. Consider the corresponding Suslin matrix $S_r(v^*, w^*) \in \mathrm{SL}_{2^r}(\mathbb{Z} \oplus I)$ described in Lemma 2.12. As $r \equiv 1 \ (mod \ 4)$, $S_r(v^*, w^*) \in \mathrm{Sp}_{J_r}(\mathbb{Z} \oplus I)$.

There is a $\beta_r(v^*, w^*) \in S_r(v^*, w^*)\mathrm{E}_{2^r}(\mathbb{Z} \oplus I)$, with $e_1\beta_r(v^*, w^*) = \chi_{r!}([v^*])$, and with $\beta_r(v^*, w^*) \equiv I_{r+1}$, modulo I. This follows from ([15], Corollary 4.3).

Since the dimension of the maximal spectrum of $(\mathbb{Z} \oplus I)$ is d, $\mathrm{Um}_r(\mathbb{Z} \oplus I) = e_1\mathrm{E}_r(\mathbb{Z} \oplus I)$, for $r \geq \max\{4, d+2\}$. By Proposition 2.2,

$$\mathrm{Um}_q(\mathbb{Z} \oplus I, (0 \oplus I)) = e_1\mathrm{E}_q(\mathbb{Z} \oplus I, (0 \oplus I)),$$

for all $q \geq \max\{4, d+2\}$. By Theorem 2.8,

$$\mathrm{Um}_q(\mathbb{Z} \oplus I, (0 \oplus I)) = e_1\mathrm{ESp}_q(\mathbb{Z} \oplus I, (0 \oplus I)),$$

for all $q \geq \max\{4, d+2\}$. (By Remark 2.9 we may take this to be the elementary symplectic group w.r.t. the symplectic form J_r.)

Let $e_1S_r(v^*, w^*) = e_1\varepsilon_1$, for some $\varepsilon_1 \in \mathrm{ESp}_{J_r}(\mathbb{Z} \oplus I, (0 \oplus I))$. Then one has

$$S_r(v^*, w^*)\varepsilon_1^{-1} \in \mathrm{Sp}_{J_r}(\mathbb{Z} \oplus I),$$

whence is equal, after a suitable modification by an element of $\mathrm{ESp}_{J_r}(R)$, to $1 \perp S'$, for some S', which also has its $\pi(1)$-row and $\pi(1)$-column to be $e_{\pi(1)}$, respectively $e_{\pi(1)}^t$, where π denotes the underlying permutation of J_r. Let us rename the matrix obtained by removing this row and column and loosely write $S_r(v^*, w^*)\varepsilon_1^{-1} = \mathrm{I}_2 \perp S$ for suitable S derived from S' by removing a suitable row and column from it. This $S \in \mathrm{Sp}_{J_r}(\mathbb{Z} \oplus I)$, with the aforementioned convention.

Continue the above process with S in place of $S_r(v^*, w^*)$ above and with the corresponding "sub form" of J_r obtained by deleting the 1-st and $\pi(1)$-th rows and columns, etc. Note that since $S_r(v^*, w^*) \in \mathrm{Sp}_{J_r}(\mathbb{Z} \oplus I)$, it follows that $\gamma \in \mathrm{Sp}_{J_r}(\mathbb{Z} \oplus I, (0 \oplus I))$. The first assertion over the ring $(\mathbb{Z} \oplus I)$ follows.

Now use the natural homomorphism $(\mathbb{Z} \oplus I) \longrightarrow R$ to deduce the first assertion over any ring R. Thus $S_r(v, w)$ contains a matrix γ of size $(r+1)$ in its elementary symplectic class.

Since γ is stably elementarily equivalent to $S_r(v, w)$, it follows that γ and $\beta_r(v, w)$ are stably elementarily equivalent, as both are stably elementarily equivalent to $S_r(v, w)$. $\qquad\square$

2.3. **Nice group structures.** In the papers [12], [13] W. van der Kallen shows that the orbit space of unimodular rows modulo the elementary action $\mathrm{Um}_n(A)/\mathrm{E}_n(A)$ has an abelian group structure if $2 \leq d \leq 2n-4$, where d is the stable dimension of

A. In fact he defines the notion of a weak Mennicke n-symbol in [13] and shows that the orbit space is bijective to the universal weak Mennicke n-symbol $\mathrm{WMS}_n(A)$, under the above conditions.

In [7] such groups were called **nice** if they satisfied 'coordinate-wise multiplication", viz. if in this group,

$$[(a_1, a_2, \ldots, a_n)] * [(b_1, a_2, \ldots, a_n)] \quad = \quad [(a_1 b_1, a_2, \ldots, a_n)].$$

The group structure is nice when

- (a) ([7], Theorem 3.9) A is an affine algebra of dimension d over a field k of cohomological dimension at most one (c.d.$_2 \leq 1$ suffices) and $n = d+1$. This result has been reproved in ([11], Proposition 3.5) by a different method when k is algebraically closed.
- (b) ([11], Proposition 3.5) A is an affine algebra of dimension d over the algebraic closure of a finite field k (whose characteristic is not 2) and $n = d$.
- (c) ([7], Theorem 5.1) $A = R[X]$, where R is a noetherian local ring of dimension d, in which 2 is invertible, and $n = d + 1$.

In case the group structure is nice, then in ([7], Theorem 2.1) it is shown that the universal weak Mennicke symbol is isomorphic to the universal Mennicke symbol.

In ([7], Corollary 2.2) it is deduced from this that if A is a commutative ring of stable dimension d, then the first row $e_1 \sigma$ of any stably elementary $\sigma \in \mathrm{SL}_{d+1}(A)$ can be completed to an elementary matrix.

3. The main theorems

We are now ready to prove the main theorems stated in the introduction. We begin with

Proposition 3.1. *Let R be a commutative ring of dimension $d > 1$, and let I be an ideal of R. Assume that $d \equiv 1$ modulo (4). Let $v \in \mathrm{Um}_{d+1}(R, I)$. If $\mathrm{Um}_{d+1}(R, I)/\mathrm{E}_{d+1}(R, I)$ has a nice group structure, then $\chi_{d!}([v])$ can be completed to a symplectic matrix which is congruent to the identity modulo I.*

Proof. Choose $w \in \mathrm{Um}_{d+1}(R, I)$ such that $\langle v, w \rangle = 1$. By Lemma 2.13 there is a $\gamma \in \mathrm{Sp}_{d+1}(R, I)$ such that γ and $\beta_d(v, w)$ are stably elementarily equivalent. By Proposition 2.2, $(1 \perp \beta_d(v, w)\gamma^{-1}) \in \mathrm{E}_{d+2}(R, I)$.

By ([13], Theorem 2.2) there is an $\varepsilon \in \mathrm{E}_{d+1}(R, I)$ such that

$$\beta_d(v, w)\gamma^{-1}\varepsilon \quad = \quad \begin{pmatrix} 1 + tx & ty \\ z^T & M \end{pmatrix},$$

for some $t, x \in R$, $y, z \in \mathrm{M}_{1,d}(R)$. $M \in \mathrm{M}_d(R)$, with $\begin{pmatrix} 1 + tx & ty \\ z^T & M \end{pmatrix} \in \mathrm{E}_{d+1}(R, I)$.

Since the group structure on the orbit space $\mathrm{Um}_{d+1}(R, I)/\mathrm{E}_{d+1}(R, I)$ is nice,

$$[(1 + tx, ty)] \quad = \quad [(1 + tx, y)] = 1.$$

By Theorem 2.8, $[(1 + tx, ty)] = [(1 + tx, y)] = 1$ in $\mathrm{Um}_{d+1}(R, I)/\mathrm{ESp}_{d+1}(R, I)$.

Hence, $e_1 \beta_d(v, w) = e_1 \gamma^{-1} \varepsilon_0$, for some $\varepsilon_0 \in \mathrm{ESp}_{d+1}(R, I)$, and with $\gamma \in \mathrm{Sp}_{d+1}(R, I)$. The result follows. $\qquad \square$

As a consequence of Proposition 3.1 we get Theorem 2:

Theorem 3.2. *Let R be an affine algebra of dimension $d > 1$ over an infinite field of cohomological dimension at most one, and let $I = (f)$ be a principal ideal of R. Assume that $d \equiv 1$ modulo (4). Let $v \in \mathrm{Um}_{d+1}(R, I)$. Then v can be completed to a symplectic matrix which is congruent to the identity modulo I.*

Proof. By ([15], Proposition 5.4), $[v] = [\chi_{d!}([v'])]$ in $\mathrm{Um}_{d+1}(R, I)/\mathrm{E}_{d+1}(R, I)$.

By Theorem 2.8, $[v] = [\chi_{d!}([v'])]$ in $\mathrm{Um}_{d+1}(R, I)/\mathrm{ESp}_{d+1}(R, I)$.

Work with v'^* (the vector corresponding to v') over the excision algebra B w.r.t. I as was done in Lemma 2.10.

Now apply Proposition 3.1 to $v'^* \in \mathrm{Um}_{d+1}(B, (Z))/\mathrm{E}_{d+1}(B, (Z))$ to deduce that it can be completed to a symplectic matrix which is congruent to the identity modulo (Z). Now come back to A and deduce the result for v.

We can now establish Theorem 1:

Without loss of generality we may assume that $2n = d$. Let $\sigma \in \mathrm{Sp}_d(A) \cap \mathrm{ESp}_{d+2}(A)$. Let $\delta(T) \in \mathrm{ESp}_{d+2}(A[T])$ be a symplectic homotopy between I_{d+2} and $\mathrm{I}_2 \perp \sigma$, i.e. $\delta(1) = \mathrm{I}_2 \perp \sigma$ and $\delta(0) = \mathrm{I}_{d+2}$. Let $v(T) = e_1\delta(T) \in \mathrm{Um}_{d+2}(A[T])$.

By Theorem 3.2, applied to $v(T)$, and w.r.t. the principal ideal $I = (T^2 - T)$, we get $v(T) = e_1\alpha(T)$, for some $\alpha(T) \in \mathrm{Sp}_{d+2}(A[T], (T^2 - T))$. Now

$$\delta(T)\alpha(T)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & * \\ * & 0 & \eta(T) \end{pmatrix},$$

for some $\eta(T) \in \mathrm{Sp}_d(A[T])$. Clearly, $\eta(T)$ is a symplectic homotopy of σ.

If A is non-singular, then by ([4], Theorem 3.8), $\mathrm{Sp}_d(A_\mathfrak{p}[T]) = \mathrm{ESp}_d(A_\mathfrak{p}[T])$, for all $\mathfrak{p} \in \mathrm{Spec}(A)$. Consequently, $\eta(T) \in \mathrm{ESp}_d(A[T])$. Hence, $\sigma = \eta(1) \in \mathrm{ESp}_d(A)$. □

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