

ON DEMAZURE CRYSTALS FOR $U_q(G_2^{(1)})$

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ABSTRACT. We show that there exist suitable sequences $\{w^{(k)}\}_{k \geq 0}$ and $\{w'^{(k)}\}_{k \geq 0}$ of Weyl group elements for a given perfect crystal of level $l \geq 1$ such that the path realizations of the Demazure crystals $B_{w^{(k)}}(l\Lambda_0)$ and $B_{w'^{(k)}}(l\Lambda_2)$ for the quantum affine algebra $U_q(G_2^{(1)})$ have tensor-product-like structures with mixing index $\kappa = 1$.

1. INTRODUCTION

Let \mathfrak{g} be an affine Lie algebra (cf. [4]) and $U_q(\mathfrak{g})$ denote the associated quantized universal enveloping algebra (or quantum affine algebra) (cf. [15]) for generic “ q ”. It is known that the combinatorial properties of an integrable representation of \mathfrak{g} are the same as that of $U_q(\mathfrak{g})$ (cf. [15]). The crystal (or crystal base) [8] associated with an integrable representation of $U_q(\mathfrak{g})$ provides a useful tool to study its combinatorial properties. The crystal can be thought of as the $q = 0$ limit of the canonical base [14] or global crystal base [10]. In order to do the combinatorial analysis of the representation one needs the explicit realization of the associated crystal. The search for explicit realizations of crystals for integrable representations of quantum affine algebras $U_q(\mathfrak{g})$ led to the theory of affine and perfect crystals [5]. A perfect crystal B of level l for the quantum affine algebra $U_q(\mathfrak{g})$ can be thought of as a crystal for a level zero representation of the derived subalgebra $U'_q(\mathfrak{g})$ with certain properties. The crystal $B(\lambda)$ for an integrable highest weight $U_q(\mathfrak{g})$ -module $V(\lambda)$ with dominant highest weight λ of level $l \geq 1$ can be realized in terms of certain combinatorial objects called “paths” which are elements in the semi-infinite tensor product $\cdots \otimes B \otimes B \otimes B$, where B is a perfect crystal of level l for $U_q(\mathfrak{g})$. Many perfect crystals of level l for quantum affine algebras $U_q(\mathfrak{g})$ have been explicitly constructed by several researchers (for example see: [1], [2], [3], [6], [7], [9], [19]). More recently, in [17], a new coordinatized realization of a coherent family of perfect crystals for $U_q(G_2^{(1)})$ has been constructed with explicit actions of all Kashiwara operators.

Let W denote the Weyl group and $V(\lambda)$ denote the integrable highest weight $U_q(\mathfrak{g})$ -module with dominant highest weight λ . For each $w \in W$, the $U_q^+(\mathfrak{g})$ -submodule $V_w(\lambda)$ generated by the one-dimensional extremal weight space $V(\lambda)_{w\lambda}$ is called a Demazure module. In [11], Kashiwara defined the Demazure crystal

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$B_w(\lambda)$ for the Demazure module $V_w(\lambda)$ as a subset of the crystal $B(\lambda)$ for $V(\lambda)$ satisfying a certain recursive property. In [12], it has been shown that under certain conditions the Demazure crystals $B_w(\Lambda)$ have tensor-product-like structures. In this context, a certain parameter “ κ ” called the “mixing index” enters the picture. In [12] it is shown that for $\mathfrak{g} = A_n^{(1)}$, $\kappa \leq 2$. It is conjectured that $\kappa \leq 2$ for other quantum affine algebras. It is known that the mixing index κ is dependent on the perfect crystal B (for example, see [18]). In [13], it is shown that for each $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, D_{n+1}^{(2)}, A_{2n}^{(2)}$ and $\lambda = l\Lambda$ (where Λ is a dominant weight of level 1), there exists a sequence $\{w^{(k)}\}_{k \geq 0}$ of Weyl group elements and a perfect crystal B of level l such that the path realizations of the Demazure crystals $B_{w^{(k)}}(\lambda)$ have tensor-product-like structures with $\kappa = 1$. In [16], it is shown that for the quantum affine algebra $U_q(D_4^{(3)})$ and the perfect crystal constructed in [9], there exists a sequence $\{w^{(k)}\}_{k \geq 0}$ of Weyl group elements such that the path realizations of the Demazure crystals $B_{w^{(k)}}(l\Lambda_0)$ have tensor-product-like structures with $\kappa = 1$.

In this paper we show that for the perfect crystal $B = B^{1,l}$ of level l constructed in [17], there exist suitable sequences $\{w^{(k)}\}_{k \geq 0}$ of Weyl group elements such that the path realizations of the Demazure crystals $B_{w^{(k)}}(l\Lambda_r)$, ($r = 0, 2$) for $U_q(G_2^{(1)})$ have tensor-product-like structures with $\kappa = 1$.

2. QUANTUM AFFINE ALGEBRAS AND THE PERFECT CRYSTALS

In this section we recall necessary facts in crystal base theory for quantum affine algebras. Our basic references for this section are [4], [15], [10], [5], and [7].

Let $I = \{0, 1, \dots, n\}$ be the index set and let $A = (a_{ij})_{i,j \in I}$ be an affine Cartan matrix and $D = \text{diag}(s_0, s_1, \dots, s_n)$ be a diagonal matrix with all $s_i \in \mathbb{Z}_{>0}$ such that DA is symmetric. The *dual weight lattice* P^\vee is defined to be the free abelian group $P^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \dots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d$ of rank $n + 2$, whose complexification $\mathfrak{h} = \mathbb{C} \otimes P^\vee$ is called the *Cartan subalgebra*. We define the linear functionals α_i and Λ_i ($i \in I$) on \mathfrak{h} by

$$\alpha_i(h_j) = a_{ji}, \quad \alpha_i(d) = \delta_{i0}, \quad \Lambda_i(h_j) = \delta_{ij}, \quad \Lambda_i(d) = 0 \quad (i, j \in I).$$

The α_i 's are called the *simple roots* and the Λ_i 's are called the *fundamental weights*. We denote by $\Pi = \{\alpha_i \mid i \in I\}$ the set of simple roots. We also define the *affine weight lattice* to be $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subset \mathbb{Z}\}$. The quadruple (A, P^\vee, Π, P) is called an *affine Cartan datum*. We denote by \mathfrak{g} the affine Kac-Moody algebra corresponding to the affine Cartan datum (A, P^\vee, Π, P) (see [4]). Let δ denote the *null root* and c denote the canonical central element for \mathfrak{g} (see [4, Ch. 4]). Now the affine weight lattice can be written as $P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \dots \oplus \mathbb{Z}\Lambda_n \oplus \mathbb{Z}\delta$. Let $P^+ = \{\lambda \in P \mid \lambda(h_i) \geq 0 \text{ for all } i \in I\}$. The elements of P are called the *affine weights* and the elements of P^+ are called the *affine dominant integral weights*.

Let $\bar{P}^\vee = \mathbb{Z}h_0 \oplus \dots \oplus \mathbb{Z}h_n$, $\bar{\mathfrak{h}} = \mathbb{C} \otimes_{\mathbb{Z}} \bar{P}^\vee$, $\bar{P} = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \dots \oplus \mathbb{Z}\Lambda_n$ and $\bar{P}^+ = \{\lambda \in \bar{P} \mid \lambda(h_i) \geq 0 \text{ for all } i \in I\}$. The elements of \bar{P} are called the *classical weights* and the elements of \bar{P}^+ are called the *classical dominant integral weights*. The *level* of a (classical) dominant integral weight λ is defined to be $\ell = \lambda(c)$. We call the quadruple $(A, \bar{P}^\vee, \Pi, \bar{P})$ the *classical Cartan datum*.

For the convenience of notation, we define $[k]_x = \frac{x^k - x^{-k}}{x - x^{-1}}$, where k is an integer and x is a symbol. We also define $\begin{bmatrix} m \\ k \end{bmatrix}_x = \frac{[m]_x!}{[k]_x! [m-k]_x!}$, where m and k are nonnegative integers, $m \geq k \geq 0$, $[k]_x! = [k]_x [k-1]_x \cdots [1]_x$ and $[0]_x! = 1$.

The *quantum affine algebra* $U_q(\mathfrak{g})$ is the quantum group associated with the affine Cartan datum (A, P^\vee, Π, P) . That is, it is the associative algebra over $\mathbb{C}(q)$ with unity generated by e_i, f_i ($i \in I$) and q^h ($h \in P^\vee$) satisfying the following defining relations:

- (i) $q^0 = 1, q^h q^{h'} = q^{h+h'}$ for all $h, h' \in P^\vee$,
- (ii) $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$ for $h \in P^\vee$,
- (iii) $e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}$ for $i, j \in I$, where $q_i = q^{s_i}$ and $K_i = q^{s_i h_i}$,
- (iv) $\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(1-a_{ij}-k)} e_j e_i^{(k)} = 0$ for $i \neq j$,
- (v) $\sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(1-a_{ij}-k)} f_j f_i^{(k)} = 0$ for $i \neq j$,

where $e_i^{(k)} = \frac{e_i^k}{[k]_{q_i}!}$ and $f_i^{(k)} = \frac{f_i^k}{[k]_{q_i}!}$. We denote by $U'_q(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by $e_i, f_i, K_i^{\pm 1}$ ($i \in I$). The algebra $U'_q(\mathfrak{g})$ can be regarded as the quantum group associated with the classical Cartan datum $(A, \bar{P}^\vee, \Pi, \bar{P})$.

Definition 2.1. An *affine crystal* (respectively, a *classical crystal*) is a set B together with the maps $\text{wt} : B \rightarrow P$ (respectively, $\text{wt} : B \rightarrow \bar{P}$), $\tilde{e}_i, \tilde{f}_i : B \rightarrow B \cup \{0\}$ and $\varepsilon_i, \varphi_i : B \rightarrow \mathbb{Z} \cup \{-\infty\}$ ($i \in I$) satisfying the following conditions:

- (i) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \text{wt}(b) \rangle$ for all $i \in I$,
- (ii) $\text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i$ if $\tilde{e}_i b \in B$,
- (iii) $\text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i$ if $\tilde{f}_i b \in B$,
- (iv) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ if $\tilde{e}_i b \in B$,
- (v) $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ if $\tilde{f}_i b \in B$,
- (vi) $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for $b, b' \in B, i \in I$,
- (vii) if $\varphi_i(b) = -\infty$ for $b \in B$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

Definition 2.2. Let B_1 and B_2 be affine or classical crystals. A *crystal morphism* (or *morphism of crystals*) $\Psi : B_1 \rightarrow B_2$ is a map $\Psi : B_1 \cup \{0\} \rightarrow B_2 \cup \{0\}$ such that

- (i) $\Psi(0) = 0$;
- (ii) if $b \in B_1$ and $\Psi(b) \in B_2$, then $\text{wt}(\Psi(b)) = \text{wt}(b), \varepsilon_i(\Psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\Psi(b)) = \varphi_i(b)$ for all $i \in I$;
- (iii) if $b, b' \in B_1, \Psi(b), \Psi(b') \in B_2$ and $\tilde{f}_i b = b'$, then $\tilde{f}_i \Psi(b) = \Psi(b')$ and $\Psi(b) = \tilde{e}_i \Psi(b')$ for all $i \in I$.

A crystal morphism $\Psi : B_1 \rightarrow B_2$ is called an *isomorphism* if it is a bijection from $B_1 \cup \{0\}$ to $B_2 \cup \{0\}$.

For crystals B_1 and B_2 , we define the *tensor product* $B_1 \otimes B_2$ to be the set $B_1 \times B_2$ whose crystal structure is given as follows:

$$\begin{aligned} \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases} \\ \text{wt}(b_1 \otimes b_2) &= \text{wt}(b_1) + \text{wt}(b_2), \\ \varepsilon_i(b_1 \otimes b_2) &= \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \text{wt}(b_1) \rangle), \\ \varphi_i(b_1 \otimes b_2) &= \max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, \text{wt}(b_2) \rangle). \end{aligned}$$

Let B be a classical crystal. For an element $b \in B$, we define

$$\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b) \Lambda_i, \quad \varphi(b) = \sum_{i \in I} \varphi_i(b) \Lambda_i.$$

Definition 2.3. Let l be a positive integer. A classical crystal B is called a *perfect crystal of level l* if

- (1) there exists a finite-dimensional $U'_q(\mathfrak{g})$ -module with a crystal basis whose crystal graph is isomorphic to B ,
- (2) $B \otimes B$ is connected,
- (3) there exists a classical weight $\lambda_0 \in \bar{P}$ such that $\text{wt}(B) \subset \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\leq 0} \alpha_i$,
 $\#(B_{\lambda_0}) = 1$, where $B_{\lambda_0} = \{b \in B \mid \text{wt}(b) = \lambda_0\}$,
- (4) for any $b \in B$, $\langle c, \varepsilon(b) \rangle \geq l$,
- (5) for any $\lambda \in \bar{P}^+$ with $\lambda(c) = l$, there exist unique $b^\lambda, b_\lambda \in B$ such that $\varepsilon(b^\lambda) = \lambda = \varphi(b_\lambda)$.

The following crystal isomorphism theorem plays a fundamental role in the theory of perfect crystals.

Theorem 2.4 ([5]). *Let B be a perfect crystal of level l ($l \in \mathbb{Z}_{\geq 0}$). For any $\lambda \in \bar{P}^+$ with $\lambda(c) = l$, there exists a unique classical crystal isomorphism*

$$\Psi : B(\lambda) \xrightarrow{\sim} B(\varepsilon(b_\lambda)) \otimes B \quad \text{given by } u_\lambda \mapsto u_{\varepsilon(b_\lambda)} \otimes b_\lambda,$$

where u_λ is the highest weight vector in $B(\lambda)$ and b_λ is the unique vector in B such that $\varphi(b_\lambda) = \lambda$.

Set $\lambda_0 = \lambda$, $\lambda_{k+1} = \varepsilon(b_{\lambda_k})$, $b_0 = b_{\lambda_0}$, $b_{k+1} = b_{\lambda_{k+1}}$. Applying the above crystal isomorphism repeatedly, we get a sequence of crystal isomorphisms

$$\begin{aligned} B(\lambda) &\xrightarrow{\sim} B(\lambda_1) \otimes B &\xrightarrow{\sim} B(\lambda_2) \otimes B \otimes B &\xrightarrow{\sim} \dots \\ u_\lambda &\mapsto u_{\lambda_1} \otimes b_0 &\mapsto u_{\lambda_2} \otimes b_1 \otimes b_0 &\mapsto \dots \end{aligned}$$

In this process, we get an infinite sequence $\mathbf{p}_\lambda = (b_k)_{k=0}^\infty \in B^{\otimes \infty}$, which is called the *ground-state path of weight λ* . Let $\mathcal{P}(\lambda) := \{\mathbf{p} = (p(k))_{k=0}^\infty \in B^{\otimes \infty} \mid p(k) \in B, p(k) = b_k \text{ for all } k \gg 0\}$. The elements of $\mathcal{P}(\lambda)$ are called the λ -paths. The following result gives the *path realization* of $B(\lambda)$.

Proposition 2.5 ([5]). *There exists an isomorphism of classical crystals*

$$\Psi_\lambda : B(\lambda) \xrightarrow{\sim} \mathcal{P}(\lambda) \quad \text{given by } u_\lambda \mapsto \mathbf{p}_\lambda,$$

where u_λ is the highest weight vector in $B(\lambda)$.

3. $U_q(G_2^{(1)})$ -PERFECT CRYSTALS

In this section we recall the perfect crystal for the quantum affine algebra $U_q(G_2^{(1)})$ of level $l > 0$ constructed in [17].

First we fix the data for $G_2^{(1)}$. Let $\{\alpha_0, \alpha_1, \alpha_2\}$, $\{h_0, h_1, h_2\}$ and $\{\Lambda_0, \Lambda_1, \Lambda_2\}$ be the set of simple roots, simple coroots and fundamental weights, respectively. The Cartan matrix $A = (a_{ij})_{i,j=0,1,2}$ is given by

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix},$$

and its Dynkin diagram is given by

$$\circ_0 - \circ_1 \Rightarrow \circ_2.$$

The standard null root δ and the canonical central element c are given by

$$\delta = \alpha_0 + 2\alpha_1 + 3\alpha_2 \quad \text{and} \quad c = h_0 + 2h_1 + h_2,$$

where $\alpha_0 = 2\Lambda_0 - \Lambda_1 + \delta$, $\alpha_1 = -\Lambda_0 + 2\Lambda_1 - 3\Lambda_2$, $\alpha_2 = -\Lambda_1 + 2\Lambda_2$.

For a positive integer l define the set

$$B = \left\{ b = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \in (\mathbb{Z}_{\geq 0}/3)^6 \left| \begin{array}{l} 3x_3 \equiv 3\bar{x}_3 \pmod{2}, \\ \sum_{i=1,2} (x_i + \bar{x}_i) + \frac{x_3 + \bar{x}_3}{2} \leq l \\ x_1, \bar{x}_1, x_2 - x_3, \bar{x}_3 - \bar{x}_2 \in \mathbb{Z} \end{array} \right. \right\}.$$

For $b = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \in B$ we denote

$$(3.1) \quad s(b) = x_1 + x_2 + \frac{x_3 + \bar{x}_3}{2} + \bar{x}_2 + \bar{x}_1, t(b) = x_2 + \frac{x_3 + \bar{x}_3}{2}$$

and

$$(3.2) \quad z_1 = \bar{x}_1 - x_1, \quad z_2 = \bar{x}_2 - \bar{x}_3, \quad z_3 = x_3 - x_2, \quad z_4 = (\bar{x}_3 - x_3)/2,$$

and define

$$(3.3) \quad A = (0, z_1, z_1 + z_2, z_1 + z_2 + 3z_4, z_1 + z_2 + z_3 + 3z_4, 2z_1 + z_2 + z_3 + 3z_4).$$

Now we define conditions (E_1) - (E_6) and (F_1) - (F_6) as follows:

$$(3.4) \quad \left\{ \begin{array}{l} (F_1) \quad z_1 + z_2 + z_3 + 3z_4 \leq 0, z_1 + z_2 + 3z_4 \leq 0, z_1 + z_2 \leq 0, z_1 \leq 0, \\ (F_2) \quad z_1 + z_2 + z_3 + 3z_4 \leq 0, z_2 + 3z_4 \leq 0, z_2 \leq 0, z_1 > 0, \\ (F_3) \quad z_1 + z_3 + 3z_4 \leq 0, z_3 + 3z_4 \leq 0, z_4 \leq 0, z_2 > 0, z_1 + z_2 > 0, \\ (F_4) \quad z_1 + z_2 + 3z_4 > 0, z_2 + 3z_4 > 0, z_4 > 0, z_3 \leq 0, z_1 + z_3 \leq 0, \\ (F_5) \quad z_1 + z_2 + z_3 + 3z_4 > 0, z_3 + 3z_4 > 0, z_3 > 0, z_1 \leq 0, \\ (F_6) \quad z_1 + z_2 + z_3 + 3z_4 > 0, z_1 + z_3 + 3z_4 > 0, z_1 + z_3 > 0, z_1 > 0. \end{array} \right.$$

(E_i) ($1 \leq i \leq 6$) is defined from (F_i) by replacing $>$ (resp. \leq) with \geq (resp. $<$).

Then for $b = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \in B$, we define $\tilde{e}_i(b), \tilde{f}_i(b), \varepsilon_i(b), \varphi_i(b)$, $i = 0, 1, 2$ as follows. We use the convention: $(a)_+ = \max(a, 0)$.

$$\tilde{e}_0(b) = \begin{cases} (x_1 - 1, \dots) & \text{if } (E_1), \\ (\dots, x_3 - 1, \bar{x}_3 - 1, \dots, \bar{x}_1 + 1) & \text{if } (E_2), \\ (\dots, x_2 - \frac{2}{3}, x_3 - \frac{2}{3}, \bar{x}_3 + \frac{4}{3}, \bar{x}_2 + \frac{1}{3}, \dots) & \text{if } (E_3) \text{ and } z_4 = -\frac{1}{3}, \\ (\dots, x_2 - \frac{1}{3}, x_3 - \frac{4}{3}, \bar{x}_3 + \frac{2}{3}, \bar{x}_2 + \frac{2}{3}, \dots) & \text{if } (E_3) \text{ and } z_4 = -\frac{2}{3}, \\ (\dots, x_3 - 2, \dots, \bar{x}_2 + 1, \dots) & \text{if } (E_3) \text{ and } z_4 \neq -\frac{1}{3}, -\frac{2}{3}, \\ (\dots, x_2 - 1, \dots, \bar{x}_3 + 2, \dots) & \text{if } (E_4), \\ (x_1 - 1, \dots, x_3 + 1, \bar{x}_3 + 1, \dots) & \text{if } (E_5), \\ (\dots, \bar{x}_1 + 1) & \text{if } (E_6), \end{cases}$$

$$\tilde{f}_0(b) = \begin{cases} (x_1 + 1, \dots) & \text{if } (F_1), \\ (\dots, x_3 + 1, \bar{x}_3 + 1, \dots, \bar{x}_1 - 1) & \text{if } (F_2), \\ (\dots, x_3 + 2, \dots, \bar{x}_2 - 1, \dots) & \text{if } (F_3), \\ (\dots, x_2 + \frac{1}{3}, x_3 + \frac{4}{3}, \bar{x}_3 - \frac{2}{3}, \bar{x}_2 - \frac{2}{3}, \dots) & \text{if } (F_4) \text{ and } z_4 = \frac{1}{3}, \\ (\dots, x_2 + \frac{2}{3}, x_3 + \frac{2}{3}, \bar{x}_3 - \frac{4}{3}, \bar{x}_2 - \frac{1}{3}, \dots) & \text{if } (F_4) \text{ and } z_4 = \frac{2}{3}, \\ (\dots, x_2 + 1, \dots, \bar{x}_3 - 2, \dots) & \text{if } (F_4) \text{ and } z_4 \neq \frac{1}{3}, \frac{2}{3}, \\ (x_1 + 1, \dots, x_3 - 1, \bar{x}_3 - 1, \dots) & \text{if } (F_5), \\ (\dots, \bar{x}_1 - 1) & \text{if } (F_6), \end{cases}$$

$$\tilde{e}_1(b) = \begin{cases} (\dots, \bar{x}_2 + 1, \bar{x}_1 - 1) & \text{if } z_2 \geq (-z_3)_+, \\ (\dots, x_3 + 1, \bar{x}_3 - 1, \dots) & \text{if } z_2 < 0 \leq z_3, \\ (x_1 + 1, x_2 - 1, \dots) & \text{if } (z_2)_+ < -z_3, \end{cases}$$

$$\tilde{f}_1(b) = \begin{cases} (x_1 - 1, x_2 + 1, \dots) & \text{if } (z_2)_+ \leq -z_3, \\ (\dots, x_3 - 1, \bar{x}_3 + 1, \dots) & \text{if } z_2 \leq 0 < z_3, \\ (\dots, \bar{x}_2 - 1, \bar{x}_1 + 1) & \text{if } z_2 > (-z_3)_+, \end{cases}$$

$$\tilde{e}_2(b) = \begin{cases} (\dots, \bar{x}_3 + \frac{2}{3}, \bar{x}_2 - \frac{1}{3}, \dots) & \text{if } z_4 \geq 0, \\ (\dots, x_2 + \frac{1}{3}, x_3 - \frac{2}{3}, \dots) & \text{if } z_4 < 0, \end{cases}$$

$$\tilde{f}_2(b) = \begin{cases} (\dots, x_2 - \frac{1}{3}, x_3 + \frac{2}{3}, \dots) & \text{if } z_4 \leq 0, \\ (\dots, \bar{x}_3 - \frac{2}{3}, \bar{x}_2 + \frac{1}{3}, \dots) & \text{if } z_4 > 0. \end{cases}$$

$$\varepsilon_1(b) = \bar{x}_1 + (\bar{x}_3 - \bar{x}_2 + (x_2 - x_3)_+)_+, \quad \varphi_1(b) = x_1 + (x_3 - x_2 + (\bar{x}_2 - \bar{x}_3)_+)_+,$$

$$\varepsilon_2(b) = 3\bar{x}_2 + \frac{3}{2}(x_3 - \bar{x}_3)_+, \quad \varphi_2(b) = 3x_2 + \frac{3}{2}(\bar{x}_3 - x_3)_+,$$

$$\varepsilon_0(b) = l - s(b) + \max A - (2z_1 + z_2 + z_3 + 3z_4), \quad \varphi_0(b) = l - s(b) + \max A.$$

For $b \in B$ if $\tilde{e}_i(b)$ or $\tilde{f}_i(b)$ does not belong to B , namely, if x_j or \bar{x}_j for some j becomes negative or $s(b)$ exceeds l , we understand it to be 0.

The following is one of the main results in [17]:

Theorem 3.1 ([17]). *For the quantum affine algebra $U_q(G_2^{(1)})$ the set $B = B^{1,l}$ equipped with the maps $\tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, i = 0, 1, 2$ is a perfect crystal of level l .*

As was shown in [17], the minimal elements are given by

$$(B)_{\min} = \{(\alpha, \beta, \beta, \beta, \beta, \alpha) \mid \alpha \in \mathbb{Z}_{\geq 0}, \beta \in (\mathbb{Z}_{\geq 0})/3, 2\alpha + 3\beta \leq l\}.$$

4. DEMAZURE MODULES AND DEMAZURE CRYSTALS

Let W denote the Weyl group of the affine Lie algebra \mathfrak{g} generated by the simple reflections $\{r_i | i \in I\}$, where $r_i(\mu) = \mu - \mu(h_i)\alpha_i$ for all $\mu \in \mathfrak{h}^*$. For $w \in W$ let $l(w)$ denote the length of w and let \prec denote the Bruhat order on W . Let $U_q^+(\mathfrak{g})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by the e_i 's. For $\lambda \in P^+$ with $\lambda(d) = 0$, consider the irreducible integrable highest weight $U_q(\mathfrak{g})$ -module $V(\lambda)$ with highest weight λ and highest weight vector u_λ . It is known that for $w \in W$, the extremal weight space $V(\lambda)_{w\lambda}$ is one-dimensional. Let $V_w(\lambda)$ denote the $U_q^+(\mathfrak{g})$ -module generated by $V(\lambda)_{w\lambda}$. These modules $V_w(\lambda)$ ($w \in W$) are called the Demazure modules. They are finite-dimensional subspaces of $V(\lambda)$ and satisfy the properties: $V(\lambda) = \bigcup_{w \in W} V_w(\lambda)$ and for $w, w' \in W$ with $w \preceq w'$, we have $V_w(\lambda) \subseteq V_{w'}(\lambda)$.

In 1993, Kashiwara [11] showed that for each $w \in W$, there exists a subset $B_w(\lambda)$ of $B(\lambda)$ such that

$$\frac{V_w(\lambda) \cap L(\lambda)}{V_w(\lambda) \cap qL(\lambda)} = \bigoplus_{b \in B_w(\lambda)} \mathbb{Q}b,$$

where $B(\lambda)$ is the crystal for $V(\lambda)$. The set $B_w(\lambda)$ is the crystal for the Demazure module $V_w(\lambda)$. The Demazure crystal $B_w(\lambda)$ has the following recursive property:

$$(4.1) \quad \text{If } r_i w \succ w, \text{ then } B_{r_i w}(\lambda) = \bigcup_{n \geq 0} \tilde{f}_i^n B_w(\lambda) \setminus \{0\}.$$

Suppose $\lambda(c) = l \geq 1$ (i.e., λ has level l) and suppose B is a perfect crystal of level l for $U_q(\mathfrak{g})$. Then the crystal $B(\lambda)$ is isomorphic to the set of paths $\mathcal{P}(\lambda) = \mathcal{P}(\lambda, B)$ (see Proposition 2.5). Under this isomorphism the highest weight element $u_\lambda \in B(\lambda)$ is identified with the ground state path $\mathbf{p}_\lambda = (\cdots \otimes b_3 \otimes b_2 \otimes b_1)$. We recall the path realizations of the Demazure crystals $B_w(\lambda)$ from [12]. Fix positive integers d and κ . For a sequence of integers $\{i_a^{(j)} \mid j \geq 1, 1 \leq a \leq d\} \subset \{0, 1, \dots, n\}$, we define the subsets $\{B_a^{(j)} \mid j \geq 1, 0 \leq a \leq d\}$ of B by

$$B_0^{(j)} = \{b_j\}, B_a^{(j)} = \bigcup_{k \geq 0} \tilde{f}_{i_a^{(j)}}^k B_{a-1}^{(j)} \setminus \{0\}.$$

Next we define $B_a^{(j+1, j)}, j \geq 1, 0 \leq a \leq d$ by

$$B_0^{(j+1, j)} = B_0^{(j+1)} \otimes B_d^{(j)}, B_a^{(j+1, j)} = \bigcup_{k \geq 0} \tilde{f}_{i_a^{(j+1)}}^k B_{a-1}^{(j+1, j)} \setminus \{0\},$$

and continue until we define

$$\begin{aligned} B_0^{(j+\kappa-1, \dots, j)} &= B_0^{(j+\kappa-1)} \otimes B_d^{(j+\kappa-2, \dots, j)}, \\ B_a^{(j+\kappa-1, \dots, j)} &= \bigcup_{k \geq 0} \tilde{f}_{i_a^{(j+\kappa-1)}}^k B_{a-1}^{(j+\kappa-1, \dots, j)} \setminus \{0\}. \end{aligned}$$

Furthermore, we define a sequence $\{w^{(k)}\}$ of elements in the Weyl group W by

$$w^{(0)} = 1, w^{(k)} = r_{i_a^{(j)}} w^{(k-1)}, \text{ for } k > 0.$$

Here, j and a are uniquely determined from k by the relation $k = (j-1)d + a, j \geq 1, 1 \leq a \leq d$. Now for $k \geq 0$, we define subsets $\mathcal{P}^{(k)}(\lambda, B)$ of $\mathcal{P}(\lambda, B)$ as follows:

$$\mathcal{P}^{(0)}(\lambda, B) = \{\mathbf{p}_\lambda\},$$

and for $k > 0$,

$$\mathcal{P}^{(k)}(\lambda, B) = \begin{cases} \cdots \otimes B_0^{(j+2)} \otimes B_0^{(j+1)} \otimes B_a^{(j, \dots, 1)} & \text{if } j < \kappa, \\ \cdots \otimes B_0^{(j+2)} \otimes B_0^{(j+1)} \otimes B_a^{(j, \dots, j-\kappa+1)} \otimes B^{\otimes(j-\kappa)} & \text{if } j > \kappa. \end{cases}$$

The following theorem shows that under certain conditions the path realizations of the Demazure crystals $B_{w^{(k)}}(\lambda)$ have tensor-product-like structures.

Theorem 4.1 ([12]). *Let $\lambda \in \bar{P}^+$ with Lie algebra $U_q(\mathfrak{g})$. For fixed positive integers d and κ , suppose we have a sequence of integers $\{i_a^{(j)} \mid j \geq 1, 1 \leq a \leq d\}$ in $\{0, 1, 2, \dots, n\}$ satisfying the conditions:*

- (1) for any $j \geq 1$, $B_d^{(j+\kappa-1, \dots, j)} = B_d^{(j+\kappa-1, \dots, j+1)} \otimes B$,
- (2) for any $j \geq 1, 1 \leq a \leq d$, $\langle \lambda_j, h_{i_a^{(j)}} \rangle \leq \varepsilon_{i_a^{(j)}}(b), b \in B_{a-1}^{(j)}$, and
- (3) the sequence of elements $\{w^{(k)}\}_{k \geq 0}$ is an increasing sequence with respect to the Bruhat order.

Then we have $B_{w^{(k)}}(\lambda) \cong \mathcal{P}^{(k)}(\lambda, B)$.

The positive integer κ in Theorem 4.1 is called the mixing index. It is conjectured that for any nonexceptional affine Lie algebra \mathfrak{g} , the mixing index $\kappa \leq 2$. It is known that the mixing index κ is dependent on the choice of the perfect crystal. For example, as seen in [18] for $U_q(C_n^{(1)})$, $\lambda = \Lambda_1$, $\kappa = 1$ when $B = B^{n,l}$, and $\kappa = 2$ when $B = B^{1,l}$. In [13], for each classical affine Lie algebra $\mathfrak{g}, \lambda = l\Lambda$ (where Λ is a dominant weight of level one), and for the coherent family of perfect crystals B given in [7], a sequence of Weyl group elements $\{w^{(k)}\}$ is given that satisfies the conditions in Theorem 4.1 with $\kappa = 1$. For $U_q(D_4^{(3)})$, $\lambda = l\Lambda_0$ and the perfect crystal B given in [9], a sequence of Weyl group elements $\{w^{(k)}\}$ is given in [16] that satisfies the conditions in Theorem 4.1 with $\kappa = 1$. In the next section, for $U_q(G_2^{(1)})$, $\lambda = l\Lambda_i, i = 0, 2$ and the perfect crystal $B = B^{1,l}$ given in [17], we give a suitable sequence of Weyl group elements $\{w^{(k)}\}$ that satisfies the conditions in Theorem 4.1 with $\kappa = 1$.

For $b \in B$, let $\tilde{f}_i^{\max}(b)$ denote $\tilde{f}_i^{\varphi_i(b)}(b)$. For $j \geq 1$, we set

$$b_0^{(j)} = b_j, \quad b_a^{(j)} = \tilde{f}_{i_a^{(j)}}^{\max}(b_{a-1}^{(j)}) \quad (a = 1, 2, \dots, d).$$

The following proposition ([12], Proposition 2) will be useful to check the validity of condition (3) in Theorem 4.1.

Proposition 4.2 ([12]). *For $w \in W$, if $\langle w\mu, h_j \rangle > 0$ for some $\mu \in \bar{P}^+$, then $r_j w \succ w$.*

5. $U_q(G_2^{(1)})$ -DEMAZURE CRYSTALS

In this section we show that for the perfect crystal of level l for the quantum affine algebra $U_q(G_2^{(1)})$ given in Section 3, there are suitable sequences of Weyl group elements $\{w^{(k)}\}$ which satisfy the conditions (1), (2) and (3) for $\lambda = l\Lambda_0$ (respectively, for $\lambda = l\Lambda_2$), and hence Theorem 4.1 holds in these cases with $\kappa = 1$. Thus we have path realizations of the corresponding Demazure crystals with tensor-product-like structures.

For $\lambda = l\Lambda_0, l \geq 1$, the $l\Lambda_0$ -minimal element in the perfect crystal B is $\bar{b} = (0, 0, 0, 0, 0, 0)$ and in this case $\lambda_j = \lambda = l\Lambda_0$ for $j \geq 1$. Hence $b_j = \bar{b}$ for all $j \geq 1$. Set $d = 6$ and choose the sequence $\{i_a^{(j)} | j \geq 1, 1 \leq a \leq 6\}$ defined by

$$(5.1) \quad i_1^{(j)} = 0, i_2^{(j)} = i_4^{(j)} = i_6^{(j)} = 1, i_3^{(j)} = i_5^{(j)} = 2.$$

Hence, by the action of \tilde{f}_i on B we have, for $j \geq 1$,

$$(5.2) \quad \begin{aligned} b_0^{(j)} &= (0, 0, 0, 0, 0, 0), \\ b_1^{(j)} &= \tilde{f}_0^{\max}(b_0^{(j)}) = (l, 0, 0, 0, 0, 0), \\ b_2^{(j)} &= \tilde{f}_1^{\max}(b_1^{(j)}) = (0, l, 0, 0, 0, 0), \\ b_3^{(j)} &= \tilde{f}_2^{\max}(b_2^{(j)}) = (0, 0, 2l, 0, 0, 0), \\ b_4^{(j)} &= \tilde{f}_1^{\max}(b_3^{(j)}) = (0, 0, 0, 2l, 0, 0), \\ b_5^{(j)} &= \tilde{f}_2^{\max}(b_4^{(j)}) = (0, 0, 0, 0, l, 0), \\ b_6^{(j)} &= \tilde{f}_1^{\max}(b_5^{(j)}) = (0, 0, 0, 0, 0, l). \end{aligned}$$

Furthermore, it can be easily seen that the subsets $\{B_a^{(j)} | j \geq 1, 0 \leq a \leq 6\}$ of B are given as follows:

$$\begin{aligned} B_0^{(j)} &= \{(0, 0, 0, 0, 0, 0)\}, \\ B_1^{(j)} &= \{(x_1, 0, 0, 0, 0, 0) | x_1 \leq l\}, \\ B_2^{(j)} &= \{(x_1, x_2, 0, 0, 0, 0) | x_1 + x_2 \leq l\}, \\ B_3^{(j)} &= \{(x_1, x_2, x_3, 0, 0, 0) | 3x_3 \text{ even}, x_1 + x_2 + \frac{1}{2}x_3 \leq l\}, \\ B_4^{(j)} &= \{(x_1, x_2, x_3, \bar{x}_3, 0, 0) | 3x_3 + 3\bar{x}_3 \text{ even}, x_1 + x_2 + \frac{1}{2}(x_3 + \bar{x}_3) \leq l\}, \\ B_5^{(j)} &= \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) | 3x_3 + 3\bar{x}_3 \text{ even}, x_1 + x_2 + \frac{1}{2}(x_3 + \bar{x}_3) + \bar{x}_2 \leq l\}, \\ B_6^{(j)} &= B. \end{aligned}$$

By direct calculation, we have the following lemma.

Lemma 5.1. *Let $k \in \mathbb{Z}_{>0}$ and $k = 6(j - 1) + a, j \geq 1, 1 \leq a \leq 6$. Then we have*

$w^{(k)}\Lambda_0 = \Lambda_0 - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2$, where

$$\begin{aligned} m_0 &= j^2, \\ m_1 &= \begin{cases} 2j^2 + 2j, & \text{if } a = 6, \\ 2j^2 - 2j, & \text{if } a = 1, \\ 2j^2 + j, & \text{if } a = 4, 5, \\ 2j^2 - j, & \text{if } a = 2, 3, \end{cases} \\ m_2 &= \begin{cases} 3j^2, & \text{if } a = 3, 4, \\ 3j^2 + 3j, & \text{if } a = 5, 6, \\ 3j^2 - 3j, & \text{if } a = 1, 2. \end{cases} \end{aligned}$$

Theorem 5.2. For $\lambda = l\Lambda_0$, $l \geq 1$ and the given perfect crystal B for the quantum affine algebra $U_q(G_2^{(1)})$ with $d = 6$ and the sequence $\{i_a^{(j)}\}$ given above, conditions (1), (2) and (3) in Theorem 4.1 hold. Hence we have path realizations of the corresponding Demazure crystals $B_{w^{(k)}}(l\Lambda_0)$ for $U_q(G_2^{(1)})$ with tensor-product-like structures.

Proof. We have already shown above by explicit descriptions of the subsets $B_a^{(j)}$ that condition (1) holds.

Since $\langle l\Lambda_0, h_{i_a^{(j)}} \rangle = 0$ for $2 \leq a \leq 6$, $\langle l\Lambda_0, h_{i_1^{(j)}} \rangle = l$ and $\varepsilon_{i_1^{(j)}}(0, 0, 0, 0, 0, 0) = l$, condition (2) holds.

To prove condition (3), we use Lemma 5.1 to obtain:

$$\langle w^{(k)}\Lambda_0, h_{i_{a+1}^{(j)}} \rangle = \begin{cases} j & \text{if } a = 1, 5, \\ 2j & \text{if } a = 3, \\ 2j + 1 & \text{if } a = 6, \\ 3j & \text{if } a = 2, 4, \end{cases}$$

where $k = 6(j - 1) + a$, $j \geq 1$. Hence by Proposition 4.2, $w^{(k+1)} = r_{i_{a+1}^{(j)}} w^{(k)} \succ w^{(k)}$, which implies that condition (3) holds. □

For $\lambda = l\Lambda_2$, $l \geq 1$, the $l\Lambda_2$ -minimal element in the perfect crystal B is $\bar{b} = (0, \frac{l}{3}, \frac{l}{3}, \frac{l}{3}, \frac{l}{3}, 0)$ and in this case, $\lambda_j = \lambda = l\Lambda_2$ for $j \geq 1$. Hence $b_j = \bar{b}$ for all $j \geq 1$. Set $d = 6$ and choose the sequence $\{i_a^{(j)} | j \geq 1, 1 \leq a \leq 6\}$ defined by

$$(5.3) \quad i_1^{(j)} = i_3^{(j)} = 2, i_2^{(j)} = i_4^{(j)} = i_6^{(j)} = 1, i_5^{(j)} = 0.$$

Hence, by the action of \tilde{f}_i on B we have for $j \geq 1$,

$$(5.4) \quad \begin{aligned} b_0^{(j)} &= (0, \frac{l}{3}, \frac{l}{3}, \frac{l}{3}, \frac{l}{3}, 0), \\ b_1^{(j)} &= \tilde{f}_2^{\max}(b_0^{(j)}) = (0, 0, l, \frac{l}{3}, \frac{l}{3}, 0), \\ b_2^{(j)} &= \tilde{f}_1^{\max}(b_1^{(j)}) = (0, 0, 0, \frac{4l}{3}, \frac{l}{3}, 0), \\ b_3^{(j)} &= \tilde{f}_2^{\max}(b_2^{(j)}) = (0, 0, 0, 0, l, 0), \\ b_4^{(j)} &= \tilde{f}_1^{\max}(b_3^{(j)}) = (0, 0, 0, 0, 0, l), \\ b_5^{(j)} &= \tilde{f}_0^{\max}(b_4^{(j)}) = (l, 0, 0, 0, 0, 0), \\ b_6^{(j)} &= \tilde{f}_1^{\max}(b_5^{(j)}) = (0, l, 0, 0, 0, 0). \end{aligned}$$

Furthermore, by direct calculation it can be seen that the subsets $\{B_a^{(j)} \mid j \geq 1, 0 \leq a \leq 6\}$ of B are as follows:

$$\begin{aligned}
 B_0^{(j)} &= \{(0, \frac{l}{3}, \frac{l}{3}, \frac{l}{3}, \frac{l}{3}, 0)\}, \\
 B_1^{(j)} &= B_0^{(j)} \cup \{(0, x_2, x_3, \frac{l}{3}, \frac{l}{3}, 0) \mid z_3 > 0, s(b) = l\}, \\
 B_2^{(j)} &= B_1^{(j)} \cup \{(0, x_2, x_3, \bar{x}_3, \frac{l}{3}, 0) \mid z_3 \geq 0, z_2 < 0, s(b) = l\}, \\
 B_3^{(j)} &= B_2^{(j)} \cup \{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid z_3 \geq 0, t(b) \leq \frac{2l}{3}, z_3 + 3z_4 \geq 0, s(b) = l\}, \\
 B_4^{(j)} &= B_3^{(j)} \cup \{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_3 \geq 0, t(b) \leq \frac{2l}{3}, \bar{x}_1 > 0, z_4 \geq 0, s(b) = l\} \\
 &\quad \cup \{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid z_3 \geq 0, t(b) \leq \frac{2l}{3}, \bar{x}_1 > 0, z_4 < 0, \\
 &\quad \quad \quad (-2z_2)_+ \leq z_3 + 3z_4, s(b) = l\}, \\
 B_5^{(j)} &= C_1 \cup C_2 \cup C_3 \cup D_1 \cup D_2 \cup \dots \cup D_{12}, \\
 B_6^{(j)} &= B,
 \end{aligned}$$

where $C_i = \{(0, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \mid (P_i) \text{ holds}\}, 1 \leq i \leq 3,$

$$\begin{aligned}
 D_j &= \{(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, 0) \mid (Q_j) \text{ holds}\}, 1 \leq j \leq 12, \\
 (P_1) \quad & z_1 + z_2 + z_3 + 3z_4 > 0, z_1 > 0, z_3 \geq 0, z_4 \geq 0, t(b) < \frac{2l}{3}, \\
 (P_2) \quad & z_1 > 0, z_3 \geq 0, z_4 < 0, (-2z_2)_+ \leq z_3 + 3z_4, t(b) < \frac{2l}{3}, \\
 (P_3) \quad & z_1 + z_2 + z_3 + 3z_4 \leq 0, z_2 < 0, z_1 > 0, z_4 \geq 0, z_1 + z_2 + 2z_3 + 3z_4 \geq 0, \\
 & \quad \quad \quad t(b) < \frac{2l}{3} - (z_1 + z_2 + z_3 + 3z_4), \\
 (Q_1) \quad & (-z_3)_+ \leq z_2, \\
 (Q_2) \quad & z_4 \leq 0, z_1 + z_2 > 0, z_3 + 2z_4 \leq -1, z_2 + z_3 < 0, t(b) < \frac{2l}{3} + z_1 - z_4, \\
 (Q_3) \quad & z_1 + z_2 + 3z_4 > 0, z_4 > 0, z_3 \leq 0, z_2 + z_3 < 0, t(b) < \frac{2l}{3} + z_1, \\
 (Q_4) \quad & z_1 + z_2 + z_3 + 3z_4 > 0, z_3 > 0, z_2 < 0, z_4 \geq 0, t(b) < \frac{2l}{3} + z_1, \\
 (Q_5) \quad & z_1 + 2z_2 + z_3 + 3z_4 > 0, z_3 > 0, z_2 < 0, z_4 < 0, t(b) < \frac{2l}{3} + z_1, \\
 (Q_6) \quad & z_1 + z_2 + z_3 + 3z_4 \leq 0, z_3 \geq 0, z_2 < 0, z_4 \geq 0, z_2 + 2z_3 + 3z_4 \geq 0, \\
 & \quad \quad \quad t(b) \leq \frac{2l}{3} + |z_2 + z_3 + 3z_4|, \\
 (Q_7) \quad & z_1 + z_2 + z_3 + 3z_4 \leq 0, z_3 \geq 0, z_2 < 0, z_4 \geq 0, z_2 + 2z_3 + 3z_4 < 0, t(b) \leq \frac{2l}{3}, \\
 (Q_8) \quad & z_3 \geq 0, z_2 < 0, z_4 < 0, z_3 + 2z_4 \leq -1, z_2 + 2z_3 + 3z_4 < 0, t(b) < \frac{2l}{3}, \\
 & \quad \quad \quad z_1 = 0 \text{ only when } s(b) < l, \\
 (Q_9) \quad & z_1 + z_2 + z_3 + 3z_4 \leq 0, z_3 \geq 0, z_2 < 0, z_4 < 0, z_1 = 0, s(b) = l, z_3 + 3z_4 \geq 0, \\
 & \quad \quad \quad t(b) \leq \frac{2l}{3}, \\
 (Q_{10}) \quad & z_1 + z_2 + z_3 + 3z_4 \leq 0, z_3 \geq 0, z_2 < 0, z_4 < 0, t(b) \leq \frac{2l}{3}, z_1 \neq 0 \text{ or } s(b) \neq l, \\
 (Q_{11}) \quad & z_1 + z_2 + 3z_4 \leq 0, z_1 + z_2 \leq 0, z_3 < 0, z_2 + z_3 < 0, z_4 \geq 0, t(b) < \frac{2l}{3}, \\
 (Q_{12}) \quad & z_1 + z_2 + 3z_4 \leq 0, z_1 + z_2 \leq 0, z_3 < 0, z_2 + z_3 < 0, z_4 < 0, \\
 & \quad \quad \quad t(b) < \frac{2l}{3} + z_1 - z_4.
 \end{aligned}$$

By direct calculation, we have the following lemma.

Lemma 5.3. *Let $k \in \mathbb{Z}_{>0}$ and $k = 6(j - 1) + a, j \geq 1, 1 \leq a \leq 6$. Then we have $w^{(k)}\Lambda_2 = \Lambda_2 - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2$, where*

$$m_0 = \begin{cases} j^2 + j & \text{if } a = 5, 6, \\ j^2 - j & \text{if } a = 1, 2, 3, 4, \end{cases}$$

$$m_1 = \begin{cases} 2j^2 + j, & \text{if } a = 6, \\ 2j^2 - j, & \text{if } a = 2, 3, \\ 2j^2, & \text{if } a = 4, 5, \\ 2j^2 - 3j + 1, & \text{if } a = 1, \end{cases}$$

$$m_2 = \begin{cases} 3j^2, & \text{if } a = 3, 4, 5, 6, \\ 3j^2 - 3j + 1, & \text{if } a = 1, 2. \end{cases}$$

Theorem 5.4. For $\lambda = l\Lambda_2$, $l \geq 1$ and the given perfect crystal B for the quantum affine algebra $U_q(G_2^{(1)})$ with $d = 6$ and the sequence $\{i_a^{(j)}\}$ given above, conditions (1), (2) and (3) in Theorem 4.1 hold. Hence we have path realizations of the corresponding Demazure crystals $B_{w^{(k)}}(l\Lambda_2)$ for $U_q(G_2^{(1)})$ with tensor-product-like structures.

Proof. We have already shown above by explicit descriptions of the subsets $B_a^{(j)}$ that condition (1) holds.

Since $\langle l\Lambda_2, h_{i_a^{(j)}} \rangle = 0$ for $a = 2, 4, 5, 6$, $\langle l\Lambda_2, h_{i_1^{(j)}} \rangle = l$, $\varepsilon_{i_1^{(j)}}((0, \frac{l}{3}, \frac{l}{3}, \frac{l}{3}, \frac{l}{3}, 0)) = l$, and $\langle l\Lambda_2, h_{i_3^{(j)}} \rangle = l$, for $b \in B_2^{(j)}$, $\varepsilon_{i_3^{(j)}}(b) \geq l$, condition (2) holds.

To prove condition (3), we use Lemma 5.3 to obtain:

$$\langle w^{(k)}\Lambda_2, h_{i_{a+1}^{(j)}} \rangle = \begin{cases} j & \text{if } a = 3, 5, \\ 2j - 1 & \text{if } a = 1, \\ 2j & \text{if } a = 4, \\ 3j - 1 & \text{if } a = 2, \\ 3j + 1 & \text{if } a = 6, \end{cases}$$

where $k = 6(j - 1) + a$, $j \geq 1$. Hence by Proposition 4.2, $w^{(k+1)} = r_{i_{a+1}^{(j)}} w^{(k)} \succ w^{(k)}$, which implies that condition (3) holds. □

6. EXAMPLE

For $\mathfrak{g} = G_2^{(1)}$, the level one perfect crystal B given in Section 3 is as shown in Figure 1, where

$$\begin{array}{lll} v_0 = (0, 0, 0, 0, 0, 0), & v_1 = (1, 0, 0, 0, 0, 0), & v_2 = (0, 1, 0, 0, 0, 0), \\ v_3 = (0, \frac{2}{3}, \frac{2}{3}, 0, 0, 0), & v_4 = (0, \frac{1}{3}, \frac{4}{3}, 0, 0, 0), & v_5 = (0, \frac{1}{3}, \frac{1}{3}, 1, 0, 0), \\ v_6 = (0, 0, 2, 0, 0, 0), & v_7 = (0, 0, 1, 1, 0, 0), & v_{\bar{7}} = (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0), \\ v_{\bar{6}} = (0, 0, 0, 2, 0, 0), & v_{\bar{5}} = (0, 0, 1, \frac{1}{3}, \frac{1}{3}, 0), & v_{\bar{4}} = (0, 0, 0, \frac{4}{3}, \frac{1}{3}, 0), \\ v_{\bar{3}} = (0, 0, 0, \frac{2}{3}, \frac{2}{3}, 0), & v_{\bar{2}} = (0, 0, 0, 0, 1, 0), & v_{\bar{1}} = (0, 0, 0, 0, 0, 1). \end{array}$$

For $\lambda = \Lambda_0$ the minimal element is v_0 and the Λ_0 -ground state path is $\mathbf{p}_{\Lambda_0} = (\cdots \otimes v_0 \otimes v_0 \otimes v_0)$. In this case we choose $d = 6$ and

$$i_1^{(j)} = 0, i_2^{(j)} = 1, i_3^{(j)} = 2, i_4^{(j)} = 1, i_5^{(j)} = 2, i_6^{(j)} = 1.$$

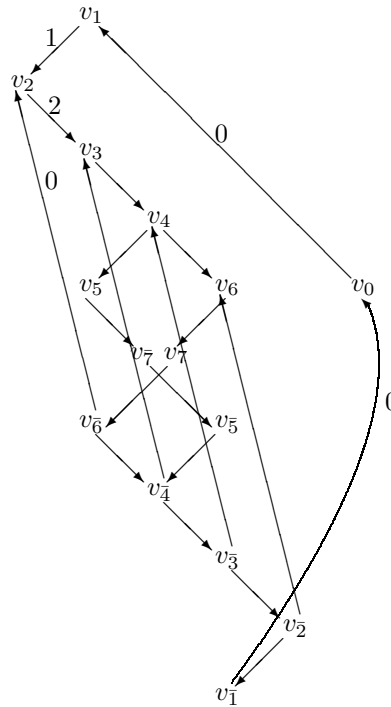


FIGURE 1

Then we have

$$B_0^{(j)} = \{v_0\}, B_1^{(j)} = B_0^{(j)} \cup \{v_1\}, B_2^{(j)} = B_1^{(j)} \cup \{v_2\}, B_3^{(j)} = B_2^{(j)} \cup \{v_3, v_4, v_6\},$$

$$B_4^{(j)} = B_3^{(j)} \cup \{v_5, v_7, v_8\}, B_5^{(j)} = B_4^{(j)} \cup \{v_9, v_{10}, v_4, v_3, v_2\}, B_6^{(j)} = B_5^{(j)} \cup \{v_1\} = B.$$

For $\lambda = \Lambda_2$ the minimal element is v_9 and the Λ_2 -ground state path is $\mathbf{p}_{\Lambda_2} = (\cdots \otimes v_9 \otimes v_9 \otimes v_9)$. In this case we choose $d = 6$ and

$$i_1^{(j)} = 2, i_2^{(j)} = 1, i_3^{(j)} = 2, i_4^{(j)} = 1, i_5^{(j)} = 0, i_6^{(j)} = 1.$$

Then we have

$$B_0^{(j)} = \{v_9\}, B_1^{(j)} = B_0^{(j)} \cup \{v_{10}\}, B_2^{(j)} = B_1^{(j)} \cup \{v_4\}, B_3^{(j)} = B_2^{(j)} \cup \{v_3, v_2\},$$

$$B_4^{(j)} = B_3^{(j)} \cup \{v_1\}, B_5^{(j)} = B_4^{(j)} \cup \{v_3, v_4, v_6, v_0, v_1\},$$

$$B_6^{(j)} = B_5^{(j)} \cup \{v_2, v_5, v_7, v_8\} = B.$$

For $k = (j - 1)6 + a, j \geq 1, 1 \leq a \leq 6$, and $\lambda = \Lambda_0$ (or Λ_2), by Theorem 4.1 we see that the Demazure crystal $B_{w^{(k)}}(\lambda)$ is isomorphic to $\mathbf{p}_\lambda \otimes B_a^{(j)} \otimes B^{\otimes(j-1)}$. In particular, if $k = 6j$, then $B_{w^{(k)}}(\lambda) \cong \mathbf{p}_\lambda \otimes B^{\otimes j}$.

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