

IRREDUCIBLE WEIGHT MODULES OVER WITT ALGEBRAS

XIANGQIAN GUO AND KAIMING ZHAO

(Communicated by Gail R. Letzter)

ABSTRACT. In 1986, Shen defined a class of modules over the Witt algebra W_d from irreducible modules over the general linear Lie algebra \mathfrak{gl}_d , which were also given by Larsson in 1992. In 1996, Eswara Rao determined the irreducibility of these modules. In this paper, we use simpler methods to give a short and straightforward proof to the results of Eswara Rao.

1. INTRODUCTION

Consider the Lie algebra W_d of all derivations of the Laurent polynomial algebra $A = C[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_d^{\pm 1}]$. This algebra is known as the Witt algebra and is isomorphic to the algebra of diffeomorphisms of the d -dimensional torus. The algebra W_d is a natural higher rank generalization of the Virasoro algebra; it has many applications to different branches of mathematics and physics (see [M2], [L1]–[L5]) and at the same time a much more complicated representation theory.

Modules over Witt algebras were used by O. Mathieu [M2] to model simple cuspidal weight modules with finite-dimensional weight spaces over simple finite-dimensional complex Lie algebras. Representations of Witt algebras are also closely connected to the representation theory of extended affine Lie algebras ([AABGP]) and toroidal Lie algebras ([B, E2, EJ]).

The representation theory of Witt algebras was intensively studied by many mathematicians and physicists; see [E1, E3, BMZ, L1, L2, L3, L4, L5, MZ, Z]. In particular, [MZ] asserted that any simple Harish-Chandra W_d -module is either dense (with uniformly bounded weight spaces) or punctured (with uniformly bounded weight spaces) or a simple quotient of some generalized Verma module. So far, the only known dense or punctured modules are those introduced and studied in ([Sh, L3]). The following conjecture is generally considered to be true.

Conjecture. *All dense or punctured modules over W_n are those mentioned in Theorem 1.*

Assuming this conjecture and using the result in [BZ], we can deduce that the third class of modules mentioned above depends only on the first two classes of modules. So it is crucial to classify dense or punctured modules over W_n .

Eswara Rao [E1] determined the irreducibility of those modules introduced and studied in ([Sh, L3]). But we found that his proofs were very technical and hard

Received by the editors April 7, 2010 and, in revised form, June 23, 2010 and June 27, 2010.
2010 *Mathematics Subject Classification.* Primary 17B10, 17B20, 17B65, 17B66, 17B68.
Key words and phrases. Witt algebra, weight module, irreducible module.

to understand. In this paper, we use much simpler and different methods to give a short and straightforward proof to the result in [E1].

We are grateful to the referee for giving a lot of suggestions to make the paper much more readable.

2. NOTATION AND PRELIMINARIES

2.1. **Witt algebras** W_d . We denote by \mathbb{Z} , \mathbb{Z}_+ , and \mathbb{C} the sets of all integers, nonnegative integers, and complex numbers, respectively.

We fix a positive integer $d > 1$ and the column vector space \mathbb{C}^d of $d \times 1$ matrices with the standard basis $\{e_1, e_2, \dots, e_d\}$. Let $(\cdot | \cdot)$ be the standard symmetric bilinear form such that $(u|v)$ is the product $u^T v \in \mathbb{C}$, where u^T is the matrix transpose.

Let $A = \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ be the Laurent polynomial algebra over \mathbb{C} and denote by W_d the algebra of all derivations of A , called the Witt algebra. We set $t^a = t_1^{a_1} t_2^{a_2} \cdots t_d^{a_d}$ for any $a \in \mathbb{Z}^d$ and $\partial_i = t_i \frac{\partial}{\partial t_i}$ for any $i \in \{1, 2, \dots, d\}$. For any $u \in \mathbb{C}^d$ and $r \in \mathbb{Z}^d$, we also set $D(u, r) = t^r \sum_{i=1}^d u_i \partial_i$ for convenience. The Lie bracket of W_d is given by

$$[D(u, r), D(v, s)] = D(w, r + s), \forall u, v \in \mathbb{C}^d, r, s \in \mathbb{Z}^d,$$

where $w = (u|s)v - (v|r)u$. Note that for any $u, v \in \mathbb{C}^d$, the product uv^T is a $d \times d$ matrix.

Let $\mathfrak{t} = \{D(u, 0) | u \in \mathbb{C}^d\}$ be the Cartan subalgebra of W_d . A W_d -module V is called a *weight* module if the action of \mathfrak{t} on V is diagonalizable. For any weight module V we have the decomposition $V = \bigoplus_{\lambda \in \mathfrak{t}^*} V_\lambda$, where $\mathfrak{t}^* = \text{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathbb{C})$ and

$$V_\lambda = \{v \in V : \partial v = \lambda(\partial)v \text{ for all } \partial \in \mathfrak{t}\}.$$

The space V_λ is called the *weight space* corresponding to the *weight* λ . The *support* of the weight module V , denoted by $\text{supp}(V)$, is defined as the set of all weights λ with $V_\lambda \neq 0$. If V is a weight W_d -module and $\dim_{\mathbb{C}} V_\lambda < \infty$ for all $\lambda \in \mathfrak{t}^*$, the module V is called a *Harish-Chandra* module.

2.2. **Finite-dimensional \mathfrak{gl}_d -modules.** Let \mathfrak{gl}_d be the Lie algebra of all $d \times d$ complex matrices, \mathfrak{sl}_d the subalgebra of \mathfrak{gl}_d consisting of all traceless matrices. For $1 \leq i, j \leq d$ we use E_{ij} to denote the matrix units. Let $\mathfrak{h} = \text{span}\{E_{ii} | 1 \leq i \leq d\}$ and $\mathfrak{h} = \text{span}\{h_i | 1 \leq i \leq d-1\}$, where $h_i = E_{ii} - E_{i+1, i+1}$, and define $\epsilon_i \in \mathfrak{h}^*$ by $\epsilon_i(\sum_{j=0}^d a_j E_{jj}) = a_i$.

Let $P^+ = \{\lambda \in \mathfrak{h}^* | \lambda(h_i) \in \mathbb{Z}_+, i = 1, 2, \dots, d-1\}$. For any $\psi \in P^+$ let $V(\psi)$ be the simple \mathfrak{sl}_d -module with highest weight ψ . We make $V(\psi)$ into a \mathfrak{gl}_d -module by defining the action of the identity matrix I as some scalar $b \in \mathbb{C}$. We denote the resulting module as $V(\psi, b)$.

Define the fundamental weights $\delta_i \in \mathfrak{h}^*$ by $\delta_i(h_j) = \delta_{i,j}$ for all $i, j = 1, 2, \dots, d-1$. It is well known that the module $V(\delta_1, 1)$ can be realized as the natural representation of \mathfrak{gl}_d on \mathbb{C}^d (the matrix product), which we can write as $E_{ji}e_l = \delta_{li}e_j$. In particular,

$$(2.1) \quad (ru^T)v = (u|v)r, \forall u, v, r \in \mathbb{C}^d.$$

The exterior product $\bigwedge^k(\mathbb{C}^d) = \mathbb{C}^d \wedge \cdots \wedge \mathbb{C}^d$ (k times) is a \mathfrak{gl}_d -module with the action given by $X(v_1 \wedge \cdots \wedge v_k) = \sum_{i=1}^k v_1 \wedge \cdots \wedge v_{i-1} \wedge Xv_i \wedge \cdots \wedge v_k$ for all $X \in \mathfrak{gl}_d$,

and the following \mathfrak{gl}_d -module isomorphism is well known:

$$(2.2) \quad \bigwedge^k (\mathbb{C}^d) \cong V(\delta_k, k), \forall 1 \leq k \leq d - 1.$$

3. SHEN'S MODULES $F^\alpha(\psi, b)$

3.1. **Defining the modules.** Suppose $\psi \in P^+$, $b \in \mathbb{C}$ and $\alpha \in \mathbb{C}^d$. Denote $F^\alpha(\psi, b) = V(\psi, b) \otimes A$. For simplicity we write $v(n) = v \otimes t^n$ for any $v \in V(\psi, b)$ and $n \in \mathbb{Z}^d$. Then $F^\alpha(\psi, b)$ becomes a W_d -module by defining

$$(3.1) \quad D(u, r)v(n) = (u|n + \alpha)v(n + r) + (ru^T)v(n + r),$$

for all $u \in \mathbb{C}^d, n, r \in \mathbb{Z}^d, v \in V(\psi, b)$. The modules $F^\alpha(\psi, b)$ were defined in [Sh], [L3] and studied in [E1]. We note that there are similarities shared by these modules and the generalized Verma modules studied in [KM].

One can easily verify that $F^\alpha(\psi, b) \simeq F^{\alpha+n}(\psi, b)$ for any $n \in \mathbb{Z}^d$. So we always assume that $\alpha_i = 0$ if $\alpha_i \in \mathbb{Z}$ in what follows. Using (2.1) and (2.2) one can also verify that the modules $F^\alpha(\delta_k, k)$ for $k = 1, 2, \dots, d - 1$ have submodules

$$W(\alpha, k) = \bigoplus_{n \in \mathbb{Z}^d} (\mathbb{C}(n + \alpha) \wedge \mathbb{C}^d \wedge \dots \wedge \mathbb{C}^d) \otimes t^n.$$

If $\alpha = 0$, then $F^0(\delta_k, k)$ for $k = 1, 2, \dots, d - 1$ have bigger submodules

$$\tilde{W}(0, k) = W(0, k) \oplus (\mathbb{C}^d \wedge \dots \wedge \mathbb{C}^d) \otimes t^0.$$

For convenience, we also denote $\tilde{W}(\alpha, k) = W(\alpha, k)$ for $\alpha \neq 0$.

3.2. **Eswara Rao's Theorem.** We will give a much simpler proof for Eswara Rao's Theorem.

Theorem 1. *Suppose $d > 1$, $\psi \in P^+ \setminus \{0\}$, $b \in \mathbb{C}$ and $k = 1, 2, \dots, d - 1$.*

- (a) *If $(\psi, b) \neq (\delta_k, k)$, then $F^\alpha(\psi, b)$ is irreducible.*
- (b) *$W(\alpha, k)$ and $F^\alpha(\delta_k, k)/\tilde{W}(\alpha, k)$ are irreducible, and*

$$F^\alpha(\delta_k, k)/\tilde{W}(\alpha, k) \simeq W(\alpha, k + 1), \forall k = 1, 2, \dots, d - 1.$$

Proof. For convenience, we define $\mathbb{Z}_*^d(\alpha) = \{n \in \mathbb{Z}^d : n + \alpha \neq 0\}$ and $\mathbb{Z}_{**}^d(\alpha) = \{n \in \mathbb{Z}^d : n_i + \alpha_i \neq 0, \forall i = 1, \dots, d\}$. Let M be a nonzero submodule of $F^\alpha(\psi, b)$ and denote $M_n = \{v \in V(\psi, b) \mid v \otimes t^n \in M\}$. Then M_n is a subspace of $V(\psi, b)$ and $M = \bigoplus_{n \in \mathbb{Z}^d} M_n \otimes t^n$ since M is a weight module. Denote $M_* = \bigcap_{n \in \mathbb{Z}_*^d(\alpha)} M_n$.

Claim 1. If $M_* \neq 0$, then $M = F^\alpha(\psi, b)$.

For any $v \in M_*, u \in \mathbb{C}^d, r \in \mathbb{Z}^d, n, n - kr \in \mathbb{Z}_*^d, k \in \mathbb{Z}$, from

$$D(u, kr)v(n - kr) = \left((u|n - kr + \alpha)v + k(ru^T)v \right)(n)$$

we see that $(ru^T)v \in M_n$; i.e., $(ru^T)v \in M_*$. So M_* is a \mathfrak{gl}_d -submodule. Thus $M_* = V(\psi, b)$. If $\alpha \neq 0$, it follows that $M = F^\alpha(\psi, b)$.

Now suppose $\alpha = 0$. For all $v \in V(\psi, b)$ and $i \neq j$ we have $D(e_j, -e_i)v(e_i) = -E_{ij}v(0)$, which implies that $\mathfrak{sl}_d V(\psi, b) \subset M_0$. Again it follows that $M = F^\alpha(\psi, b)$. Claim 1 is proved.

We first prove (b). We identify $V(\delta_k, k) = \bigwedge^k(\mathbb{C}^d)$. For any $1 \leq k < d$ it is easy to see that the linear map

$$\begin{aligned} \pi_k : F^\alpha(\delta_k, k) &\rightarrow F^\alpha(\delta_{k+1}, k+1), \\ (v_1 \wedge v_2 \wedge \cdots \wedge v_k)(n) &\mapsto ((n + \alpha) \wedge v_1 \wedge v_2 \wedge \cdots \wedge v_k)(n) \end{aligned}$$

is a W_d -module homomorphism. It is obvious that

$$\begin{aligned} \ker(\pi_k) &= \tilde{W}(\alpha, k), \quad \text{Im}(\pi_k) = W(\alpha, k+1), \\ F^\alpha(\delta_k, k)/\tilde{W}(\alpha, k) &\simeq W(\alpha, k+1) \end{aligned}$$

for all $k = 1, \dots, d-1$.

Recall that $W(\alpha, 1) = \bigoplus_{n \in \mathbb{Z}^d} \mathbb{C}((n + \alpha) \otimes t^n)$. Then by (3.1), we deduce that $D(u, r)((n + \alpha) \otimes t^n) = (u|n + \alpha)((n + r + \alpha) \otimes t^{n+r})$, which implies the irreducibility of $W(\alpha, 1)$. Note also that $\tilde{W}(\alpha, d) = F^\alpha(\delta_d, d)$. Thus to obtain (b), we only need to show that $F^\alpha(\delta_k, k)/\tilde{W}(\alpha, k)$ are irreducible for $k = 1, \dots, d-1$.

Assume that M properly contains $\tilde{W}(\alpha, k), 1 \leq k < d$. We may choose $(v_1 \wedge v_2 \wedge \cdots \wedge v_k)(n) \in M_n(n) \setminus \tilde{W}(\alpha, k)$ for some $n \in \mathbb{Z}_*^d(\alpha)$, since $\tilde{W}(0, k)(0) = F^0(\delta_k, k)(0)$. It is clear that $n + \alpha, v_1, \dots, v_k$ are linearly independent over \mathbb{C} . Take any $u \in \mathbb{C}^d$ such that $(u|n + \alpha) \neq 0$ and $(u|v_i) = 0$ for all $i = 1, \dots, k$. Then

$$D(u, r)(v_1 \wedge v_2 \wedge \cdots \wedge v_k)(n) = (u|n + \alpha)(v_1 \wedge v_2 \wedge \cdots \wedge v_k)(n+r),$$

which implies that $v_1 \wedge v_2 \wedge \cdots \wedge v_k \in M_{n+r}$, i.e., $v_1 \wedge v_2 \wedge \cdots \wedge v_k \in M_*$. From Claim 1 we know that $M = F^\alpha(\psi, b)$. This proves (b).

Now we prove (a). Assume that $F^\alpha(\psi, b)$ is reducible. Then it suffices to show that $(\psi, b) = (\delta_k, k)$ for some $k = 1, \dots, d-1$. Recall that M is a nonzero submodule of $F^\alpha(\psi, b)$.

To make the proof self-contained we include the following claim and its proof, which belong to Eswara Rao [E1, Lemma 3.2] with slight modifications.

Claim 2. If $v \in M_n$, then M_n contains the following vectors: $(E_{ii}^2 - E_{ii})v, E_{ij}^2v, E_{ij}E_{ii}v, (n_i + \alpha_i)E_{jj}v - (n_j + \alpha_j)E_{ji}v$ for all $i \neq j$.

From (3.1) we see that

$$\begin{aligned} (3.2) \quad D(w, -r)D(u, r)v \otimes t^n &= ((w|n + \alpha + r)((u|n + \alpha)v + (ru^T)v) \\ &\quad - (u|n + \alpha)(rw^T)v - (rw^T)(ru^T)v) \otimes t^n \in M_n(n), \end{aligned}$$

for $u, w \in \mathbb{C}^d, r, n \in \mathbb{Z}^d, v \in M_n$.

Letting $w = u = r = e_i$ in (3.2), we obtain $(n_i + \alpha_i + 1)(n_i + \alpha_i)v + E_{ii}v - E_{ii}^2v \in M_n$, yielding $(E_{ii} - E_{ii}^2)v \in M_n$.

Letting $w = u = e_j$ and $r = e_i$ with $i \neq j$ in (3.2), we obtain that $(n_j + \alpha_j)^2v + E_{ij}^2v \in M_n$, yielding $E_{ij}^2v \in M_n$.

Letting $w = u = e_i + e_j$ and $r = e_i$ with $i \neq j$ in (3.2), we simplify it to give

$$((E_{ii} + E_{ij}) - (E_{ii} + E_{ij})^2)v = (E_{ii} + E_{ij} - E_{ii}^2 + E_{ij}^2 - E_{ii}E_{ij} - E_{ij}E_{ii})v \in M_n.$$

Using the above-established results and the formula $E_{ij}v = (E_{ii}E_{ij} - E_{ij}E_{ii})v$, we deduce that $E_{ij}E_{ii}v \in M_n$.

Letting $w = e_i$ and $u = r = e_j$ with $i \neq j$ in (3.2), and using the above-established results, we deduce that $(n_i + \alpha_i)E_{jj}v - (n_j + \alpha_j)E_{ji}v \in M_n$. Thus, Claim 2 follows.

Claim 3. If for some $n \in \mathbb{Z}_{**}^d(\alpha)$, M_n contains a nonzero weight vector with respect to \mathfrak{H} , then $M = F^\alpha(\psi, b)$.

For any weight vector $v \in M_n$, from Claim 2 we know that $(n_i + \alpha_i)E_{jj}v - (n_j + \alpha_j)E_{ji}v \in M_n$ for all $i \neq j$. But $(n_i + \alpha_i)E_{jj}v \in \mathbb{C}v$. Then $E_{ji}v \in M_n$ for all $i \neq j$. Thus M_n is a nonzero \mathfrak{sl}_d -submodule, yielding that $M_n = V(\psi, b)$.

For any fixed $m \in \mathbb{Z}_*^d$, choose u such that $(u|m + \alpha)(u|n + \alpha) \neq 0$. Then the algebra $\text{Vir}(u, m - n) = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}D(u, i(m - n))$ is isomorphic to the centerless Virasoro algebra and $\bigoplus_{i \in \mathbb{Z}} M_{n+i(m-n)}(n + i(m - n))$ is a uniformly bounded $\text{Vir}(u, m - n)$ module. Notice that the weights of M_m and M_n relative to the algebra $\text{Vir}(u, m - n)$ are $(u|m + \alpha) \neq 0$ and $(u|n + \alpha) \neq 0$ respectively. Then from the representation theory of the Virasoro algebra (see [M1]), we see that $\dim M_m = \dim M_n$. Thus $M_m = V(\psi, b)$ for all $m \in \mathbb{Z}_*^d$, and hence $M_* = V(\psi, b)$. Claim 3 follows from Claim 1.

For any $0 \neq v \in V(\psi, b)$ we can uniquely write $v = \sum_{\lambda \in \mathfrak{H}^*} v_\lambda$ such that $hv_\lambda = \lambda(h)v, \forall h \in \mathfrak{H}$. Denote $\text{Supp}(v) = \{\lambda : v_\lambda \neq 0\}$ and $\lambda_i = \lambda(E_{ii}), \forall \lambda \in \mathfrak{H}^*$. Define $S_v = \{i : \lambda_i = \mu_i, \forall \lambda, \mu \in \text{Supp}(v)\}$. It is clear that $E_{ii}v \in \mathbb{C}v, \forall i \in S_v$.

Take any nonzero $v \in M_n, n \in \mathbb{Z}_{**}^d(\alpha)$ such that $|\text{Supp}(v)|$ is minimal. Denote $p = |\text{Supp}(v)|$. Let

$$K = \{u \in M_n : n \in \mathbb{Z}_{**}^d(\alpha), |\text{Supp}(u)| = p\}.$$

From Claim 3 we know that $p \geq 2$ and $S_u \neq \{1, 2, \dots, d\}$ for any $u \in K$.

Claim 4. For any $v \in K$ and $\mu \in \text{Supp}(v)$, the following holds:

- (1) $\mu_i \in \{0, 1\}$ for any i and $1 \leq |\{i : \mu_i = 1\}| \leq d - 1$;
- (2) $E_{ij}v_\mu \neq 0$ for some $i \neq j$ implies $i \in S_v$.

From Claim 2, we have $(E_{ii}^2 - E_{ii})v = \sum_{\mu \in \text{Supp}(v)} (\mu_i^2 - \mu_i)v_\mu \in M_n$. By the minimality of $|\text{Supp}(v)|$, we must have that $\mu_i = \nu_i$ or $\mu_i + \nu_i = 1, \forall \mu, \nu \in \text{Supp}(v), i = 1, 2, \dots, d$.

For any $i \neq j$ with $i \notin S_v$, we claim that $E_{ij}^2v = E_{ji}^2v = 0$. We have $E_{ij}^2v, E_{ji}^2v \in M_n$ by Claim 2. If $E_{ij}^2v \neq 0$, we would have $\text{Supp}(E_{ij}^2v) = \{\mu' = \mu + 2\alpha_{i,j} : \mu \in \text{Supp}(v)\}$ by the minimality of $\text{Supp}(v)$. Since $i \notin S_v$ there are $\mu, \nu \in \text{Supp}(v)$ with $\mu_i \neq \nu_i$ and hence $\mu'_i \neq \nu'_i$. On the other hand, $\mu'_i + \nu'_i = \mu_i + 2 + \nu_i + 2 = 5 \neq 1$, contradicting the result in the last paragraph (with E_{ij}^2v replacing v). So we have $E_{ij}^2v = 0$. Similarly, we get $E_{ji}^2v = 0$.

For any $i \neq j$ with $i \notin S_v$, we claim that $E_{ij}v_\mu = 0$ if $\mu_i \neq 0$ and $E_{ji}v_\mu = 0$ if $\mu_j \neq 0$, for any $\mu \in \text{Supp}(v)$. We know that $E_{ij}E_{ii}v \in M_n$ from Claim 2. Similarly to the previous paragraph, one shows that $E_{ij}E_{ii}v = 0$ and hence $E_{ij}E_{ii}v_\mu = 0$, which implies that $E_{ij}v_\mu = 0$ if $\mu_i \neq 0$. We can deduce that $E_{ji}v_\mu = 0$ if $\mu_j \neq 0$ in the same way. Part (2) follows.

Since $p \geq 2$, for any $\lambda \in \text{Supp}(v)$ with $\lambda_i \neq 0$ for some $i \notin S_v$ (such a λ exists), we have the following arguments.

For any $i \neq j$ with $i \notin S_v$ and $\lambda_i \lambda_j \neq 0$, we have $E_{ij}v_\lambda = E_{ji}v_\lambda = 0$ and hence $[E_{ij}, E_{ji}]v_\lambda = (E_{ii} - E_{jj})v_\lambda = (\lambda_i - \lambda_j)v_\lambda = 0$. Thus $\lambda_i = \lambda_j$. Since $p \geq 2$, in the case $\lambda_1 = \lambda_2 = \dots = \lambda_d \neq 0$, there exists $\mu \in \text{Supp}(v)$ with $\mu_i = 0, i \notin S_v$ (otherwise $\mu = \lambda$). Thus $\lambda_i = 1$ by $\mu_i + \lambda_i = 1$ and hence $\lambda_1 = \lambda_2 = \dots = \lambda_d = 1$. Then $\mu_1 + \dots + \mu_d = d$ implies that $\mu_j \geq 2$ for some $j \neq i$, contradicting the fact that $\lambda_j = \mu_j$ or $\lambda_j + \mu_j = 1$. So there exists some $\lambda_j = 0$.

For $i \neq j$ with $i \notin S_v$ and $\lambda_i \neq 0, \lambda_j = 0$, the facts that $E_{j,i}^2v_\lambda = 0$ and $E_{i,j}v_\lambda = 0$ imply that $0 = [E_{i,j}, E_{j,i}^2]v_\lambda = 2(\lambda_i - 1)E_{j,i}v_\lambda = 0$. Since $[E_{i,j}, E_{j,i}]v_\lambda = \lambda_i v_\lambda \neq 0$,

then $E_{j,i}v_\lambda \neq 0$. We have $\lambda_i = 1$. That is, $\lambda_k \in \{0, 1\}$ for any $k = 1, \dots, d$. Since $\lambda_k = \mu_k$ or $\lambda_k + \mu_k = 1$, we see that $\mu_k \in \{0, 1\}$ for all $\mu \in \text{Supp}(v)$. Claim 4 follows.

We define a total ordering $<$ on $\Lambda = \bigoplus_{i=1}^d \mathbb{Z}\epsilon_i$ as $k_1\epsilon_1 + \dots + k_d\epsilon_d < l_1\epsilon_1 + \dots + l_d\epsilon_d$ if there exists an i such that $k_j = l_j$ for $j < i$ and $k_i < l_i$. Now fix any $v \in K$ such that $\lambda = \max(\text{Supp}(v))$ is maximal with respect to this order.

We claim that λ is the highest weight of $V(\psi, b)$, that is, $\psi = \lambda$. Otherwise, there is some i such that $E_{i,i+1}v_\lambda \neq 0$. By Claim 4(2), we know that $i \in S_v$ and $E_{ii}v \in \mathbb{C}v \subset M_n$. Then Claim 2 ensures that $(n_{i+1} + \alpha_{i+1})E_{ii}v - (n_i + \alpha_i)E_{i,i+1}v \in M_n$ and hence $w = E_{i,i+1}v \in M_n$. By the minimality of $|\text{Supp}(v)|$, we have $|\text{Supp}(w)| = p$; that is, $w \in K$. However, it is clear that $\max(\text{Supp}(w)) = \lambda + \epsilon_i - \epsilon_{i+1}$, which is larger than $\max(\text{Supp}(v)) = \lambda$, contradicting the choice of v . So $\lambda = \psi$.

From Claim 4, we know that $\lambda_i \in \{0, 1\}$. Since λ is dominant, we must have $\psi = \lambda = \delta_k$, where k is the number of i with $\lambda_i = 1$. Finally, $(\sum_{i=1}^d E_{ii})v_\lambda = kv_\lambda$ implies $b = k$. This gives $(\psi, b) = (\delta_k, k)$ and completes the proof. \square

ACKNOWLEDGMENTS

The second author was partially supported by NSERC and NSF of China (Grant 10871192).

REFERENCES

- [AABGP] B. N. Allison, S. Azam, S. Berman, Y. Gao, and A. Pianzola, *Extended affine Lie algebras and their root systems*, Mem. Amer. Math. Soc. 126 (605) (1997). MR1376741 (97i:17015)
- [B] Y. Billig, A category of modules for the full toroidal Lie algebra, Int. Math. Res. Not. 2006, Art. ID 68395, 46 pp. MR2272091 (2007h:17030)
- [BMZ] Y. Billig, A. Molev, and R. Zhang, Differential equations in vertex algebras and simple modules for the Lie algebra of vector fields on a torus, Adv. Math. 218(2008), no. 6, 1972-2004. MR2431666 (2009e:17042)
- [BZ] Y. Billig and K. Zhao, Weight modules over exp-polynomial Lie algebras, J. Pure Appl. Algebra 191 (2004), no. 1-2, 23-42. MR2048305 (2005a:17021)
- [E1] S. Eswara Rao, Irreducible representations of the Lie-algebra of the diffeomorphisms of a d -dimensional torus, J. Algebra 182 (1996), no. 2, 401-421. MR1391590 (97f:17007)
- [E2] S. Eswara Rao, Irreducible representations for toroidal Lie-algebras, Journal of Pure and Applied Algebra 202 (2005), 102-117. MR2163403 (2006e:17011)
- [E3] S. Eswara Rao, Partial classification of modules for Lie algebra of diffeomorphisms of d -dimensional torus, J. Math. Phys. 45 (8) (2004), 3322-3333. MR2077512 (2005d:17028)
- [EJ] S. Eswara Rao and C. Jiang, Classification of irreducible integrable representations for the full toroidal Lie algebras, J. Pure Appl. Algebra 200 (2005), no. 1-2, 71-85. MR2142351 (2006b:17038)
- [KR] V. Kac and A. Raina, *Bombay lectures on highest weight representations of infinite dimensional Lie algebras*, World Sci., Singapore, 1987. MR1021978 (90k:17013)
- [KM] O. Koshnikov and V. Mazorchuk, Generalized Verma modules induced from $\mathfrak{sl}(2, \mathbb{C})$ and associated Verma modules. *J. Algebra* 242 (2001), no. 2, 561-576. MR1848960 (2002f:17009)
- [L1] T. A. Larsson, Multi-dimensional Virasoro algebra, Phys. Lett. B 231 (1989), 94-96. MR1023577 (90k:17050)
- [L2] T. A. Larsson, Central and non-central extensions of multi-graded Lie algebras, J. Phys. A 25 (1992), 1177-1184. MR1154861 (93c:17047)
- [L3] T. A. Larsson, Conformal fields: A class of representations of $\text{Vect}(N)$, Int. J. Mod. Phys. A 7 (1992), 6493-6508. MR1186508 (93h:17053)
- [L4] T. A. Larsson, Lowest-energy representations of non-centrally extended diffeomorphism algebras, Commun. Math. Phys. 201 (1999), 461-470. MR1682285 (2000c:17042)

- [L5] T. A. Larsson, Extended diffeomorphism algebras and trajectories in jet space, *Commun. Math. Phys.* 214 (2000), 469-491. MR1796031 (2001j:58017)
- [M1] O. Mathieu, Classification of Harish-Chandra modules over the Virasoro Lie algebra, *Invent. Math.* 107 (1992), no. 2, 225-234. MR1144422 (93d:17034)
- [M2] O. Mathieu, Classification of irreducible weight modules, *Ann. Inst. Fourier (Grenoble)* 50 (2000), no. 2, 537-592. MR1775361 (2001h:17017)
- [MZ] V. Mazorchuk and K. Zhao, Supports of weight modules over Witt algebras, *Proc. R. Soc. Edinb. Sect. A-Math.*, in press.
- [Sh] G. Shen, Graded modules of graded Lie algebras of Cartan type. I. Mixed products of modules, *Sci. Sinica Ser. A* 29 (1986), no. 6, 570-581. MR862418 (88d:17014)
- [Z] K. Zhao, Weight modules over generalized Witt algebras with 1-dimensional weight spaces, *Forum Math.* 16 (2004), no. 5, 725-748. MR2096685 (2005h:17034)

DEPARTMENT OF MATHEMATICS, ZHENGZHOU UNIVERSITY, ZHENGZHOU 450001, HENAN, PEOPLE'S REPUBLIC OF CHINA

E-mail address: guoxq@amss.ac.cn

DEPARTMENT OF MATHEMATICS, WILFRID LAURIER UNIVERSITY, WATERLOO, ON, CANADA N2L 3C5 – AND – COLLEGE OF MATHEMATICS AND INFORMATION SCIENCE, HEBEI NORMAL UNIVERSITY, SHIJIAZHUANG, HEBEI 050016, PEOPLE'S REPUBLIC OF CHINA

E-mail address: kzhao@wlu.ca