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IRREDUCIBLE WEIGHT MODULES OVER WITT ALGEBRAS

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Abstract. In 1986, Shen defined a class of modules over the Witt algebra W_d from irreducible modules over the general linear Lie algebra $\mathfrak{gl}_d$, which were also given by Larsson in 1992. In 1996, Eswara Rao determined the irreducibility of these modules. In this paper, we use simpler methods to give a short and straightforward proof to the results of Eswara Rao.

1. Introduction

Consider the Lie algebra W_d of all derivations of the Laurent polynomial algebra $A = C[t_1^{\pm 1}, t_2^{\pm 1}, ..., t_d^{\pm 1}]$. This algebra is known as the Witt algebra and is isomorphic to the algebra of diffeomorphisms of the d-dimensional torus. The algebra W_d is a natural higher rank generalization of the Virasoro algebra; it has many applications to different branches of mathematics and physics (see [M2], [L1]–[L5]) and at the same time a much more complicated representation theory.

Modules over Witt algebras were used by O. Mathieu [M2] to model simple cuspidal weight modules with finite-dimensional weight spaces over simple finite-dimensional complex Lie algebras. Representations of Witt algebras are also closely connected to the representation theory of extended affine Lie algebras ([AABGP]) and toroidal Lie algebras ([B, E2, EJ]).

The representation theory of Witt algebras was intensively studied by many mathematicians and physicists; see [E1, E3, BMZ, L1, L2, L3, L4, L5, MZ, Z]. In particular, [MZ] asserted that any simple Harish-Chandra W_d-module is either dense (with uniformly bounded weight spaces) or punctured (with uniformly bounded weight spaces) or a simple quotient of some generalized Verma module. So far, the only known dense or punctured modules are those introduced and studied in ([Sh, L3]). The following conjecture is generally considered to be true.

Conjecture. *All dense or punctured modules over W_n are those mentioned in Theorem* 1.

Assuming this conjecture and using the result in [BZ], we can deduce that the third class of modules mentioned above depends only on the first two classes of modules. So it is crucial to classify dense or punctured modules over W_n.

Eswara Rao [E1] determined the irreducibility of those modules introduced and studied in ([Sh, L3]). But we found that his proofs were very technical and hard

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to understand. In this paper, we use much simpler and different methods to give a short and straightforward proof to the result in [E1].

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2. Notation and preliminaries

2.1. **Witt algebras W_d.** We denote by $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{C}$ the sets of all integers, nonnegative integers, and complex numbers, respectively.

We fix a positive integer $d > 1$ and the column vector space $\mathbb{C}^d$ of $d \times 1$ matrices with the standard basis $\{e_1, e_2, ..., e_d\}$. Let $(\cdot\,|\,\cdot)$ be the standard symmetric bilinear form such that $(u|v)$ is the product $u^T v \in \mathbb{C}$, where u^T is the matrix transpose.

Let $A = \mathbb{C}[t_1^{\pm 1}, ..., t_d^{\pm 1}]$ be the Laurent polynomial algebra over $\mathbb{C}$ and denote by W_d the algebra of all derivations of A, called the Witt algebra. We set $t^a = t_1^{a_1} t_2^{a_2} \cdots t_d^{a_d}$ for any $a \in \mathbb{Z}^d$ and $\partial_i = t_i \frac{\partial}{\partial t_i}$ for any $i \in \{1, 2, \ldots, d\}$. For any $u \in \mathbb{C}^d$ and $r \in \mathbb{Z}^d$, we also set $D(u, r) = t^r \sum_{i=1}^d u_i \partial_i$ for convenience. The Lie bracket of W_d is given by

$$[D(u, r), D(v, s)] = D(w, r + s), \forall\, u, v \in \mathbb{C}^d, r, s \in \mathbb{Z}^d,$$

where $w = (u|s)v - (v|r)u$. Note that for any $u, v \in \mathbb{C}^d$, the product uv^T is a $d \times d$ matrix.

Let $\mathfrak{t} = \{D(u, 0) | u \in \mathbb{C}^d\}$ be the Cartan subalgebra of W_d. A W_d-module V is called a *weight* module if the action of $\mathfrak{t}$ on V is diagonalizable. For any weight module V we have the decomposition $V = \bigoplus_{\lambda \in \mathfrak{t}^*} V_\lambda$, where $\mathfrak{t}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathbb{C})$ and

$$V_\lambda = \{v \in V : \partial v = \lambda(\partial) v \text{ for all } \partial \in \mathfrak{t}\}.$$

The space V_λ is called the *weight space* corresponding to the *weight* λ. The *support* of the weight module V, denoted by $\mathrm{supp}(V)$, is defined as the set of all weights λ with $V_\lambda \neq 0$. If V is a weight W_d-module and $\dim_{\mathbb{C}} V_\lambda < \infty$ for all $\lambda \in \mathfrak{t}^*$, the module V is called a *Harish-Chandra* module.

2.2. **Finite-dimensional $\mathfrak{gl}_d$-modules.** Let $\mathfrak{gl}_d$ be the Lie algebra of all $d \times d$ complex matrices, $\mathfrak{sl}_d$ the subalgebra of $\mathfrak{gl}_d$ consisting of all traceless matrices. For $1 \le i, j \le d$ we use E_{ij} to denote the matrix units. Let $\mathfrak{H} = \mathrm{span}\{E_{ii} \,|\, 1 \le i \le d\}$ and $\mathfrak{h} = \mathrm{span}\{h_i \,|\, 1 \le i \le d - 1\}$, where $h_i = E_{ii} - E_{i+1,i+1}$, and define $\epsilon_i \in \mathfrak{H}^*$ by $\epsilon_i(\sum_{j=0}^d a_j E_{jj}) = a_i$.

Let $P^+ = \{\lambda \in \mathfrak{h}^* \,|\, \lambda(h_i) \in \mathbb{Z}_+, i = 1, 2, ..., d - 1\}$. For any $\psi \in P^+$ let $V(\psi)$ be the simple $\mathfrak{sl}_d$-module with highest weight ψ. We make $V(\psi)$ into a $\mathfrak{gl}_d$-module by defining the action of the identity matrix I as some scalar $b \in \mathbb{C}$. We denote the resulting module as $V(\psi, b)$.

Define the fundamental weights $\delta_i \in \mathfrak{h}^*$ by $\delta_i(h_j) = \delta_{i,j}$ for all $i, j = 1, 2, ..., d-1$. It is well known that the module $V(\delta_1, 1)$ can be realized as the natural representation of $\mathfrak{gl}_d$ on $\mathbb{C}^d$ (the matrix product), which we can write as $E_{ji} e_l = \delta_{li} e_j$. In particular,

$$(ru^T)v = (u|v)r,\ \forall\ u, v, r \in \mathbb{C}^d. \tag{2.1}$$

The exterior product $\bigwedge^k(\mathbb{C}^d) = \mathbb{C}^d \wedge \cdots \wedge \mathbb{C}^d$ (k times) is a $\mathfrak{gl}_d$-module with the action given by $X(v_1 \wedge \cdots \wedge v_k) = \sum_{i=1}^k v_1 \wedge \cdots v_{i-1} \wedge X v_i \cdots \wedge v_k$ for all $X \in \mathfrak{gl}_d$, and the following $\mathfrak{gl}_d$-module isomorphism is well known:

$$\bigwedge^k(\mathbb{C}^d) \cong V(\delta_k, k),\ \forall\ 1 \leqslant k \leqslant d-1. \tag{2.2}$$

3. Shen's modules $F^\alpha(\psi, b)$

3.1. **Defining the modules.** Suppose $\psi \in P^+$, $b \in \mathbb{C}$ and $\alpha \in \mathbb{C}^d$. Denote $F^\alpha(\psi, b) = V(\psi, b) \otimes A$. For simplicity we write $v(n) = v \otimes t^n$ for any $v \in V(\psi, b)$ and $n \in \mathbb{Z}^d$. Then $F^\alpha(\psi, b)$ becomes a W_d-module by defining

$$D(u, r)v(n) = (u|n + \alpha)v(n + r) + (ru^T)v(n + r), \tag{3.1}$$

for all $u \in \mathbb{C}^d, n, r \in \mathbb{Z}^d, v \in V(\psi, b)$. The modules $F^\alpha(\psi, b)$ were defined in [Sh], [L3] and studied in [E1]. We note that there are similarities shared by these modules and the generalized Verma modules studied in [KM].

One can easily verify that $F^\alpha(\psi, b) \simeq F^{\alpha+n}(\psi, b)$ for any $n \in \mathbb{Z}^d$. So we always assume that $\alpha_i = 0$ if $\alpha_i \in \mathbb{Z}$ in what follows. Using (2.1) and (2.2) one can also verify that the modules $F^\alpha(\delta_k, k)$ for $k = 1, 2, ..., d-1$ have submodules

$$W(\alpha, k) = \bigoplus_{n \in \mathbb{Z}^d} \left(\mathbb{C}(n + \alpha) \wedge \mathbb{C}^d \wedge \cdots \wedge \mathbb{C}^d\right) \otimes t^n.$$

If $\alpha = 0$, then $F^0(\delta_k, k)$ for $k = 1, 2, ..., d-1$ have bigger submodules

$$\tilde{W}(0, k) = W(0, k) \oplus \left(\mathbb{C}^d \wedge \cdots \wedge \mathbb{C}^d\right) \otimes t^0.$$

For convenience, we also denote $\tilde{W}(\alpha, k) = W(\alpha, k)$ for $\alpha \neq 0$.

3.2. **Eswara Rao's Theorem.** We will give a much simpler proof for Eswara Rao's Theorem.

Theorem 1. *Suppose $d > 1$, $\psi \in P^+ \setminus \{0\}$, $b \in \mathbb{C}$ and $k = 1, 2, ..., d-1$.*

(a) *If $(\psi, b) \neq (\delta_k, k)$, then $F^\alpha(\psi, b)$ is irreducible.*
(b) *$W(\alpha, k)$ and $F^\alpha(\delta_k, k)/\tilde{W}(\alpha, k)$ are irreducible, and*

$$F^\alpha(\delta_k, k)/\tilde{W}(\alpha, k) \simeq W(\alpha, k+1),\ \forall\ k = 1, 2, ..., d-1.$$

Proof. For convenience, we define $\mathbb{Z}^d_*(\alpha) = \{n \in \mathbb{Z}^d : n + \alpha \neq 0\}$ and $\mathbb{Z}^d_{**}(\alpha) = \{n \in \mathbb{Z}^d : n_i + \alpha_i \neq 0, \forall\ i = 1, ..., d\}$. Let M be a nonzero submodule of $F^\alpha(\psi, b)$ and denote $M_n = \{v \in V(\psi, b)\,|\, v \otimes t^n \in M\}$. Then M_n is a subspace of $V(\psi, b)$ and $M = \bigoplus_{n \in \mathbb{Z}^d} M_n \otimes t^n$ since M is a weight module. Denote $M_* = \bigcap_{n \in \mathbb{Z}^d_*(\alpha)} M_n$.

Claim 1. If $M_* \neq 0$, then $M = F^\alpha(\psi, b)$.

For any $v \in M_*$, $u \in \mathbb{C}^d, r \in \mathbb{Z}^d, n, n - kr \in \mathbb{Z}^d_*, k \in \mathbb{Z}$, from

$$D(u, kr)v(n - kr) = \Big((u|n - kr + \alpha)v + k(ru^T)v\Big)(n)$$

we see that $(ru^T)v \in M_n$; i.e., $(ru^T)v \in M_*$. So M_* is a $\mathfrak{gl}_d$-submodule. Thus $M_* = V(\psi, b)$. If $\alpha \neq 0$, it follows that $M = F^\alpha(\psi, b)$.

Now suppose $\alpha = 0$. For all $v \in V(\psi, b)$ and $i \neq j$ we have $D(e_j, -e_i)v(e_i) = -E_{ij}v(0)$, which implies that $\mathfrak{sl}_d V(\psi, b) \subset M_0$. Again it follows that $M = F^\alpha(\psi, b)$. Claim 1 is proved.

We first prove (b). We identify $V(\delta_k,k)=\bigwedge^k(\mathbb{C}^d)$. For any $1\leqslant k<d$ it is easy to see that the linear map

$$\pi_k : F^\alpha(\delta_k,k)\to F^\alpha(\delta_{k+1},k+1),$$
$$\big(v_1\wedge v_2\wedge\cdots\wedge v_k\big)(n)\mapsto\big((n+\alpha)\wedge v_1\wedge v_2\wedge\cdots\wedge v_k\big)(n)$$

is a W_d-module homomorphism. It is obvious that

$$\ker(\pi_k)=\tilde{W}(\alpha,k),\ \mathrm{Im}(\pi_k)=W(\alpha,k+1),$$
$$F^\alpha(\delta_k,k)/\tilde{W}(\alpha,k)\simeq W(\alpha,k+1)$$

for all $k=1,...,d-1$.

Recall that $W(\alpha,1)=\bigoplus_{n\in\mathbb{Z}^d}\mathbb{C}\big((n+\alpha)\otimes t^n\big)$. Then by (3.1), we deduce that $D(u,r)\big((n+\alpha)\otimes t^n\big)=(u|n+\alpha)\big((n+r+\alpha)\otimes t^{n+r}\big)$, which implies the irreducibility of $W(\alpha,1)$. Note also that $\tilde{W}(\alpha,d)=F^\alpha(\delta_d,d)$. Thus to obtain (b), we only need to show that $F^\alpha(\delta_k,k)/\tilde{W}(\alpha,k)$ are irreducible for $k=1,...,d-1$.

Assume that M properly contains $\tilde{W}(\alpha,k),1\leqslant k<d$. We may choose $\big(v_1\wedge v_2\wedge ...\wedge v_k\big)(n)\in M_n(n)\setminus\tilde{W}(\alpha,k)$ for some $n\in\mathbb{Z}^d_*(\alpha)$, since $\tilde{W}(0,k)(0)=F^0(\delta_k,k)(0)$. It is clear that $n+\alpha,v_1,...,v_k$ are linearly independent over $\mathbb{C}$. Take any $u\in\mathbb{C}^d$ such that $(u|n+\alpha)\neq 0$ and $(u|v_i)=0$ for all $i=1,...,k$. Then

$$D(u,r)\Big(v_1\wedge v_2\wedge...\wedge v_k\Big)(n)=(u|n+\alpha)\Big(v_1\wedge v_2\wedge...\wedge v_k\Big)(n+r),$$

which implies that $v_1\wedge v_2\wedge...\wedge v_k\in M_{n+r}$, i.e., $v_1\wedge v_2\wedge...\wedge v_k\in M_*$. From Claim 1 we know that $M=F^\alpha(\psi,b)$. This proves (b).

Now we prove (a). Assume that $F^\alpha(\psi,b)$ is reducible. Then it suffices to show that $(\psi,b)=(\delta_k,k)$ for some $k=1,...,d-1$. Recall that M is a nonzero submodule of $F^\alpha(\psi,b)$.

To make the proof self-contained we include the following claim and its proof, which belong to Eswara Rao [E1, Lemma 3.2] with slight modifications.

Claim 2. If $v\in M_n$, then M_n contains the following vectors: $(E_{ii}^2-E_{ii})v, E_{ij}^2v$, $E_{ij}E_{ii}v$, $(n_i+\alpha_i)E_{jj}v-(n_j+\alpha_j)E_{ji}v$ for all $i\neq j$.

From (3.1) we see that

$$D(w,-r)D(u,r)v\otimes t^n=\Big((w|n+\alpha+r)\big((u|n+\alpha)v+(ru^T)v\big)$$
$$-(u|n+\alpha)(rw^T)v-(rw^T)(ru^T)v\Big)\otimes t^n\in M_n(n),\tag{3.2}$$

for $u,w\in\mathbb{C}^d,r,n\in\mathbb{Z}^d,v\in M_n$.

Letting $w=u=r=e_i$ in (3.2), we obtain $(n_i+\alpha_i+1)(n_i+\alpha_i)v+E_{ii}v-E_{ii}^2v\in M_n$, yielding $(E_{ii}-E_{ii}^2)v\in M_n$.

Letting $w=u=e_j$ and $r=e_i$ with $i\neq j$ in (3.2), we obtain that $(n_j+\alpha_j)^2v+E_{ij}^2v\in M_n$, yielding $E_{ij}^2v\in M_n$.

Letting $w=u=e_i+e_j$ and $r=e_i$ with $i\neq j$ in (3.2), we simplify it to give

$$((E_{ii}+E_{ij})-(E_{ii}+E_{ij})^2)v=(E_{ii}+E_{ij}-E_{ii}^2+E_{ij}^2-E_{ii}E_{ij}-E_{ij}E_{ii})v\in M_n.$$

Using the above-established results and the formula $E_{ij}v=(E_{ii}E_{ij}-E_{ij}E_{ii})v$, we deduce that $E_{ij}E_{ii}v\in M_n$.

Letting $w=e_i$ and $u=r=e_j$ with $i\neq j$ in (3.2), and using the above-established results, we deduce that $(n_i+\alpha_i)E_{jj}v-(n_j+\alpha_j)E_{ji}v\in M_n$. Thus, Claim 2 follows.

Claim 3. If for some $n \in \mathbb{Z}^d_{**}(\alpha)$, M_n contains a nonzero weight vector with respect to $\mathfrak{H}$, then $M = F^\alpha(\psi, b)$.

For any weight vector $v \in M_n$, from Claim 2 we know that $(n_i+\alpha_i)E_{jj}v-(n_j+\alpha_j)E_{ji}v \in M_n$ for all $i \neq j$. But $(n_i+\alpha_i)E_{jj}v \in \mathbb{C}v$. Then $E_{ji}v \in M_n$ for all $i \neq j$. Thus M_n is a nonzero $\mathfrak{sl}_d$-submodule, yielding that $M_n = V(\psi, b)$.

For any fixed $m \in \mathbb{Z}^d_*$, choose u such that $(u|m+\alpha)(u|n+\alpha) \neq 0$. Then the algebra $\mathrm{Vir}(u, m-n) = \bigoplus_{i\in\mathbb{Z}} \mathbb{C}D(u, i(m-n))$ is isomorphic to the centerless Virasoro algebra and $\bigoplus_{i\in\mathbb{Z}} M_{n+i(m-n)}(n+i(m-n))$ is a uniformly bounded $\mathrm{Vir}(u, m-n)$ module. Notice that the weights of M_m and M_n relative to the algebra $\mathrm{Vir}(u, m-n)$ are $(u|m+\alpha) \neq 0$ and $(u|n+\alpha) \neq 0$ respectively. Then from the representation theory of the Virasoro algebra (see [M1]), we see that $\dim M_m = \dim M_n$. Thus $M_m = V(\psi, b)$ for all $m \in \mathbb{Z}^d_*$, and hence $M_* = V(\psi, b)$. Claim 3 follows from Claim 1.

For any $0 \neq v \in V(\psi, b)$ we can uniquely write $v = \sum_{\lambda\in\mathfrak{H}^*} v_\lambda$ such that $hv_\lambda = \lambda(h)v, \forall\, h \in \mathfrak{H}$. Denote $\mathrm{Supp}(v) = \{\lambda : v_\lambda \neq 0\}$ and $\lambda_i = \lambda(E_{ii}), \forall\, \lambda \in \mathfrak{H}^*$. Define $S_v = \{i : \lambda_i = \mu_i, \forall\, \lambda, \mu \in \mathrm{Supp}(v)\}$. It is clear that $E_{ii}v \in \mathbb{C}v, \forall\, i \in S_v$.

Take any nonzero $v \in M_n, n \in \mathbb{Z}^d_{**}(\alpha)$ such that $|\,\mathrm{Supp}(v)|$ is minimal. Denote $p = |\,\mathrm{Supp}(v)|$. Let

$$K = \{u \in M_n : n \in \mathbb{Z}^d_{**}(\alpha),\ |\,\mathrm{Supp}(u)| = p\}.$$

From Claim 3 we know that $p \geqslant 2$ and $S_u \neq \{1, 2, ..., d\}$ for any $u \in K$.

Claim 4. For any $v \in K$ and $\mu \in \mathrm{Supp}(v)$, the following holds:

(1) $\mu_i \in \{0, 1\}$ for any i and $1 \leqslant |\{i : \mu_i = 1\}| \leqslant d-1$;

(2) $E_{ij}v_\mu \neq 0$ for some $i \neq j$ implies $i \in S_v$.

From Claim 2, we have $(E^2_{ii} - E_{ii})v = \sum_{\mu\in\mathrm{Supp}(v)}(\mu_i^2 - \mu_i)v_\mu \in M_n$. By the minimality of $|\,\mathrm{Supp}(v)|$, we must have that $\mu_i = \nu_i$ or $\mu_i + \nu_i = 1$, $\forall\, \mu, \nu \in \mathrm{Supp}(v), i = 1, 2, ..., d$.

For any $i \neq j$ with $i \notin S_v$, we claim that $E^2_{ij}v = E^2_{ji}v = 0$. We have $E^2_{ij}v, E^2_{ji}v \in M_n$ by Claim 2. If $E^2_{ij}v \neq 0$, we would have $\mathrm{Supp}(E^2_{ij}v) = \{\mu' = \mu + 2\alpha_{i,j} : \mu \in \mathrm{Supp}(v)\}$ by the minimality of $\mathrm{Supp}(v)$. Since $i \notin S_v$ there are $\mu, \nu \in \mathrm{Supp}(v)$ with $\mu_i \neq \nu_i$ and hence $\mu'_i \neq \nu'_i$. On the other hand, $\mu'_i + \nu'_i = \mu_i + 2 + \nu_i + 2 = 5 \neq 1$, contradicting the result in the last paragraph (with $E^2_{ij}v$ replacing v). So we have $E^2_{ij}v = 0$. Similarly, we get $E^2_{ji}v = 0$.

For any $i \neq j$ with $i \notin S_v$, we claim that $E_{ij}v_\mu = 0$ if $\mu_i \neq 0$ and $E_{ji}v_\mu = 0$ if $\mu_j \neq 0$, for any $\mu \in \mathrm{Supp}(v)$. We know that $E_{ij}E_{ii}v \in M_n$ from Claim 2. Similarly to the previous paragraph, one shows that $E_{ij}E_{ii}v = 0$ and hence $E_{ij}E_{ii}v_\mu = 0$, which implies that $E_{ij}v_\mu = 0$ if $\mu_i \neq 0$. We can deduce that $E_{ji}v_\mu = 0$ if $\mu_j \neq 0$ in the same way. Part (2) follows.

Since $p \geqslant 2$, for any $\lambda \in \mathrm{Supp}(v)$ with $\lambda_i \neq 0$ for some $i \notin S_v$ (such a λ exists), we have the following arguments.

For any $i \neq j$ with $i \notin S_v$ and $\lambda_i\lambda_j \neq 0$, we have $E_{ij}v_\lambda = E_{ji}v_\lambda = 0$ and hence $[E_{ij}, E_{ji}]v_\lambda = (E_{ii} - E_{jj})v_\lambda = (\lambda_i - \lambda_j)v_\lambda = 0$. Thus $\lambda_i = \lambda_j$. Since $p \geqslant 2$, in the case $\lambda_1 = \lambda_2 = \cdots = \lambda_d \neq 0$, there exists $\mu \in \mathrm{Supp}(v)$ with $\mu_i = 0$, $i \notin S_v$ (otherwise $\mu = \lambda$). Thus $\lambda_i = 1$ by $\mu_i + \lambda_i = 1$ and hence $\lambda_1 = \lambda_2 = \cdots = \lambda_d = 1$. Then $\mu_1 + \cdots + \mu_d = d$ implies that $\mu_j \geq 2$ for some $j \neq i$, contradicting the fact that $\lambda_j = \mu_j$ or $\lambda_j + \mu_j = 1$. So there exists some $\lambda_j = 0$.

For $i \neq j$ with $i \notin S_v$ and $\lambda_i \neq 0, \lambda_j = 0$, the facts that $E^2_{j,i}v_\lambda = 0$ and $E_{i,j}v_\lambda = 0$ imply that $0 = [E_{i,j}, E^2_{j,i}]v_\lambda = 2(\lambda_i - 1)E_{j,i}v_\lambda = 0$. Since $[E_{i,j}, E_{j,i}]v_\lambda = \lambda_i v_\lambda \neq 0$, then $E_{j,i}v_\lambda \neq 0$. We have $\lambda_i = 1$. That is, $\lambda_k \in \{0,1\}$ for any $k = 1, ..., d$. Since $\lambda_k = \mu_k$ or $\lambda_k + \mu_k = 1$, we see that $\mu_k \in \{0,1\}$ for all $\mu \in \mathrm{Supp}(v)$. Claim 4 follows.

We define a total ordering $<$ on $\Lambda = \bigoplus_{i=1}^d \mathbb{Z}\epsilon_i$ as $k_1\epsilon_1+\cdots+k_d\epsilon_d < l_1\epsilon_1+\cdots+l_d\epsilon_d$ if there exists an i such that $k_j = l_j$ for $j < i$ and $k_i < l_i$. Now fix any $v \in K$ such that $\lambda = \max(\mathrm{Supp}(v))$ is maximal with respect to this order.

We claim that λ is the highest weight of $V(\psi, b)$, that is, $\psi = \lambda$. Otherwise, there is some i such that $E_{i,i+1}v_\lambda \neq 0$. By Claim 4(2), we know that $i \in S_v$ and $E_{ii}v \in \mathbb{C}v \subset M_n$. Then Claim 2 ensures that $(n_{i+1}+\alpha_{i+1})E_{ii}v-(n_i+\alpha_i)E_{i,i+1}v \in M_n$ and hence $w = E_{i,i+1}v \in M_n$. By the minimality of $|\mathrm{Supp}(v)|$, we have $|\mathrm{Supp}(w)| = p$; that is, $w \in K$. However, it is clear that $\max(\mathrm{Supp}(w)) = \lambda + \epsilon_i - \epsilon_{i+1}$, which is larger than $\max(\mathrm{Supp}(v)) = \lambda$, contradicting the choice of v. So $\lambda = \psi$.

From Claim 4, we know that $\lambda_i \in \{0,1\}$. Since λ is dominant, we must have $\psi = \lambda = \delta_k$, where k is the number of i with $\lambda_i = 1$. Finally, $(\sum_{i=1}^d E_{ii})v_\lambda = kv_\lambda$ implies $b = k$. This gives $(\psi, b) = (\delta_k, k)$ and completes the proof. $\square$

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