

MULTIPLIERS AND ESSENTIAL NORM ON THE DRURY-ARVESON SPACE

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ABSTRACT. It is well known that for multipliers f of the Drury-Arveson space H_n^2 , $\|f\|_\infty$ does not dominate the operator norm of M_f . We show that in general $\|f\|_\infty$ does not even dominate the essential norm of M_f . A consequence of this is that there exist multipliers f of H_n^2 for which M_f fails to be essentially hyponormal; i.e., if K is any compact, self-adjoint operator, then the inequality $M_f^* M_f - M_f M_f^* + K \geq 0$ does not hold.

1. INTRODUCTION

Let \mathbf{B} denote the open unit ball $\{z : |z| < 1\}$ in \mathbf{C}^n . In this paper, the complex dimension n is assumed to be greater than or equal to 2. An analogue of the classic Hardy space is the space H_n^2 of analytic functions on \mathbf{B} introduced by Drury [8] and Arveson [2]. Because of its connection to various topics in operator theory, e.g. the von Neumann inequality for commuting row contractions, H_n^2 has been the subject of intense recent studies [1]–[7], [9], [10], [12].

Recall that the Drury-Arveson space H_n^2 is a reproducing kernel Hilbert space with

$$K(z, w) = \frac{1}{1 - \langle z, w \rangle}, \quad z, w \in \mathbf{B},$$

as its kernel [2, 8]. Note that $K(z, w)$ is a multivariable-generalization of the one-variable Szegő kernel. An orthonormal basis of H_n^2 is given by $\{(|\alpha|!/\alpha!)^{1/2}\zeta^\alpha : \alpha \in \mathbf{Z}_+^n\}$, where we use the standard multi-index notation. Thus for functions $f, g \in H_n^2$ with Taylor expansions

$$f(\zeta) = \sum_{\alpha \in \mathbf{Z}_+^n} c_\alpha \zeta^\alpha \quad \text{and} \quad g(\zeta) = \sum_{\alpha \in \mathbf{Z}_+^n} d_\alpha \zeta^\alpha,$$

the inner product is given by

$$\langle f, g \rangle = \sum_{\alpha \in \mathbf{Z}_+^n} \frac{\alpha!}{|\alpha|!} c_\alpha \overline{d_\alpha}.$$

With the identification of each variable ζ_i with each multiplication operator M_{ζ_i} , H_n^2 is a free Hilbert module over the polynomial ring $\mathbf{C}[\zeta_1, \dots, \zeta_n]$. See [2].

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An analytic function f on \mathbf{B} is said to be a *multiplier* of the Drury-Arveson space H_n^2 if $fg \in H_n^2$ for every $g \in H_n^2$. Throughout the paper, we denote the collection of multipliers of H_n^2 by \mathcal{M} . For each $f \in \mathcal{M}$, the multiplication operator M_f defined by $M_f g = fg$ is necessarily bounded on H_n^2 [2], and the operator norm $\|M_f\|$ is also called the multiplier norm of f .

Multipliers are an important part of operator theory on H_n^2 . For example, if \mathcal{E} is a closed linear subspace of H_n^2 which is invariant under $M_{\zeta_1}, \dots, M_{\zeta_n}$, then there exist $\{f_1, \dots, f_k, \dots\} \subset \mathcal{M}$ such that the operator

$$M_{f_1} M_{f_1}^* + \dots + M_{f_k} M_{f_k}^* + \dots$$

is the orthogonal projection from H_n^2 onto \mathcal{E} (see page 191 in [3]).

Among the recent results related to multipliers, we would like to mention the following developments. Interpolation problems for multipliers and model theory related to the Drury-Arveson space have been intensely studied over the past decade or so [4, 5, 10, 12]. Recently, Arcozzi, Rochberg and Sawyer gave a characterization of the multipliers in terms of Carleson measures for H_n^2 [1]. In [7], Costea, Sawyer and Wick proved a corona theorem for \mathcal{M} . More recently, we showed in [9] that for each $f \in \mathcal{M}$ and each $1 \leq i \leq n$, the commutator $[M_f^*, M_{\zeta_i}]$ belongs to the Schatten class \mathcal{C}_p , $p > 2n$.

Of particular relevance to this paper is the fact that under the assumption $n \geq 2$, \mathcal{M} is strictly smaller than H^∞ [2]. Moreover, Arveson showed in [2] that, even for polynomials q , $\|q\|_\infty$ in general does not dominate the operator norm of M_q on H_n^2 . This naturally brings up the question, what about the essential norm of M_f on H_n^2 for general $f \in \mathcal{M}$?

Recall that the *essential norm* of a bounded operator A on a Hilbert space \mathcal{H} is

$$\|A\|_{\mathcal{Q}} = \inf\{\|A + K\| : K \in \mathcal{K}(\mathcal{H})\},$$

where $\mathcal{K}(\mathcal{H})$ is the collection of compact operators on \mathcal{H} . Alternately, $\|A\|_{\mathcal{Q}} = \|\pi(A)\|$, where π denotes the quotient homomorphism from $\mathcal{B}(\mathcal{H})$ to the Calkin algebra $\mathcal{Q} = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

Let \mathcal{T}_n be the C^* -algebra generated by $M_{\zeta_1}, \dots, M_{\zeta_n}$ on H_n^2 , which was introduced by Arveson in [2]. In more ways than one, \mathcal{T}_n is the analogue of the C^* -algebra generated by Toeplitz operators with *continuous* symbols. Indeed Arveson showed that there is an exact sequence

$$(1.1) \quad \{0\} \rightarrow \mathcal{K}(H_n^2) \rightarrow \mathcal{T}_n \xrightarrow{\tau} C(S) \rightarrow \{0\},$$

where the homomorphism τ is an extension of the map

$$\tau(M_{\zeta_j}) = \zeta_j,$$

$j = 1, \dots, n$, and $S = \{z \in \mathbf{C}^n : |z| = 1\}$. It follows that if q is a *polynomial*, then

$$(1.2) \quad \|M_q\|_{\mathcal{Q}} = \|q\|_\infty.$$

This equality can also be understood from a slightly different point of view. Indeed by Proposition 5.3 in [2], for each polynomial q , the operator M_q is *essentially normal*, i.e., $[M_q^*, M_q]$ is compact. On the other hand, by Proposition 2.12 in [2], if q is a polynomial, then the spectral radius of M_q equals $\|q\|_\infty$. Since the norm and the spectral radius of any normal element in any C^* -algebra coincide, it follows that $\|M_q\|_{\mathcal{Q}} \leq \|q\|_\infty$ whenever q is a polynomial. The reverse inequality, $\|M_q\|_{\mathcal{Q}} \geq \|q\|_\infty$, is easy once M_q^* is applied to the normalized reproducing kernel of H_n^2 .

Equality (1.2) is particularly interesting in view of the fact that, even for polynomials, $\|q\|_\infty$ in general does not dominate the operator norm of M_q . The obvious question is, what happens if we consider a general $f \in \mathcal{M}$?

We report that (1.2) in general fails if we consider multipliers which are not polynomials. More specifically, the main results of the paper are Theorems 1.1 and 1.2 below.

Theorem 1.1. *There exists a sequence $\{\psi_k\} \subset \mathcal{M}$ such that*

$$\inf_{k \geq 1} \|M_{\psi_k}\|_{\mathcal{Q}} > 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \|\psi_k\|_\infty = 0.$$

This has implications for other essential properties of multipliers.

Recall that an operator T is said to be *hyponormal* if $T^*T - TT^* \geq 0$. It is well known that the norm of a hyponormal operator coincides with its spectral radius. As we mentioned earlier, by Proposition 2.12 in [2], if q is a polynomial, then the spectral radius of M_q equals $\|q\|_\infty$. Therefore if q is a polynomial such that $\|M_q\| > \|q\|_\infty$, then M_q is not hyponormal. Thus there are plenty of multipliers $f \in \mathcal{M}$ for which M_f fails to be hyponormal on H_n^2 . This is one phenomenon that sets the Drury-Arveson space apart from the Hardy space and the Bergman space. We will show that this phenomenon persists under compact perturbation.

Definition. An operator T is said to be *essentially hyponormal* if there is a compact self-adjoint operator K such that

$$T^*T - TT^* + K \geq 0.$$

Obviously, T is essentially hyponormal if and only if $\pi(T)$ is a hyponormal element in the Calkin algebra \mathcal{Q} , i.e., $\pi(T^*)\pi(T) - \pi(T)\pi(T^*) \geq 0$.

Theorem 1.2. *There exists a $\psi \in \mathcal{M}$ such that the multiplication operator M_ψ on H_n^2 is not essentially hyponormal.*

Having introduced our results, the rest of this short paper consists of their proofs.

2. ESTIMATES FOR CERTAIN MULTIPLIERS

The proof of Theorem 1.1 involves Möbius transform. For each $z \in \mathbf{B} \setminus \{0\}$, let

$$(2.1) \quad \varphi_z(\zeta) = \frac{1}{1 - \langle \zeta, z \rangle} \left\{ z - \frac{\langle \zeta, z \rangle}{|z|^2} z - (1 - |z|^2)^{1/2} \left(\zeta - \frac{\langle \zeta, z \rangle}{|z|^2} z \right) \right\}.$$

Then φ_z is an involution, i.e., $\varphi_z \circ \varphi_z = \text{id}$ [13, Theorem 2.2.2]. Recall that the normalized reproducing kernel for H_n^2 is given by

$$(2.2) \quad k_z(\zeta) = \frac{(1 - |z|^2)^{1/2}}{1 - \langle \zeta, z \rangle}, \quad z, \zeta \in \mathbf{B}.$$

For each $z \in \mathbf{B} \setminus \{0\}$, define the operator U_z by the formula

$$(2.3) \quad (U_z g)(\zeta) = g(\varphi_z(\zeta))k_z(\zeta), \quad g \in H_n^2.$$

It follows easily from Theorem 2.2.2 in [13] that if $z \in \mathbf{B} \setminus \{0\}$ and $x, y \in \mathbf{B}$, then

$$\langle U_z k_x, U_z k_y \rangle = \frac{(1 - |x|^2)^{1/2}(1 - |y|^2)^{1/2}}{1 - \langle y, x \rangle} = \langle k_x, k_y \rangle.$$

Hence each U_z is a unitary operator on H_n^2 . Moreover, we have

$$(2.4) \quad U_z M_f U_z^* = M_{f \circ \varphi_z}$$

for all $z \in \mathbf{B} \setminus \{0\}$ and $f \in \mathcal{M}$.

For each $j \in \mathbf{N}$, let E_j be the linear span of $\{\zeta^\alpha : |\alpha| \leq j\}$ in H_n^2 , and let $P_j : H_n^2 \rightarrow E_j$ be the orthogonal projection. Moreover, denote

$$Q_j = 1 - P_j.$$

Obviously, we have the strong convergence $Q_j \rightarrow 0$ as $j \rightarrow \infty$.

Lemma 2.1. *For each $j \in \mathbf{N}$, there is a constant $1 \leq C_j < \infty$ such that*

$$\limsup_{|z| \uparrow 1} \|M_{f \circ \varphi_z} P_j\| \leq C_j \|M_f\|_{\mathcal{Q}}$$

for every $f \in \mathcal{M}$.

Proof. For each $j \in \mathbf{N}$, since $\dim(E_j) < \infty$, any two norms on E_j are equivalent. Since E_j consists of polynomials, we have $\|M_g\| < \infty$ for each $g \in E_j$. Hence there is a C_j such that

$$(2.5) \quad \|M_g\| \leq C_j \|g\|_{H_n^2}$$

for every $g \in E_j$. Now let $f \in \mathcal{M}$. Using the unitary U_z , we have $\|f \circ \varphi_z\|_{H_n^2} = \|U_z(f \circ \varphi_z)\|_{H_n^2} = \|fk_z\|_{H_n^2}$. Since $k_z \rightarrow 0$ weakly as $|z| \uparrow 1$, we have

$$\limsup_{|z| \uparrow 1} \|f \circ \varphi_z\|_{H_n^2} = \limsup_{|z| \uparrow 1} \|fk_z\|_{H_n^2} \leq \limsup_{|z| \uparrow 1} \|(M_f + K)k_z\|_{H_n^2}$$

for every compact operator K . Consequently,

$$(2.6) \quad \limsup_{|z| \uparrow 1} \|f \circ \varphi_z\|_{H_n^2} \leq \|M_f\|_{\mathcal{Q}}.$$

Now if $g \in E_j$, then, using (2.5), we have

$$\|M_{f \circ \varphi_z} g\|_{H_n^2} = \|M_g(f \circ \varphi_z)\|_{H_n^2} \leq \|M_g\| \|f \circ \varphi_z\|_{H_n^2} \leq C_j \|f \circ \varphi_z\|_{H_n^2} \|g\|_{H_n^2}.$$

Hence $\|M_{f \circ \varphi_z} P_j\| \leq C_j \|f \circ \varphi_z\|_{H_n^2}$. Combining this with (2.6), the lemma follows. \square

The next lemma is so elementary that its proof will be omitted.

Lemma 2.2. *For each bounded operator A on H_n^2 , we have*

$$\limsup_{j \rightarrow \infty} \|Q_j A\| \leq \|A\|_{\mathcal{Q}} \quad \text{and} \quad \limsup_{j \rightarrow \infty} \|A Q_j\| \leq \|A\|_{\mathcal{Q}}.$$

Proof of Theorem 1.1. By Theorem 3.3 in [2], there is a sequence of polynomials $\{p_i\}$ such that

$$(2.7) \quad \|M_{p_i}\| = 1$$

for every i and

$$(2.8) \quad \lim_{i \rightarrow \infty} \|p_i\|_{\infty} = 0.$$

We will find a sequence of natural numbers $\{i(j)\}_{j=1}^{\infty}$ and a sequence $\{z_j\} \subset \mathbf{B} \setminus \{0\}$ such that the desired multipliers $\{\psi_k\}$ will have the form

$$(2.9) \quad \psi_k = \sum_{j=k}^{\infty} p_{i(j)} \circ \varphi_{z_j},$$

$k \in \mathbf{N}$. To do this, we note that (2.8) enables us to inductively select an ascending sequence of natural numbers

$$\ell(1) < \ell(2) < \dots < \ell(m) < \dots$$

such that

$$(2.10) \quad C_m \|p_{\ell(m)}\|_{\infty} \leq \frac{1}{2^m}$$

for each $m \in \mathbf{N}$, where C_m is the constant provided by Lemma 2.1. Since each p_i is a polynomial, by (1.2) this implies

$$(2.11) \quad C_m \|M_{p_{\ell(m)}}\|_{\mathcal{Q}} \leq \frac{1}{2^m}.$$

By (2.11) and Lemma 2.1, for each $m \in \mathbf{N}$ there is a $w_m \in \mathbf{B} \setminus \{0\}$ such that

$$(2.12) \quad \|M_{p_{\ell(m)} \circ \varphi_{w_m}} P_m\| \leq \frac{2}{2^m}.$$

It follows from Lemma 2.2 that for each $m \in \mathbf{N}$ there is a natural number $r(m) > m$ such that

$$\|M_{p_{\ell(m)} \circ \varphi_{w_m}} Q_{r(m)}\| \leq 2 \|M_{p_{\ell(m)} \circ \varphi_{w_m}}\|_{\mathcal{Q}} = 2 \|M_{p_{\ell(m)}}\|_{\mathcal{Q}},$$

where the = sign is a consequence of (2.4). By (2.11) and the fact that $C_m \geq 1$, we have

$$(2.13) \quad \|M_{p_{\ell(m)} \circ \varphi_{w_m}} Q_{r(m)}\| \leq \frac{2}{2^m}.$$

By a similar argument, for each $m \in \mathbf{N}$, there is an $s(m) > m$ such that

$$(2.14) \quad \|Q_{s(m)} M_{p_{\ell(m)} \circ \varphi_{w_m}}\| \leq \frac{2}{2^m}.$$

Note that for each m , the subspace $Q_m H_n^2$ is invariant under $\{M_f : f \in \mathcal{M}\}$. That is,

$$M_f g = Q_m M_f g \quad \text{if } g \in Q_m H_n^2 \quad \text{and } f \in \mathcal{M}.$$

Using this fact and the relation $P_i = 1 - Q_i$, it follows from simple algebra that

$$\begin{aligned} M_{p_{\ell(m)} \circ \varphi_{w_m}} &= M_{p_{\ell(m)} \circ \varphi_{w_m}} P_m + Q_{s(m)} M_{p_{\ell(m)} \circ \varphi_{w_m}} (Q_m - Q_{r(m)}) \\ &\quad + M_{p_{\ell(m)} \circ \varphi_{w_m}} Q_{r(m)} + (P_{s(m)} - P_m) M_{p_{\ell(m)} \circ \varphi_{w_m}} (P_{r(m)} - P_m). \end{aligned}$$

Thus if we set

$$Y_m = M_{p_{\ell(m)} \circ \varphi_{w_m}} P_m + Q_{s(m)} M_{p_{\ell(m)} \circ \varphi_{w_m}} (Q_m - Q_{r(m)}) + M_{p_{\ell(m)} \circ \varphi_{w_m}} Q_{r(m)},$$

then

$$(2.15) \quad M_{p_{\ell(m)} \circ \varphi_{w_m}} = Y_m + (P_{s(m)} - P_m) M_{p_{\ell(m)} \circ \varphi_{w_m}} (P_{r(m)} - P_m),$$

$m \in \mathbf{N}$. Note that by (2.12), (2.13) and (2.14), we have

$$(2.16) \quad \|Y_m\| \leq \frac{6}{2^m}.$$

Set $m_1 = 5$. We then inductively select a sequence of integers $m_1 < m_2 < \dots < m_j < \dots$ such that the inequality

$$(2.17) \quad m_{j+1} > \max \{r(m_j), s(m_j)\}$$

holds for every $j \geq 1$. Now set

$$(2.18) \quad i(j) = \ell(m_j) \quad \text{and} \quad z_j = w_{m_j}$$

for each $j \in \mathbf{N}$. With this notation, from (2.15) we obtain

$$(2.19) \quad M_{p_{i(j)} \circ \varphi_{z_j}} = Y_{m_j} + (P_{s(m_j)} - P_{m_j})M_{p_{i(j)} \circ \varphi_{z_j}}(P_{r(m_j)} - P_{m_j}).$$

With $i(j)$ and z_j determined as above, we now define ψ_k by (2.9) for each $k \geq 1$.

Next we show that the sequence $\{\psi_k\}$ has the desired properties. First we need to show that $\psi_k \in \mathcal{M}$ for each k . By the relations $s(m) > m, r(m) > m$ and (2.17), we have

$$(2.20) \quad P_{s(m_j)} - P_{m_j} \perp P_{s(m_{j'})} - P_{m_{j'}} \quad \text{and} \quad P_{r(m_j)} - P_{m_j} \perp P_{r(m_{j'})} - P_{m_{j'}}$$

whenever $j < j'$. Recall that $\|M_{p_i \circ \varphi_z}\| = \|M_{p_i}\|$ by (2.4) and that $\|M_{p_i}\| = 1$ by choice. Combining these facts with (2.20), we see that the norm of the operator

$$B_k = \sum_{j=k}^{\infty} (P_{s(m_j)} - P_{m_j})M_{p_{i(j)} \circ \varphi_{z_j}}(P_{r(m_j)} - P_{m_j})$$

does not exceed 1. By (2.16) and the choice that $m_1 = 5$, the norm of the operator

$$A_k = \sum_{j=k}^{\infty} Y_{m_j}$$

does not exceed 1/2. By (2.9) and (2.19), $M_{\psi_k} = A_k + B_k$. Thus the norm of the operator M_{ψ_k} is at most 3/2. That is, $\psi_k \in \mathcal{M}$ for each $k \in \mathbf{N}$. Applying (2.18) and (2.10), we have

$$\|\psi_k\|_{\infty} \leq \sum_{j=k}^{\infty} \|p_{i(j)} \circ \varphi_{z_j}\|_{\infty} = \sum_{j=k}^{\infty} \|p_{i(j)}\|_{\infty} \leq \sum_{j=k}^{\infty} \frac{1}{2^{m_j}}.$$

Hence

$$\lim_{k \rightarrow \infty} \|\psi_k\|_{\infty} = 0.$$

What remains for the proof is the inequality

$$\inf_{k \geq 1} \|M_{\psi_k}\|_{\mathcal{Q}} > 0.$$

Since $M_{\psi_k} = A_k + B_k$ and $\|A_k\| \leq 1/2$, it suffices to show that

$$\inf_{k \geq 1} \|B_k\|_{\mathcal{Q}} \geq 1.$$

Since $\|M_{p_{i(j)} \circ \varphi_{z_j}}\| = \|M_{p_{i(j)}}\| = 1$ and $\lim_{j \rightarrow \infty} \|Y_{m_j}\| = 0$, by (2.19) we have

$$\lim_{j \rightarrow \infty} \|(P_{s(m_j)} - P_{m_j})M_{p_{i(j)} \circ \varphi_{z_j}}(P_{r(m_j)} - P_{m_j})\| = 1.$$

Therefore there exists a sequence of unit vectors $\{g_j\}$ such that

$$(2.21) \quad g_j \in (P_{r(m_j)} - P_{m_j})H_n^2$$

for every j and

$$(2.22) \quad \lim_{j \rightarrow \infty} \|(P_{s(m_j)} - P_{m_j})M_{p_{i(j)} \circ \varphi_{z_j}}(P_{r(m_j)} - P_{m_j})g_j\|_{H_n^2} = 1.$$

By (2.21) and (2.20), $g_j \rightarrow 0$ weakly as $j \rightarrow \infty$. By this weak convergence and (2.22), we have

$$\lim_{j \rightarrow \infty} \|(B_k + K)g_j\|_{H_n^2} = 1$$

for every compact operator K . This implies $\|B_k\|_{\mathcal{Q}} \geq 1$, completing the proof of Theorem 1.1. □

3. SPECTRAL RADIUS

To prove Theorem 1.2, we begin with a simple fact about multipliers.

Lemma 3.1. *Let $f \in \mathcal{M}$. If there is a $c > 0$ such that $|f(z)| \geq c$ for every $z \in \mathbf{B}$, then $1/f$ is also a multiplier of H_n^2 .*

Proof. This certainly follows from the recently proved corona theorem for \mathcal{M} [7]. But it also follows from an earlier, much simpler result due to Chen [6]. By Theorem 2 in [6], there are constants $0 < A \leq B < \infty$ such that $A\|g\|_{H_n^2} \leq \|g\|_{\#} \leq B\|g\|_{H_n^2}$ for every $g \in H_n^2$, where

$$(3.1) \quad \|g\|_{\#}^2 = |g(0)|^2 + \iint \frac{|g(z) - g(w)|^2}{|1 - \langle z, w \rangle|^{2n+1}} dv(z)dv(w).$$

Let $f \in \mathcal{M}$ be such that $|f| \geq c > 0$ on \mathbf{B} . Then for each $g \in H_n^2$,

$$(3.2) \quad \frac{g(z)}{f(z)} - \frac{g(w)}{f(w)} = \frac{g(z) - g(w)}{f(z)} + \frac{g(z) - g(w)}{f(w)} + \frac{f(w)g(w) - f(z)g(z)}{f(z)f(w)},$$

$z, w \in \mathbf{B}$. From (3.1) and (3.2) we see that $1/f \in \mathcal{M}$. □

From this lemma we immediately obtain

Proposition 3.2. *For each $f \in \mathcal{M}$, the spectrum of the operator M_f on H_n^2 is contained in the closure of $\{f(z) : z \in \mathbf{B}\}$. Consequently the spectral radius of M_f does not exceed $\|f\|_{\infty}$.*

Remark 1. In the case where f has the property that there is a sequence of polynomials $\{p_k\}$ such that $\lim_{k \rightarrow \infty} \|M_f - M_{p_k}\| = 0$, Proposition 3.2 was proved by Arveson. See Proposition 2.12 in [2].

Remark 2. It follows from (3.1) and (3.2) that if $f \in \mathcal{M}$ and if $\inf_{z \in \mathbf{B}} |f(z)| > 0$, then

$$\|M_{1/f}\| \leq C(\|1/f\|_{\infty} + \|1/f\|_{\infty}^2 \|M_f\|).$$

Surprisingly, Chen himself did not seem to notice this fact in [6].

Remark 3. The referee observes that an alternate proof for Lemma 3.1 is to use the Carleson-measure characterization for functions in \mathcal{M} [1]. In this approach, what replaces (3.2) is the formula for high-order derivatives of $1/f$.

Proposition 3.3. *Let $f \in \mathcal{M}$. If f has the property that $\|M_f\|_{\mathcal{Q}} > \|f\|_{\infty}$, then the operator M_f on H_n^2 is not essentially hyponormal.*

Proof. Recall that we denote the quotient map from $\mathcal{B}(H_n^2)$ to the Calkin algebra \mathcal{Q} by π . Let Φ be the GNS representation of \mathcal{Q} on a Hilbert space \mathcal{H} . If M_f were essentially hyponormal, then $\{\pi(M_f)\}^* \pi(M_f) - \pi(M_f) \{\pi(M_f)\}^* \geq 0$ in \mathcal{Q} . Consequently $\Phi(\pi(M_f))$ would be a hyponormal operator on \mathcal{H} .

Write $\text{rad}(T)$ for the spectral radius of any operator T . It is well known that if T is a hyponormal operator, then $\|T\| = \text{rad}(T)$. See Problem 205 in [11]. Thus we would have

$$\|\Phi(\pi(M_f))\| = \text{rad}(\Phi(\pi(M_f))).$$

Since $\Phi \circ \pi$ is a C^* -algebraic homomorphism, the spectrum of $\Phi(\pi(M_f))$ is contained in the spectrum of M_f . Therefore $\text{rad}(\Phi(\pi(M_f))) \leq \text{rad}(M_f)$. Applying Proposition 3.2, we have

$$\text{rad}(\Phi(\pi(M_f))) \leq \text{rad}(M_f) \leq \|f\|_{\infty}.$$

On the other hand, since Φ is a faithful representation, we have

$$\|\Phi(\pi(M_f))\| = \|\pi(M_f)\| = \|M_f\|_{\mathcal{Q}}.$$

These three displayed lines together contradict the assumption $\|M_f\|_{\mathcal{Q}} > \|f\|_{\infty}$. \square

Proof of Theorem 1.2. By Theorem 1.1, there is a $\psi \in \mathcal{M}$ with $\|M_{\psi}\|_{\mathcal{Q}} = c$ for some $c > 0$ while $\|\psi\|_{\infty} < c/2$. That is, we have a $\psi \in \mathcal{M}$ with $\|M_{\psi}\|_{\mathcal{Q}} > \|\psi\|_{\infty}$. For such a ψ , Proposition 3.3 tells us that M_{ψ} is not essentially hyponormal on H_n^2 . \square

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