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CANONICAL FILTRATIONS OF GORENSTEIN INJECTIVE MODULES

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Abstract. The principle "Every result in classical homological algebra should have a counterpart in Gorenstein homological algebra" was given by Henrik Holm. There is a remarkable body of evidence supporting this claim. Perhaps one of the most glaring exceptions is provided by the fact that tensor products of Gorenstein projective modules need not be Gorenstein projective, even over Gorenstein rings. So perhaps it is surprising that tensor products of Gorenstein injective modules over Gorenstein rings of finite Krull dimension are Gorenstein injective.

Our main result is in support of the principle. Over commutative, noetherian rings injective modules have direct sum decompositions into indecomposable modules. We will show that Gorenstein injective modules over Gorenstein rings of finite Krull dimension have filtrations analogous to those provided by these decompositions. This result will then provide us with the tools to prove that all tensor products of Gorenstein injective modules over these rings are Gorenstein injective.

1. Introduction

Throughout this paper R will denote a commutative and noetherian ring and $Spec(R)$ will denote the set of its prime ideals. The term *module* will then mean an R-module. An injective envelope of the module M will be denoted by $E(M)$ and

$$0 \to M \to E^0(M) \to \cdots \to E^n(M) \to \cdots$$

will denote a minimal injective resolution of M.

We will now give several definitions and results. For ease in referring to these later in the paper they will be numbered.

(1) Every injective module is uniquely up to isomorphism the direct sum of modules, each of which is isomorphic to $E(R/P)$ for some $P \in Spec(R)$ [6, Theorem 2.5 and Proposition 3.1].
(2) We say R is a Gorenstein ring if $inj, dim_{R_P} R_P < \infty$ for each $P \in Spec(R)$. If in fact $inj.dim_R R < \infty$, then R is Gorenstein and the Krull dimension of R equals $inj.dim_R R$ [1, Corollary 3.4].
(3) In $inj.dim_R R < \infty$ (and so R is Gorenstein) and if $0 \to R \to E^0(R) \to \cdots E^n(R) \to 0$ is a minimal injective resolution of R as a module, then for

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$0 \le k \le n$, $E^k(R) = \bigoplus_{ht(P)=k} E(R/P)$, where these P are in $Spec(R)$ [1, Proposition 3.6]. Furthermore when $ht(P) = k$, we have $fl.dimE(R/P) = k$ [7, Theorem 5.1.2].

(4) If R is Gorenstein and E, E' are injective modules, then for any $k \ge 0$ the module $Tor_k(E, E')$ is injective. More precisely, if $P, Q \in Spec(R)$, then $Tor_k(E(R/P), E(R/Q)) = 0$ unless both $P = Q$ and $k = ht(P)$. Also in this case we have that $Tor_k(E(R/P), E(R/P)) \cong E(R/P)$ [5, Lemma 2.1 and Theorem 4.1]. So using (1) we see that $Tor_k(E(R/P), E) = 0$ when E is injective and $k \ne ht(P)$.

(5) If $P \in Spec(R)$ a module S will be said to have property $t(P)$ if for each $r \in R - P$ we have that $S \xrightarrow{r} S$ is an isomorphism and if for each $x \in S$ we have that $P^m x = 0$ for some $m \ge 1$. If S has property $t(P)$ and property $t(Q)$ with $P \ne Q$, then it is easy to see that $S = 0$. If S has property $t(P)$ and if N is any module, then $Tor_k(S, N)$ has property $t(P)$ for all $k \ge 0$. This can be seen by using a projective resolution of N to compute this Tor. Consequently, if S has property $t(P)$ and T has property $t(Q)$ where $P \ne Q$, we get $Tor_k(S, T) = 0$ for all $k \ge 0$. For any $P \in Spec(R)$ the module $E(R/P)$ has property $t(P)$ [6, Lemma 3.2].

(6) We now argue that if S has property $t(P)$, then so does $E(S)$. By (1) above $E(S)$ is a direct sum of copies of $E(R/Q)$ for various $Q \in Spec(R)$. If $r \in R-P$, then since $S \xrightarrow{r} S$ is an isomorphism, then so is $E(S) \xrightarrow{r} E(S)$. Now assume that $E(R/Q)$ is a summand of $E(S)$. Then for $r \in R - P$ we have that $E(R/Q) \xrightarrow{r} E(R/Q)$ is an isomorphism. Hence $r \in R - Q$. So we get $Q \subset P$. We want to argue that $Q = P$. If not, let $r \in P - Q$. The extension $S \to E(S)$ is essential, so the module $S' = E(R/Q) \cap S$ is non-zero. Let $x \in S'$, $x \ne 0$. Then since $x \in S$ and since S has property $t(P)$, we get $P^m x = 0$ for some $m \ge 1$. So $r^m x = 0$. But $E(R/Q)$ has property $t(Q)$ and $r \in R - Q$. Hence $E(R/Q) \xrightarrow{r} E(R/Q)$ is an isomorphism. But then since $S' \subset E(R/Q)$ we get that $S' \xrightarrow{r} S'$ is an injection. But this is not possible if $r^m x = 0$, where $x \in S'$ and $x \ne 0$. So we get $Q = P$.

So $E(S)$ is a direct sum of copies of $E(R/P)$, and so by (5) we see that $E(S)$ has property $t(P)$. But then the quotient module $E(S)/S$ will have property $t(P)$. So continuing we see that all the terms $E^i(S)$ $(i \ge 1)$ in a minimal injective resolution of S have property $t(P)$.

(7) If S has property $t(P)$ and T has property $t(Q)$ and if $P \not\subset Q$, then $Hom(S, T) = 0$. If $r \in P - Q$ and if $f(x) = y$ for some $f \in Hom(S, T)$, we have $r^n x = 0$ for some $n \ge 1$, and so $r^n y = 0$. But since $r \notin Q$ this is possible only if $y = 0$. So we get $f = 0$.

(8) We recall that a module G is said to be Gorenstein injective if and only if there is an exact sequence

$$\cdots \to E_2 \to E_1 \to E_0 \to E^0 \to E^1 \to E^2 \to \cdots$$

of injective modules with $G = Ker(E^0 \to E^1)$ and such that $Hom(E, -)$ leaves the sequence exact whenever E is an injective module. For the rest of (8) we assume that R is a Gorenstein ring of finite Krull dimension n. If $n \ge 1$, a module G is Gorenstein injective if and only if there is an exact sequence

$$E_{n-1} \to \cdots \to E_1 \to E_0 \to G \to 0$$

with $E_{n-1}, \ldots, E_0$ injective modules. This result gives that the class of Gorenstein injective modules over such R is closed under arbitrary direct sums. Also if $n = 0$, then every module G is Gorenstein injective (see [4, Theorem 4.2] for both these claims). As a consequence we get that if P is a minimal prime ideal of R and if G is an R_P-module, then G is a Gorenstein injective R-module. This follows from the observation that R_P is a flat R-module, so any injective R_p module is also an injective R-module. Hence an exact sequence $\cdots E_2 \to E_1 \to E_0 \to G \to 0$ of R_P-modules with the E_k injective R_P-modules gives us an exact sequence of R-modules with the E_k injective R-modules.

We need a slightly stronger version of the result above. So again we suppose R is Gorenstein and of Krull dimension n but with $n \geq 1$. We claim that if G is such that there is an exact sequence

$$G_{n-1} \to \cdots \to G_0 \to G \to 0$$

with $G_{n-1}, \ldots, G_0$ all Gorenstein injective, then G is Gorenstein injective. By [6, Proposition 1.11], G is Gorenstein injective if and only if $Ext^1(L, G) = 0$ whenever $proj.dimL < \infty$. By [4, Corollary 4.4], we have that $proj.dimL < \infty$ implies $proj.dimL \leq n$. So now using dimension shifting and these results we get that G is Gorenstein injective.

(9) If G is Gorenstein injective over any Gorenstein R and $r \in R$ is R-regular, then $proj.dimR/(r) = 1$. So $Ext^1(R/(r), G) = 0$ by (8). This gives that $Hom(R, G) \xrightarrow{r} Hom(R, G) \to 0$ is exact. This means that $G \xrightarrow{r} G$ is surjective. So for every $x \in G$ there is a $y \in G$ with $ry = x$. Consequently we get that $G \otimes T = 0$ if T has property $t(P)$ and if $r \in P$. Also, if $x \in G$ and $y \in T$ and $n \geq 1$ we have that $x = r^n\overline{x}$ for some $\overline{x} \in G$. So $x \otimes y = r^n\overline{x} \otimes y = \overline{x} \otimes r^n y$. But if n is sufficiently large we have $r^n y = 0$. Hence $x \otimes y = 0$.

Now if $P \in X$ and if $ht(P) \geq 1$, then since R is Gorenstein (and so Cohen-Macaulay) there is an R-regular $r \in P$. Consequently we get that $G \otimes T = 0$ whenever G is Gorenstein injective and when T has property $t(P)$ with $ht(P) \geq 1$.

2. Torsion products of injective and Gorenstein injective modules

In this section R will be a Gorenstein ring of finite Krull dimension n. We let $X = Spec(R)$. When we refer to (1),(2),...,(9) we mean the corresponding result in the preceding section.

Lemma 2.1. *If $P \in X$ and $ht(P) \geq 1$, then for any Gorenstein injective module G we have $E(R/P) \otimes G = 0$.*

Proof. By (5) we know that $E(R/P)$ has property $t(P)$. So this result is a special case of (9). □

Proposition 2.2. *If G is Gorenstein injective and $P \in X$, then $Tor_k(E(R/P), G) = 0$ if $ht(P) \neq k$.*

Proof. By (3) we know that $fl.dimE(R/P) = ht(P)$, so $Tor_k(E(R/P), -) = 0$ if $k > ht(P)$. Therefore, we only need prove that $Tor_k(E(R/P), G) = 0$ when G is Gorenstein injective and $k < ht(P)$. We prove this by induction on k. If $k = 0$, then

$Tor_k(E(R/P), G) = E(R/P) \otimes G = 0$ if $ht(P) \geq 1$ and G is Gorenstein injective by Lemma 2.1.

So now we make an induction hypothesis and let $ht(P) > k$ and let G be Gorenstein injective. We have an exact sequence $0 \to H \to E \to G \to 0$ with E injective and H Gorenstein injective. We have the exact sequence $Tor_k(E(R/P), E) \to Tor_k(E(R/P), G) \to Tor_{k-1}(E(R/P), H)$. By the induction hypothesis and the fact that $ht(P) > k > k-1$, we have that $Tor_{k-1}(E(R/P), H) = 0$. But $Tor_k(E(R/P), E) = 0$ by (4), and so $Tor_k(E(R/P), G) = 0$. □

Corollary 2.3. *If $0 \to G' \to G \to G'' \to 0$ is an exact sequence of Gorenstein injective modules and if E is an injective module, then for any $k \geq 0$ the sequence $0 \to Tor_k(E, G') \to Tor_k(E, G) \to Tor_k(E, G'') \to 0$ is exact.*

Proof. By (1) E is a direct sum of submodules isomorphic to $E(R/P)$ with $P \in X$; it suffices to prove the claim when $E = E(R/P)$ for any P. In this case the claim follows by considering the long exact sequence of $Tor(E(R/P), -)$ associated with $0 \to G' \to G \to G'' \to 0$ and Proposition 2.2. □

Proposition 2.4. *If G is Gorenstein injective and E is injective, then for any $k \geq 0$ the module $Tor_k(E, G)$ is a Gorenstein injective module.*

Proof. By (8) we have an exact sequence $\cdots \to E_2 \to E_1 \to E_0 \to G \to 0$ with all the E_i injective modules, where the kernels of $E_0 \to G$, $E_1 \to E_0, \ldots$ are Gorenstein injective. Therefore, we can split the exact sequence into short exact sequences $0 \to G_1 \to E_0 \to G \to 0$, $0 \to G_2 \to E_1 \to G_1 \to 0, \ldots$ with each G_k and G Gorenstein injective. We then apply Corollary 2.3 and splice the resulting short exact sequences together to get the exact sequence $\cdots \to Tor_k(E, E_1) \to Tor_k(E, E_0) \to Tor_k(E, G) \to 0$. Since each $Tor_k(E, E_n)$ is injective we get that $Tor_k(E, G)$ is Gorenstein injective by the Proposition (8). □

3. Filtrations of Gorenstein injective modules

We again let R be a Gorenstein ring of finite Krull dimension n and let $X = Spec(R)$ and let $X_k \subset X$ for $k \geq 0$ consist of the $P \in X$ such that $ht(P) = k$. In this section we will also appeal to the results (1) , . . . , (9) of the first section.

The main contribution of this paper is the following result.

Theorem 3.1. *If G is a Gorenstein injective module, then G has a filtration $0 = G_{n+1} \subset G_n \subset \cdots \subset G_2 \subset G_1 \subset G_0 = G$ where each G_k/G_{k+1} $(0 \leq k \leq n)$ is Gorenstein injective and has a direct sum decomposition indexed by $P \in X_k$ such that the summand, say S, corresponding to P has the property $t(P)$ (see (5)). Furthermore, such filtrations and direct sum decompositions are unique and functorial in G.*

Proof. We first comment that "functorial in G" means that if H is another Gorenstein injective module with such a filtration $0 = H_{n+1} \subset H_n \subset \cdots \subset H_1 \subset H_0 = H$ where T is the summand of H_k/H_{k+1} corresponding to $P \in X_k$ and if $f : G \to H$ is linear, then $f(G_k) \subset H_k$ for each k and the induced map $G_k/G_{k+1} \to H_k/H_{k+1}$ maps S (as in the theorem) into T.

Now let $0 \to R \to E^0(R) \to \cdots \to E^n(R) \to 0$ be the minimal injective resolution of R and let $\cdots \to P_2 \to P_1 \to P_0 \to G \to 0$ be a projective resolution of G. We form the double complex

$$\begin{array}{ccccccccc} & & 0 & & & & 0 & & \\ & & \uparrow & & & & \uparrow & & \\ 0 & \longrightarrow & E^0(R)\otimes P_0 & \longrightarrow & \cdots\cdots\cdots & \longrightarrow & E^n(R)\otimes P_0 & \longrightarrow & 0 \\ & & \uparrow & & & & \uparrow & & \\ 0 & \longrightarrow & E^0(R)\otimes P_1 & \longrightarrow & \cdots\cdots\cdots & \longrightarrow & E^n(R)\otimes P_1 & \longrightarrow & 0 \\ & & \uparrow & & & & \uparrow & & \\ & & \vdots & & & & \vdots & & \end{array}$$

We now use a simple spectral sequence argument. First we note that this double complex can be regarded as a third quadrant double complex (using a shift in indices). This will guarantee convergence of our spectral sequences. For the E^1 term of our first spectral sequence we compute homology of this double complex by using the horizontal arrows. Since each P_n is projective, and so flat, we now get the transpose of the diagram

$$\cdots \to R \otimes P_1 \to R \otimes P_0 \to 0,$$

where all the missing terms are 0. But now when we compute homology we just get G (in the $(0,0)$ position).

We now first use the vertical arrows to compute homology. The terms we get will all be of the form $Tor_i(E^j(R), G)$. By Proposition 2.2 and (3) these are 0 unless $i = j$. Therefore, we get a diagonal double complex. Hence the horizontal differentials will be 0, and when we compute homology again we get $\bigoplus_{k=0}^n Tor_k(E^k(R), G)$. This means that G has a filtration $0 = G_{n+1} \subset G_n \subset \cdots \subset G_1 \subset G_0 = G$ with $G_k/G_{k+1} \cong Tor_k(E^k(R), G)$ for $0 \le k \le n$. By Proposition 2.4 we know that each of these terms is Gorenstein injective.

By (3) $E^k(R) = \bigoplus_{P \in X_k} E(R/P)$ and so we have that $Tor_k(E^k(R), G) = \bigoplus_{P \in X_k} Tor_k(E(R/P), G)$. Since each $E(R/P)$ has property $t(P)$ by (5), so does $Tor_k(E(R/P), G)$.

The uniqueness and functoriality will now follow from (7); i.e. if P, Q are prime ideals of R with $P \not\subset Q$, then $Hom(S, T) = 0$ whenever S and T have properties $t(P)$ and $t(Q)$ respectively.

We now indicate how this observation gives us the functoriality and uniqueness. Let $0 \subset G_n \subset \cdots G_1 \subset G$ and $0 \subset H_n \subset \cdots \subset H_1 \subset H$ be filtrations of the Gorenstein injective modules G and H satisfying the conclusion of the theorem. Let $S \subset G_n$ be the summand of G_n corresponding to a given maximal ideal P of R. Assume $n \ge 1$. Then we use the observation that $Hom(S, U) = 0$ if $U \subset H/H_1$ is the summand corresponding to some $Q \in X_0$. Since this holds for all such U we get that $S \hookrightarrow G \to H/H_1$ is 0. Therefore, $f(S) \subset H_1$. Since this is true for all the summands S of G_n, we get that $f(G_n) \subset H_1$. But then we use this argument to get $f(G_n) \subset H_2, \ldots$ and finally that $f(G_n) \subset H_n$.

Repeating the argument but applied to $G/G_n \to H/H_n$ with the induced filtration, we get that $f(G_{n-1}) \subset H_{n-1}$ and then by the induction hypothesis that $f(G_k) \subset H_k$ for $0 \leq k \leq n$.

Now if $P \in X_k$ and if S and T are the summands of G_k/G_{k+1} and H_k/H_{k+1} corresponding to P respectively, then the same type of argument gives that $G_k/G_{k+1} \to H_k/H_{k+1}$ maps S into T.

The uniqueness of the filtrations and direct sum decompositions can be argued by assuming $G = H$ (with possibly different filtrations and direct sum decompositions) and letting $f = 1_G$. So the above would give $G_k \subset H_k$. Then similarly we get $H_k \subset G_k$, and so $G_k = H_k$ for all k. Likewise we get the uniqueness of the direct sum decompositions. □

Remark 3.2. We would like to thank the referee for help in writing this paper. The referee has pointed out that the G_k of Theorem 3.1 can be described by the formulas $G_k/G_{k+1} = \bigoplus_{P \in X_k} \Gamma_P(G/G_{k+1})$ for $k = 0, \ldots, n$, where for a module M we have that $\Gamma_P(M)$ consists of all $x \in M$ such that $P^n x = 0$ for some $n \geq 1$. The referee also suggested that Theorem 3.1 might hold when we assume only the ring R is Cohen-Macaulay, admitting a canonical module. We do not know if this is the case.

4. Tensor products of Gorenstein injective modules

We let R be a Gorenstein ring of finite Krull dimension n. We want to show that over such an R all tensor products of Gorenstein injective modules are Gorenstein injective. If G (or H) is a Gorenstein injective module and $0 \leq k \leq n+1$, then G_k (or H_k) will denote the k-th submodule of G (or H) that is part of the filtration provided by Theorem 3.1.

Theorem 4.1. *If G and H are Gorenstein injective modules, then so is $G \otimes H$.*

Proof. If S and T are Gorenstein injective R-modules having properties $t(P)$ and $t(Q)$ respectively, then $S \otimes T = 0$ if $P \neq Q$ (by (5)) and if $P = Q$ and $ht(P) \geq 1$ (by (9)). We use this to argue that $G \otimes H = G/G_1 \otimes H/H_1$. This claim is trivial if $n = 0$, since then $G_1 = H_1 = 0$. So suppose $n \geq 1$. Then using the above and Theorem 3.1 we see that $G_n \otimes H_k/H_{k+1} = 0$ for $k = 0, \ldots, n$. Hence $G_n \otimes H = 0$. Then tensoring the exact $0 \to G_n \to G \to G/G_n \to 0$ with H we get that $G \otimes H = G/G_n \otimes H$.

If $n \geq 2$ (i.e. $n - 1 \geq 1$), then the same argument gives that $G_{n-1}/G_n \otimes H = 0$ and then that $G \otimes H = G/G_{n-1} \otimes H$. Continuing, we get that $G \otimes H = G/G_1 \otimes H$. But then the same type argument gives that $G/G_1 \otimes H = G/G_1 \otimes H/H_1$ and so that $G \otimes H = G/G_1 \otimes H/H_1$.

Now by Theorem 3.1 and (5) we see that $G \otimes H = G/G_1 \otimes H/H_1$ will be a direct sum of modules of the form $S \otimes T$, where S and T both have property $t(P)$ for a minimal prime ideal P of R. But such an S and T are naturally modules over R_P, and hence $S \otimes T$ is an R_P-module. Then by (8) $S \otimes T$ is a Gorenstein injective R-module. So finally noting that the class of Gorenstein injective modules is closed under direct sums (by (8)), we get that $G \otimes H$ is a Gorenstein injective R-module. □

Remark 4.2. With the same hypothesis as in Theorem 4.1, we do not know if each $Tor_k(G, H)$ is also Gorenstein injective when $k > 0$.

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References

[1] H. Bass, On the ubiquity of Gorenstein rings, Math. Z. 82 (1963), 8-28. MR0153708 (27:3669)
[2] L.W. Christensen, Gorenstein Dimensions, Lecture Notes in Mathematics, 1747, Springer-Verlag, Berlin, 2000. MR1799866 (2002e:13032)
[3] L.W. Christensen, H.-B. Foxby, and H. Holm, Beyond totally reflexive modules and back—a survey of Gorenstein dimensions, in “Commutative Algebra—Noetherian and non-Noetherian Perspectives”, Springer-Verlag, New York, 2010, pp. 101-143.
[4] E.E. Enochs and O.M.G. Jenda, Gorenstein injective and projective modules, Math. Z. 220 (1995), 611-635. MR1363858 (97c:16011)
[5] E.E. Enochs and O.M.G. Jenda, Tensor and torsion products of injective modules, Journal of Pure and Applied Algebra 76 (1991), 143-149. MR1145862 (93b:13013)
[6] E. Matlis, Injective modules over Noetherian rings, Pacific J. Math. 8 (1958), 511-528. MR0099360 (20:5800)
[7] J. Xu, Flat covers of modules, Lecture Notes in Mathematics, 1634, Springer-Verlag, Berlin, 1996. MR1438789 (98b:16003)

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