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MULTIPLICATIVE STRONG UNIMODALITY FOR POSITIVE STABLE LAWS

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Abstract. It is known that real non-Gaussian stable laws are unimodal, not additive strongly unimodal, multiplicative strongly unimodal in the symmetric case, and that the only remaining relevant situation for the multiplicative strong unimodality is the one-sided case. It is shown here that positive α-stable distributions are multiplicative strongly unimodal if and only if $\alpha \leq 1/2$.

1. The MSU property and stable laws

A real random variable X is said to be unimodal (or quasi-concave) if there exists $a \in \mathbb{R}$ such that the functions $\mathbb{P}[X \leq x]$ and $\mathbb{P}[X > x]$ are convex in $(-\infty, a)$ and $(a, +\infty)$ respectively. If X is absolutely continuous, this means that its density is non-decreasing on $(-\infty, a]$ and non-increasing on $[a, +\infty)$. The number a is called a mode of X and might not be unique. A well-known example due to Chung shows that unimodality is not stable under convolution, and for this reason the notion of strong unimodality had been introduced in [8]: a real variable X is said to be strongly unimodal if the independent sum $X + Y$ is unimodal for any unimodal variable Y (in particular X itself is unimodal, choosing Y degenerated at zero). In [8] Ibragimov also obtained the celebrated criterion that X is strongly unimodal iff it is absolutely continuous with a log-concave density.

Proving unimodality or strong unimodality properties is simple for variables with given densities, but the problem might turn out complicated when these densities are not explicit. In this paper we will deal with real (strictly) stable variables, where very few closed formulae (given e.g. in [14, p. 66]) are known for the densities. A classical theorem of Yamazato shows that they are all unimodal with a unique mode – see Lemma 1 and the proof of the Theorem in [4] for the previously shown one-sided case and Theorem 53.1 in [12] for the general result. But except in the Gaussian situation it is easy to see that stable laws are not strongly unimodal, because their heavy tails prevent the densities from being everywhere log-concave – see Remark 52.8 in [12].

Having infinitely divisible distributions, stable variables appear naturally in additive identities, a framework where they hence may not preserve unimodality. Stable variables also occur in multiplicative factorizations as a by-product of the

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so-called M-scheme (or M-infinite divisibility), a feature which has been studied extensively by Zolotarev – see Chapter 3 in [14] and also [11] for the one-sided case. Another concrete example of a multiplicative identity involving a stable law is the following. Suppose that X, Y are two positive variables whose Laplace transforms are connected through the identity $\mathbb{E}[e^{-\lambda X}] = \mathbb{E}[e^{-\lambda^\alpha Y}]$ for some $\alpha \in (0,1)$ (up to a reformulation one could also take $\alpha > 1$). Then one has

$$X \stackrel{d}{=} Z_\alpha \times Y^{1/\alpha}$$

where Z_α is an independent standard positive α-stable variable. In such identities, one can be interested in understanding whether the factorization through Z_α does or does not modify some basic distributional properties.

From the point of view of unimodality, it is therefore natural to ask whether stable variables are multiplicative strongly unimodal, in other words, whether their independent product with any unimodal variable remains unimodal or not. A quick answer can be given in the symmetric case because the mode of a symmetric stable variable is obviously zero: Khintchine's theorem entails then that its product with *any* independent variable will be unimodal with mode at zero – see Proposition 3.6 in [5] for details. However, in the non-symmetric case the mode of a stable variable is never zero (this is obvious in the drifted Cauchy case $\alpha = 1$, and we refer to [14, p. 140] for the case $\alpha \neq 1$), so that such a simple argument cannot be applied. The following criterion established in [5, Theorem 3.7] is a multiplicative counterpart to Ibragimov's theorem:

Theorem (Cuculescu-Theodorescu)**.** *Let X be a unimodal random variable such that 0 is not a mode of X. Then X is multiplicative strongly unimodal if and only if it is one-sided and absolutely continuous, with a density f_X having the property that*

$$t \mapsto f_X(e^t) \quad \text{is log-concave in } \mathbb{R} \tag{1.1}$$

when X is non-negative (resp. $t \mapsto f_X(-e^t)$ is log-concave in $\mathbb{R}$ when X is non-positive).

With a slight abuse of notation, in the following we will say that a positive random variable is *MSU* if and only if (1.1) holds. Cases of multiplicative strongly unimodal, positive variables with mode at zero and such that (1.1) does not hold are hence excluded in this definition. But these cases are particular and by the above remark no more relevant to the content of this paper which deals with stable laws. Besides, a change of variable and Ibragimov's theorem entail the following useful characterization for positive variables:

$$X \text{ is MSU} \iff \log X \text{ is strongly unimodal.} \tag{1.2}$$

In particular the MSU property is stable by inversion and also, from Prékopa's theorem, by independent multiplication. Another important feature coming from (1.2) is that the MSU property remains unchanged under rescaling and power transformations, which also comes from the obvious analytical fact that (1.1) holds iff $t \mapsto K_1 e^{a_1 t} f_X(K_2 e^{a_2 t})$ is log-concave for some $a_1 \in \mathbb{R}, a_2 \neq 0$ and $K_1, K_2 > 0$. Notice however that the MSU property is barely connected to the strong unimodality of X itself (several examples of this difference are given in [5]).

In this paper we are interested in the MSU property for positive α-stable laws. For every $\alpha \in]0,1[$, consider f_α the positive α-stable density and Z_α the corresponding random variable, normalized such that

$$\int_0^\infty e^{-\lambda t} f_\alpha(t)dt \;=\; \mathbb{E}\left[e^{-\lambda Z_\alpha}\right] \;=\; e^{-\lambda^\alpha}, \qquad \lambda \geq 0. \tag{1.3}$$

Before studying the MSU property for Z_α, in view of (1.2) one must first ask if $\log Z_\alpha$ is simply unimodal. A positive answer for all $\alpha \in]0,1[$ had been given by Kanter – see Theorem 4.1 in [9] – who also deduced from [4] the decomposition

$$\log Z_\alpha \;=\; \alpha^{-1} \log b_\alpha(U) \;+\; (\alpha-1)/\alpha \log L$$

where L is a standard exponential variable, U an independent uniform variable over $[0,\pi]$, and $b_\alpha(u) = (\sin(\alpha u)/\sin(u))^\alpha (\sin((1-\alpha)u)/\sin(u))^{1-\alpha}$. The random variable $\log L$ is easily seen to be strongly unimodal, but $\log b_\alpha(U)$ is not, at least for α in a neighbourhood of $1/2$ – see the Remark before Theorem 4.1 in [10]. It can also be seen from the beginning of Section 5 in [10] that there exists a sequence $\alpha_n \to 1$ such that Z_{α_n} is not MSU. This leaves the question of the MSU property for positive stable laws unanswered, and our result aims at filling this gap:

Theorem. *The variable Z_α is MSU if and only if $\alpha \leq 1/2$.*

To conclude this introduction, let us give two further reformulations of (1.1) in the positive stable case. The first one lies at the core of our proof, whereas the second one is probably nothing but a mere curiosity. Since f_α is smooth, differentiating twice the logarithm entails that (1.1) is equivalent to the inequality

$$(x^2 f_\alpha''(x) + x f_\alpha'(x)) f_\alpha(x) \;\leq\; x^2 (f_\alpha'(x))^2, \quad x \geq 0. \tag{1.4}$$

If m_α stands now for the mode of Z_α and if $x_\alpha = \inf\{x > m_\alpha,\ f_\alpha''(x) = 0\}$, then we know from (53.13) in [12] (an identity which is proved there for a certain class of positive self-decomposable distributions, but can also be obtained for positive stable laws after taking the weak limit) and the discussion thereafter that (1.4) is true for any $x \in [0, x_\alpha]$. Hence, the MSU property amounts to the fact that it remains true for all $x > x_\alpha$. From (53.13) in [12] we also know that (1.4) is equivalent to the positivity everywhere of the function

$$x \;\mapsto\; \int_0^x (f_\alpha'(x-y) f_\alpha(x) - f_\alpha(x-y) f_\alpha'(x)) y^{-\alpha} dy,$$

but this criterion is not very tractable because of the memory involved in the integral. Thanks to the Humbert-Pollard representation for f_α which will be recalled at the beginning of Section 3, we finally mention that (1.4) is equivalent to

$$\left(\sum_{n\geq 1} (1+\alpha n)^2 \frac{(-1)^n x^{-(1+\alpha n)}}{\Gamma(-n\alpha) n!}\right)\left(\sum_{n\geq 1} \frac{(-1)^n x^{-(1+\alpha n)}}{\Gamma(-n\alpha) n!}\right) \leq \left(\sum_{n\geq 1} (1+\alpha n) \frac{(-1)^n x^{-(1+\alpha n)}}{\Gamma(-n\alpha) n!}\right)^2,$$

a strange inequality which would plainly hold in the opposite direction if the terms of the series had constant signs.

2. Some particular cases

In this section we depict some situations where the MSU property can be proved or disproved directly, thanks to more or less explicit representations for the density f_α or the variable Z_α. First of all, in the case $\alpha = 1/2$ it is readily seen from the known formula

$$f_{1/2}(x) \; = \; \frac{1}{2\sqrt{\pi x^3}} e^{-1/4x}$$

that $Z_{1/2}$ is MSU. When $\alpha = 1/3$, formula (2.8.31) in [14] yields

$$f_{1/3}(x) \; = \; \frac{1}{3\pi x^{3/2}} K_{1/3}\left(\frac{2}{3\sqrt{3}} x^{-1/2}\right)$$

where $K_{1/3}$ is the Macdonald function of order $1/3$, so that (1.1) amounts to showing that $t \mapsto K_{1/3}(e^t)$ is log-concave. From (1.4) and since $K_{1/3}$ is a solution to the modified Bessel equation

$$x^2 K''_{1/3} \; + \; x K'_{1/3} \; = \; (x^2 + 1/9) K_{1/3}$$

over $\mathbb{R}$, this is equivalent to $(x^2 + 1/9) K^2_{1/3}(x) \; \leq \; x^2 (K'_{1/3}(x))^2$ for every $x \geq 0$. Because $K'_{1/3}(x) < 0$ for every $x \geq 0$, the latter is now an immediate consequence of a Turán-type inequality for modified Bessel functions recently established in [1] (see (2.2) therein), so that $Z_{1/3}$ is MSU.

With a probabilistic and more concise argument, one can also show that property (1.1) holds for every $\alpha = 1/p, \; p \geq 2$. A classical representation originally due to E. J. Williams (see Section 2 in [13]) shows indeed that after rescaling $Z_{1/p}^{-1}$ is an independent product of p Gamma variables:

$$Z_{1/p}^{-1} \; \stackrel{d}{=} \; p^p \, Y_{1/p} \; \times \; Y_{2/p} \; \times \; \cdots \; \times \; Y_{(p-1)/p}$$

where each $Y_{k/p}$ is MSU since its density is $\Gamma(k/p)^{-1} y^{k/p-1} e^{-y} \mathbf{1}_{\{y>0\}}$. Hence, since the MSU property is stable by inversion and independent multiplication, it follows that $Z_{1/p}$ is MSU for every $p \geq 2$.

As a matter of fact, a much stronger property is known to hold when $\alpha = 1/p$ for some $p \geq 2$. In these cases, it had been noticed in [10] thanks to the classical computation of the fractional moments

$$\mathbb{E}[Z_\alpha^s] \; = \; \frac{\Gamma(1 - s/\alpha)}{\Gamma(1-s)} \tag{2.1}$$

for every $s < \alpha$, and the duplication formula for the Gamma function, that the kernel $(x, y) \mapsto f_\alpha(e^{x-y})$ is totally positive – see pp. 121-122 and 390 in [10] as well as pp. 11-12 therein for the definition of total positivity. In particular, it is totally positive of order 2 (TP_2), which means precisely that $x \mapsto f_\alpha(e^x)$ is a log-concave function – see e.g. Theorem 4.1.9 in [10, Chap. 4]. However, when α is not the reciprocal of an integer, the function

$$s \; \mapsto \; \frac{\Gamma(1-s)}{\Gamma(1 - s/\alpha)}$$

is not an entire function of the type $\mathcal{E}_2^*$ (see e.g. [10, p. 336] for a definition) and by Theorem 7.3.2 in [10], this entails that $(x, y) \mapsto f_\alpha(e^{x-y})$ is no longer a totally positive kernel. Karlin then raised the question of whether or not it should be totally positive of some finite order (see [10, p. 390]), a problem which seems as yet unaddressed.

Let us finally show that $Z_{2/3}$ is not MSU, in other words that $(x,y) \mapsto f_{2/3}(e^{x-y})$ is not a TP_2 kernel. Formula (2.8.33) in [14] (with a slight normalizing correction therein) yields first the expression

$$f_{2/3}(x) \;=\; \sqrt{\frac{3}{\pi x}}\, e^{-2/27x^2} W_{1/2,1/6}\left(\frac{4}{27}x^{-2}\right),$$

where $W_{1/2,1/6}$ is a Whittaker function. Hence, from formulæ (6.9.2) and (6.5.2) in [6] we see that (1.1) is equivalent to the log-concavity of $t \mapsto g(e^t)$ with $g(x) = e^{-x}U_4(x)$ and the notation $U_\lambda(x) = \Psi(1/6, \lambda/3, x)$ for all $\lambda > 1/2$, where

$$\Psi(a,c,x) \;=\; \frac{1}{\Gamma(a)} \int_0^\infty e^{-xs} s^{a-1}(1+s)^{c-a-1} ds$$

is a confluent hypergeometric function ($c > a > 0$). We readily see that $g'(x) = -e^{-x}U_7(x)$ and $g''(x) = e^{-x}U_{10}(x)$, so that by (1.4) the MSU property for $X_{2/3}$ amounts to

$$(x^2 U_{10}(x) - xU_7(x))U_4(x) \;\le\; x^2 U_7^2(x), \quad x \ge 0.$$

Using the contiguity relation (6.6.5) in [6] twice and some simple transformations, we then find the equivalence condition

$$(xU_4(x) - U_1(x)/6)(U_7(x) - U_4(x)) \;\ge\; -5U_4^2(x)/6, \quad x \ge 0. \tag{2.2}$$

Notice that (2.2) is true as soon as $x \ge 1/6$ thanks to the obvious inequalities $U_7(x) \ge U_4(x) \ge U_1(x)$. However, some easy computations yield the asymptotics

$$U_7(x) \sim \frac{\Gamma(4/3)}{\Gamma(1/6)} x^{-4/3}, \;\; U_4(x) \sim \frac{\Gamma(1/3)}{\Gamma(1/6)} x^{-1/3} \;\text{ and }\; U_1(x) \to \frac{\Gamma(2/3)}{\Gamma(5/6)}$$

when $x \to 0^+$, so that (2.2) does not hold anymore. This last discussion about the case $\alpha = 2/3$ may look tedious, all the more that the proof that the MSU property does not hold for any $\alpha > 1/2$ is quite simple as we will soon see. But this must be read as a preparatory example for the sequel, since we will need inequalities such as (2.2) for some confluent hypergeometric functions in order to show the MSU property when $\alpha \le 1/2$, a fact which is more involved.

3. Proof of the Theorem

We begin with the only if part, and we will give three different arguments. The first one relies on the aforementioned Humbert-Pollard representation for f_α (see e.g. formula (14.31) in [12]):

$$f_\alpha(x) \;=\; \frac{1}{\pi} \sum_{n \ge 1} \frac{(-1)^{n-1}}{n!} \sin(\pi\alpha n)\Gamma(1+\alpha n) x^{-(1+\alpha n)} \;=\; \sum_{n\ge 1} \frac{(-1)^n}{\Gamma(-n\alpha)n!} x^{-(1+\alpha n)}.$$

Because $\alpha < 1$ this expansion may be differentiated term by term on $(0, +\infty)$, yielding

$$x f_\alpha'(x) \;=\; \sum_{n\ge 1} \frac{(-1)^{n-1}}{\Gamma(-n\alpha)n!}(1+\alpha n) x^{-(1+\alpha n)}$$

and

$$x^2 f_\alpha''(x) \;+\; x f_\alpha'(x) \;=\; \sum_{n\ge 1} \frac{(-1)^n}{\Gamma(-n\alpha)n!}(1+\alpha n)^2 x^{-(1+\alpha n)}$$

for every $x > 0$ (these three series representations explain the reformulation of (1.4) in terms of a reverse Cauchy-Schwarz inequality mentioned at the end of Section 1).

Using the expansions up to $n = 2$ and the concatenation formula $\Gamma(z+1) = z\Gamma(z)$, we obtain, after some simplifications,

$$x^2(f'_\alpha)^2(x) \;-\; (x^2 f''_\alpha(x) + x f'_\alpha(x)) f_\alpha(x) \;=\; \frac{\alpha^2 x^{-(2+3\alpha)}}{2\Gamma(-\alpha)\Gamma(-2\alpha)} \;+\; \mathrm{o}(x^{-(2+3\alpha)})$$

in the neighbourhood of infinity, which entails that (1.4) does not hold when $\alpha > 1/2$, since the leading term is then negative.

The second argument hinges upon an expansion for the density g_α of the random variable $Y_\alpha = \log Z_\alpha$, which had been obtained in [2] (see (3.5) and (6.4) therein) independently of the Humbert-Pollard formula:

$$g_\alpha(t) \;=\; e^{-\alpha t - e^{-\alpha t}} \sum_{j \geq 0} b_j \alpha^{j+1} (-1)^j R_j(-e^{-\alpha t})$$

for every $t \in \mathbb{R}$, where the coefficients b_j and $R_j(x)$ are defined through the entire series

$$\frac{1}{\Gamma(1+z)} \;=\; \sum_{j \geq 0} b_j z^j \quad \text{and} \quad e^{z + xe^z} \;=\; \sum_{j \geq 0} \left(\frac{R_j(x) e^x}{j!} \right) z^j.$$

Besides, setting

$$P_\alpha(x) \;=\; \sum_{j \geq 0} b_j \alpha^{j+1} (-1)^j R_j(-x)$$

for any $x > 0$, we know from (6.3) in [2] that the series converges absolutely, and with exactly the same argument one can show that it can be differentiated term-by-term. On the other hand, simple computations entail that g_α is log-concave over $\mathbb{R}$ if and only if

$$P_\alpha(x)^2 + x^2(P'_\alpha(x))^2 \;\geq\; P_\alpha(x)(xP'_\alpha(x) + x^2 P''_\alpha(x))$$

for every $x > 0$. Letting $x \to 0^+$ yields

$$P_\alpha(x)^2 + x^2(P'_\alpha(x))^2 \;\sim\; xP_\alpha(0)^2 \;=\; x\left(\frac{\alpha}{\Gamma(1-\alpha)}\right)^2$$

and

$$P_\alpha(x)(xP'_\alpha(x) + x^2 P''_\alpha(x)) \;\sim\; xP_\alpha(0)P'_\alpha(0) \;=\; x\left(\frac{\alpha}{\Gamma(1-\alpha)}\right)^2 \left(1 - \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}\right)$$

where the evaluations of $P_\alpha(0)$ and $P'_\alpha(0)$ rest upon the definition of b_j and the fact that $R_j(0) = 1$ and $R'_j(0) = 2^j - 1$. Similarly as above, we see that the second asymptotic is larger than the first one when $\alpha > 1/2$.

For the third and simplest argument, we will invoke an identity in law connecting two independent copies Y_α and $\tilde{Y}_\alpha$ of the random variable $\log Z_\alpha$, which can be readily obtained in changing the variable in Exercise 4.21 (3) of [3]:

$$Y_\alpha \;-\; \tilde{Y}_\alpha \;\stackrel{d}{=}\; U_\alpha \tag{3.1}$$

where U_α is a real random variable with density

$$u_\alpha(x) \;=\; \frac{\sin \pi\alpha}{\pi(e^{\alpha x} + 2\cos \pi\alpha + e^{-\alpha x})}.$$

We compute then the second derivative of $x \mapsto \log(e^{\alpha x} + 2\cos \pi\alpha + e^{-\alpha x})$, which is

$$\frac{4\alpha^2 + 4\alpha^2 \cos \pi\alpha \cosh \alpha x}{(e^{\alpha x} + 2\cos \pi\alpha + e^{-\alpha x})^2}.$$

Hence, we see that U_α has a log-concave density over $\mathbb{R}$ iff $\alpha \leq 1/2$. By Prékopa's theorem this entails that $\log Z_\alpha$ does not have a log-concave density over $\mathbb{R}$ when $\alpha > 1/2$, which means that Z_α is not MSU.

Remarks. (a) This negative result shows that the kernel $(x, y) \mapsto f_\alpha(e^{x-y})$ is not TP_2 when $\alpha > 1/2$, which contradicts the affirmation made in Lemma 1(iv) of [7] that this kernel is strictly totally positive for every $0 < \alpha < 1$ (actually the contradiction could already have been seen in reading [10, p. 390] carefully). Notice that this latter affirmation seems crucial to obtain the so-called bell-shape property for all α-stable variables with index $\alpha < 1$ – see p. 237 in [7]. However, since this question is quite different from the topic of the present paper, we plan to tackle the problem (if really any) in some further research.

(b) Though somewhat more technical, the methods resting upon Humbert-Pollard's and Brockwell-Brown's expansions give some insight into the location where the inequality (1.4) breaks down, information which could not have been obtained by the third argument. I had also believed for a long time that these two expansions would give the if part, but this still eludes me because of the alternate signs.

(c) It would be interesting to see if the third argument could not give the if part either. From the analytical viewpoint this would be the consequence of a positive answer to the following question. If X is a real random variable with density such that the independent difference $X - X$ has a log-concave density, does X have a log-concave density as well? This assertion, a kind of reverse to Prékopa's theorem for which we found neither references nor counterexamples in the literature, goes somehow in the opposite direction to the central limit theorem (which leads to a log-concave density after enough convolutions on any probability distribution with finite variance), and the same direction as Crámer's theorem (which entails that if $X - X$ is Gaussian, then X is Gaussian). Franck Barthe wrote me that he would be surprised if it held true in full generality. In our positive α-stable framework, the fact that $\log Z_\alpha$ is unimodal (and log-concave at both infinities when $\alpha \leq 1/2$) might add some crucial properties, but overall I could not find any clue in the direction of this statement.

We now consider the if part, using yet another argument since the three above methods turned out fruitless. We first prove two lemmas which are of independent interest. The second one is a generalisation of Williams's result and could be formulated in several ways (we chose the one tailored to our purposes).

Lemma 1. *Let X be a $\mathrm{B}(\alpha, \beta)$ variable and Y an independent $\gamma(c)$ variable such that $\beta \leq 1$ and $\alpha + \beta \geq c$. Then the product $X \times Y$ is MSU.*

Proof. When $\alpha + \beta = c$, the result follows easily without further assumption on β because

$$X \times Y \stackrel{d}{=} \gamma(\alpha), \tag{3.2}$$

a fact which can be found e.g. in [3, Exercise 4.2]. When $\alpha + \beta \geq c$, we first compute the density function of $X \times Y$: two changes of variable entail

$$\begin{aligned} f_{X\times Y}(x) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(c)} \int_0^1 e^{-x/u}(x/u)^{c-1}u^{\alpha-1}(1-u)^{\beta-1}\frac{du}{u} \\ &= \frac{\Gamma(\alpha+\beta)x^{c-1}}{\Gamma(\alpha)\Gamma(\beta)\Gamma(c)} \left(e^{-x}\int_0^\infty e^{-xu}u^{\beta-1}(u+1)^{c-(\alpha+\beta)}du\right) \end{aligned}$$

(notice that this recovers (3.2) when $\alpha+\beta=c$), so that we will be done as soon as the function $t\mapsto g_{\alpha,\beta,c}(e^t)$ is log-concave, with the notation

$$g_{\alpha,\beta,c}(x) \;=\; e^{-x}\int_0^\infty e^{-xu}u^{\beta-1}(u+1)^{c-(\alpha+\beta)}du.$$

Now using exactly the same discussion made at the end of Section 2 for the case $\alpha=2/3$ (with adaptated computations) entails that this log-concavity property is equivalent to

$$(xg_{\alpha,\beta,c}(x)+(\alpha+\beta-c)g_{\alpha,\beta,c-1}(x))(g_{\alpha,\beta,c+1}(x)-g_{\alpha,\beta,c}(x)) \;\geq\; (\beta-1)(g_{\alpha,\beta,c}(x))^2$$

for every $x\geq 0$. But in the above, the right-hand side is negative because $\beta<1$, whereas the left-hand side is positive from the obvious inequality $g_{\alpha,\beta,c+1}(x)>g_{\alpha,\beta,c}(x)$, and since by assumption $\alpha+\beta\geq c$. □

Lemma 2. *For all integers $p,n\geq 2$ such that $n>2p$, one has the following representation as an independent product:*

$$\begin{aligned}Z_{p/n}^{-p} \;&\stackrel{d}{=}\; \frac{n^n}{p^p}\,\mathrm{B}\,(2/n,1/p-2/n)\gamma(1/n)\;\times\;\mathrm{B}\,(4/n,2/p-4/n)\,\gamma(3/n)\;\times\;\cdots\\ &\times\;\mathrm{B}\,(2(p-1)/n,(p-1)/p-2(p-1)/n)\,\gamma((2p-3)/n)\\ &\times\;\gamma\,((2p-1)/n)\;\times\;\cdots\;\times\;\gamma\,((n-1)/n).\end{aligned}$$

Proof. We first evaluate the fractional moments of $Z_{p/n}^{-p}$ using (2.1), the duplication formula for the Gamma function (see e.g. formula (1.2.11) in [6]), and some rearrangement involving the crucial assumption $n>2p$: for every $s>-1/n$ one obtains

$$\begin{aligned}\mathbb{E}[(Z_{p/n}^{-p})^s] \;&=\; \frac{\Gamma(ns+1)}{\Gamma(s+1)}\;\times\;\frac{\Gamma(s+1)}{\Gamma(ps+1)}\\ &=\; \left(\frac{n^n}{p^p}\right)^s\frac{\Gamma(s+1/n)\;\ldots\;\Gamma(s+(n-1)/n)\;\Gamma(1/p)\;\ldots\;\Gamma((p-1)/p)}{\Gamma(s+1/p)\;\ldots\;\Gamma(s+(p-1)/p)\;\Gamma(1/n)\;\ldots\;\Gamma((n-1)/n)}\\ &=\; \left(\frac{n^n}{p^p}\right)^s\left(\frac{\Gamma(s+2/n)\Gamma(1/p)}{\Gamma(s+1/p)\Gamma(2/n)}\right)\left(\frac{\Gamma(s+1/n)}{\Gamma(1/n)}\right)\\ &\quad\times\left(\frac{\Gamma(s+4/n)\Gamma(2/p)}{\Gamma(s+2/p)\Gamma(4/n)}\right)\left(\frac{\Gamma(s+3/n)}{\Gamma(3/n)}\right)\;\times\;\cdots\\ &\quad\quad\times\left(\frac{\Gamma(s+2(p-1)/n)\Gamma((p-1)/p)}{\Gamma(s+(p-1)/p)\Gamma(2(p-1)/n)}\right)\left(\frac{\Gamma(s+(2p-3)/n)}{\Gamma((2p-3)/n)}\right)\\ &\quad\quad\quad\times\left(\frac{\Gamma(s+(2p-1)/n)}{\Gamma((2p-1)/n)}\right)\;\times\;\cdots\;\times\;\left(\frac{\Gamma(s+(n-1)/n)}{\Gamma((n-1)/n)}\right).\end{aligned}$$

On the other hand, it is well-known and easy to see that the fractional moments of the $\mathrm{B}\,(\alpha,\beta)$ and $\gamma(c)$ variables are given by

$$\mathbb{E}\left[(\mathrm{B}\,(\alpha,\beta))^s\right] \;=\; \frac{\Gamma(s+\alpha)\Gamma(\alpha+\beta)}{\Gamma(s+\alpha+\beta)\Gamma(\alpha)}\quad\text{and}\quad\mathbb{E}\left[(\gamma(c))^s\right] \;=\; \frac{\Gamma(s+c)}{\Gamma(c)}.$$

The claim follows now by identification of the Mellin transform. □

Remark. As mentioned before, we see from this proof that analogous product representations for positive α-stable laws with any α rational can be obtained accordingly. This might be useful to some other problems. Compare also with Theorem 2.8.4 in [14] where transforms of the densities $f_{p/q}$ are given as solutions to differential equations of higher order, an analytical representation which seems less tractable than Williams-type representations.

End of the proof. We need to show (1.4) for any $\alpha \leq 1/2$ and $x \geq 0$. Setting $g_\alpha(x) = (x^2 f''_\alpha(x) + x f'_\alpha(x)) f_\alpha(x) - x^2 (f'_\alpha(x))^2$, we see from the Humbert-Pollard decomposition and its differentiations that the application $\alpha \mapsto g_\alpha(x)$ is continuous on $(0,1)$ for every fixed $x \geq 0$. By a density argument, it is therefore sufficient to prove (1.4) for any $\alpha = p/n$ with p, n integers greater than two such that $n > 2p$ and every $x \geq 0$. This amounts to the MSU property for $Z_{p/n}^{-p}$, which now comes easily from Lemmas 1 & 2, the MSU property for Gamma variables and the stability of the MSU property by independent multiplication. $\square$

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