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# PICARD-VESSIOT EXTENSIONS FOR REAL FIELDS

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Abstract. We define a notion of Picard-Vessiot extension for a homogeneous linear differential equation $\mathcal{L} = 0$ defined over a real differential field $K$ with a real closed field of constants $C_K$. When $\mathcal{L}$ has differential Galois group $GL_n$ over the complexification of $K$, we prove that a Picard-Vessiot extension for $\mathcal{L}$ exists over $K$.

## 1. Introduction

Picard-Vessiot theory can be described as Galois theory of linear differential equations. A derivation of a field $K$ is defined as a map $d : K \to K$ satisfying $d(a+b) = d(a)+d(b)$ and $d(ab) = d(a)b+ad(b)$, for all $a, b$ in $K$. A differential field is a field endowed with a derivation. We shall use the usual notation $a', a'', \ldots, a^{(n)}$ for the successive derivations of the element $a$. We denote by $C_K$ the field of constants of a differential field $K$. We shall consider homogeneous linear differential equations over a differential field $K$ of the form

$$\mathcal{L}(Y) := Y^{(n)} + a_{n-1}Y^{(n-1)} + \ldots + a_1Y' + a_0Y = 0,$$

where $a_i \in K$ for $i \in \{0, 1, \ldots, n-1\}$. If $L$ is a differential field extension of $K$, i.e. a differential field containing $K$ with derivation extending the derivation in $K$, the set of solutions of $\mathcal{L}(Y) = 0$ in $L$ is a $C_L$-vector space of dimension $\leq n$. A fundamental system of solutions of $\mathcal{L}(Y) = 0$ is a set of $n$ solutions of the equation in some differential extension $L$ of $K$, linearly independent over $C_L$.

A Picard-Vessiot extension for $\mathcal{L}(Y) = 0$ over $K$ is a differential field extension of $K$ differentially generated by a fundamental system of solutions of $\mathcal{L}(Y) = 0$, i.e. generated by the elements in the fundamental system and their derivatives, and not adding constants. Picard-Vessiot theory is due to E. Picard and E. Vessiot and in rigorous form to E. Kolchin, who built on the work of J.F. Ritt in differential algebra. It was made more accessible by the book of I. Kaplansky [7]. We also refer the reader to [4], [9] and [11] for the results of Picard-Vessiot theory used in this paper.

Picard-Vessiot theory has been built under the hypothesis that the field of constants $C_K$ of the differential field $K$ is algebraically closed. In this case, one obtains existence and uniqueness, up to $K$-differential isomorphisms, of the Picard-Vessiot extension of the differential equation and that the differential Galois group of the

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differential equation, defined as the group of $K$-differential automorphisms of its Picard-Vessiot extension, is a linear algebraic group of rank $n$ over $C_K$. It is worth considering whether the condition $C_K$ algebraically closed can be weakened. In particular, the case of real fields is interesting due to the application of Picard-Vessiot theory to the integrability of hamiltonian systems (see [10] or [4]).

In this paper we consider homogeneous linear differential equations $\mathcal{L}(Y) = 0$ defined over a real field $K$ with a real closed field of constants. We prove that, in case $\mathcal{L}$ has differential Galois group $GL_n$ over the complexification of $K$, there exists a Picard-Vessiot extension for $\mathcal{L}(Y) = 0$ over $K$, which moreover is a real field. For the results of the theory of real fields used in this paper, we refer the reader to [3].

## 2. Auxiliary results

We consider a homogeneous linear differential equation of order $n$ over a real differential field $K$, with real closed field of constants $C_K$

$$\mathcal{L}(Y) := Y^{(n)} + a_{n-1}Y^{(n-1)} + \ldots + a_1Y' + a_0Y = 0, \tag{2.1}$$

where $a_i \in K$ for $i \in \{0, 1, \ldots, n-1\}$.

We recall the definition of a Picard-Vessiot extension.

**Definition 2.1.** A differential field extension $K \subset L$ is a Picard-Vessiot extension for $\mathcal{L}$ if

1. $L$ is differentially generated over $K$ by the set of solutions of $\mathcal{L}(Y) = 0$ in $L$,
2. $\mathcal{L}(Y) = 0$ has in $L$ exactly $n$ solutions lineary independent over $C_K$,
3. every constant of $L$ lies in $K$, i.e. $C_K = C_L$.

In the case in which $C_K$ is an algebraically closed field, a Picard-Vessiot extension for $\mathcal{L}$ over $K$ is obtained by constructing the full universal solution algebra $R$ (see below) and considering a maximal differential ideal $M$ of $R$, i.e. a maximal element in the set of proper differential ideals of $R$, which is proved to be prime. Then one can prove that the field of fractions of the integer domain $R/M$ fulfills the conditions required to be a Picard-Vessiot extension for $\mathcal{L}$ over $K$.

In this paper we consider the case in which $K$ is a real field and $C_K$ is real closed. We are interested in real differential ideals of the full universal solution algebra $R$, and a crucial point for our construction will be to prove that a maximal real differential ideal of $R$, i.e. a maximal element in the set of proper real differential ideals of $R$, is prime. To this end we shall use a theorem of Ritt, of which we give the statement for the convenience of the reader (see [12], chapter I.16, or [2], 1.3).

**Theorem 2.2** ([12])**.** *Let $K$ be a differential field of characteristic zero. Let $R$ be a differential $K$-algebra finitely differentially generated and let $I$ be a proper radical differential ideal of $R$. Then there exist finitely many prime differential ideals $P_1, \ldots, P_s$ of $R$ such that*

$$I = P_1 \cap \ldots \cap P_s.$$

*Moreover, when $P_i \nsubseteq P_j$ for all $i \neq j$, $i, j \in \{1, \ldots, s\}$, then $\{P_1, \ldots, P_s\}$ is unique.*

**Proposition 2.3.** *Let $K$ be a differential field of characteristic zero and let $R$ be a noetherian differential $K$-algebra finitely differentially generated. Let $I$ be a maximal real differential ideal of $R$. Then $I$ is prime.*

*Proof.* $I$ is radical because it is real (see [3], Lemma 4.1.5). Then, by Theorem 2.2, $I$ is an intersection of finite number of prime differential ideals, i.e.

$$I = P_1 \cap \ldots \cap P_s. \tag{2.2}$$

Moreover, we can assume that $P_i \not\subseteq P_j$ for all $i \neq j$. Indeed, if some $P_i \subset P_j$ for $i \neq j$, we can omit $P_j$ and reduce the decomposition. Therefore (2.2) is a primary decomposition of the ideal $I$ with

$$rad(P_i) = P_i \neq rad(P_j) = P_j \quad \forall\, i \neq j.$$

Hence, by unicity in Theorem 2.2, it is a reduced primary decomposition (see [1], chap. 4). So the $P_i$'s are exactly the minimal prime ideals containing $I$.

Now, minimal prime ideals containing the real ideal $I$ are as well real (see [3], Lemma 4.1.5). But $I$ is a maximal real differential ideal, so $s = 1$ and $I = P_1$. Therefore $I$ is prime. □

## 3. Main result

We consider the homogeneous linear differential equation (2.1) defined over the real field $K$. We construct a differential $K$-algebra containing a full set of solutions of the equation and prove that it can be ordered.

Let us consider the ring $K[Y_{ij}]$, where $0 \leq i \leq n-1$ and $1 \leq j \leq n$. It is a polynomial ring in $n^2$ indeterminates. We extend the derivation of $K$ to $K[Y_{ij}]$ by defining

$$Y_{ij}' = Y_{i+1,j} \quad \text{for} \quad 0 \leq i \leq n-2,$$
$$Y_{n-1,j}' = -a_{n-1}Y_{n-1,j} - \ldots - a_1 Y_{1j} - a_0 Y_{0j}.$$

We denote $W = \det(Y_{ij})$. We have

$$W = \det \begin{pmatrix} Y_{01} & \ldots & Y_{0n} \\ Y_{11} & \ldots & Y_{1n} \\ \ldots & \ldots & \ldots \\ Y_{n-1,1} & \ldots & Y_{n-1,n} \end{pmatrix} = \det \begin{pmatrix} Y_{01} & \ldots & Y_{0n} \\ Y_{01}' & \ldots & Y_{0n}' \\ \ldots & \ldots & \ldots \\ Y_{01}^{(n-1)} & \ldots & Y_{0n}^{n-1} \end{pmatrix}.$$

So $W$ is the wronskian (determinant) of $Y_{01}, \ldots, Y_{0n}$.

Let $\mathcal{W} = \{W^n\}_{n>0}$ be the multiplicative system of the powers of $W$. Let $R := K[Y_{ij}]_{\mathcal{W}}$ be the localization of $K[Y_{ij}]$ in $\mathcal{W}$. The derivation of $K[Y_{ij}]$ extends to $R$ in a unique way (see e.g. [4], Remark 2.1).

The $K$-algebra $R$ is called the *full universal solution algebra* (see [9], Definition 2.12). By construction, it contains $n$ solutions of equation (2.1) linearly independent over constants. Forgetting the differential structure, $R$ is a subring of the field of rational functions in $n^2$ variables over $K$, so it can be ordered (see [3], example 1.1.2) and it is an integral domain. Hence 0 is a real differential ideal of $R$, and the set of real differential ideals of $R$ is not empty.

Let $P$ be a maximal real differential ideal of $R$. Then, by Proposition 2.3, $P$ is prime. So the quotient ring $R/P$ is an integral domain. The field of fractions $L = Fr(R/P)$ is a real field, because $P$ is a real differential ideal (see [3], Lemma 4.1.6). By our construction $L$ is a real field differentially generated over $K$ by a fundamental system of solutions of equation (2.1). Concerning the field of constants $C_L$ of the real differential field $L$, we obtain the following result.

**Lemma 3.1.** *Let $K$ and $L$ be as above and assume that the constant field $C_K$ of $K$ is real closed. If $c \in C_L \setminus C_K$, then $c$ is transcendent over $K$.*

*Proof.* If the constant $c$ were algebraic over $K$, it would be algebraic over $C_K$ (see e.g. [4], proof of Prop. 3.5). In this case, it would be a real algebraic element. But $C_K$ is a real closed field, so $c \in C_K$. □

*Remark* 3.2. If the construction above leads to an algebraic extension $K \subset L$, then $L$ is a Picard-Vessiot extension of $K$ since $C_L = C_K$.

Let $\bar{C}_K$ denote the algebraic closure of $C_K$. Let $\hat{K} := K \otimes_{C_K} \bar{C}_K$. We have $\bar{C}_K = C_K(i)$ and $\hat{K} = K(i)$ with $i^2 = -1$.

The derivation extends from $K$ to its complexification $K(i)$ and is given by the formula

$$(a+bi)' = a' + b'i \qquad \forall a, b \in K.$$

$C_K(i)$ is the field of constants of $K(i)$, and it is algebraically closed. So there exists a Picard-Vessiot extension for equation (2.1) over $K(i)$.

Let $R = K[Y_{ij}]_{\mathcal{W}}$ be the full universal solution algebra of equation (2.1) over $K$ constructed above. Let $\hat{R} = R \otimes_{C_K} \bar{C}_K$. Then $\hat{R}$ is isomorphic to the full universal solution algebra of equation (2.1) over $\hat{K}$.

For a maximal real differential ideal $P$ of $R$, the extended ideal $P^e$ is clearly a proper differential ideal of $\hat{R}$. Let $M$ be a maximal differential ideal of $\hat{R}$ containing $P^e$. We then have an epimorphism $R/P \to R/P^{ec}$, where $c$ denotes the contraction of ideals, inducing an epimorphism

$$R/P \otimes_{C_K} \bar{C}_K \to R/P^{ec} \otimes_{C_K} \bar{C}_K,$$

and also an epimorphism $\hat{R}/P^e \to \hat{R}/M$. Let us now prove

$$R/P^{ec} \otimes_{C_K} \bar{C}_K \simeq \hat{R}/P^e.$$

Indeed, the kernel of the epimorphism

$$\hat{R} = R \otimes_{C_K} \bar{C}_K \to R/P^{ec} \otimes_{C_K} \bar{C}_K$$

is $P^{ece} = P^e$. We then have the following inequalities:

$$trdeg(L|K) = trdeg(R/P|K) \geq trdeg(\hat{R}/P^e|\hat{K}) \geq trdeg(\hat{R}/M|\hat{K}). \tag{3.1}$$

Moreover, by construction, $trdeg(L|K) \leq n^2$. We obtain

**Theorem 3.3.** *Let $K$ be a real differential field with a real closed field of constants $C_K$. Let $\mathcal{L}(Y) = 0$ be a homogeneous linear differential equation of order $n$ defined over $K$. Let $\hat{K} = K \otimes_{C_K} \bar{C}_K$ and assume that the differential Galois group of $\mathcal{L}$ over $\hat{K}$ is $Gl_n(\bar{C}_K)$. Then there exists a Picard-Vessiot extension $L$ for $\mathcal{L}(Y) = 0$ over $K$, which moreover is a real field.*

*Proof.* In the above construction the fraction field of $\hat{R}/M$ is a Picard-Vessiot extension for $\mathcal{L}$ over $\hat{K}$. So $trdeg(\hat{R}/M|\hat{K}) = n^2$ (see e.g. [4], corollary 4.1). Hence inequalities (3.1) are all equalities. We want to prove that the real field $L$ constructed above is a Picard-Vessiot extension for $\mathcal{L}(Y) = 0$ over $K$. To this end, it remains to prove $C_L = C_K$. Let us assume the contrary. By Lemma 3.1, an element $a \in C_L \setminus C_K$ is transcendent over $K$. Let $S := R/P$ be the quotient of the full universal solution algebra by a maximal real differential ideal constructed above. Let $I_1$ (resp. $I_2$) be the ideal of $S$ of denominators (resp. of numerators) of $a$, i.e. $I_1 := \{b \in S | ba \in S\}, I_2 := \{b \in S | ba^{-1} \in S\}$. Then both ideals are differential ideals and at least one of them contains elements transcendent over $K$. Let us denote it by $I$. We assume first that $I_1$ and $I_2$ are both proper, so $I$ is as well. Let us denote by $J$ the inverse image of $I$ in $R$. $J$ is proper since $I$ is proper. The extended ideal $P^e$ will be contained strictly in the extension of the differential ideal $J$, which will be contained in a maximal differential ideal $M$ of $\hat{R}$. Then we would have $trdeg(\hat{R}/P^e|\hat{K}) > trdeg(\hat{R}/M|\hat{K})$, which gives a contradiction. We now assume that only one of $I_1$ and $I_2$ is proper, say $I_2$. Then $a$ is a noninvertible element in $S$. The ideal $(a)$ of $S$ is a proper differential ideal containing at least one transcendent element, so we arrive at a contradiction as before. If the ideals $I_1$ and $I_2$ are both equal to $S$ for all elements $a \in C_L \setminus C_K$, we have $C_L \subset S$. In this case, by applying Lemma 3.4 below, for each $a \in C_L \setminus C_K$, there is a constant $c \in C_K$ such that $a - c$ is not invertible in $S$. Hence, the ideal $(a - c)$ of $S$ is a proper differential ideal containing at least one transcendent element, so we again arrive at a contradiction. We have then obtained that the real field $L$ is a Picard-Vessiot extension for $\mathcal{L}$ over $K$. □

**Lemma 3.4.** *Let $K$ be a real differential field with real closed field of constants $C_K$. Let $A$ be a finitely generated $K$-algebra without zero divisors and let $a$ be an element of $A$. Then either $a$ is algebraic over $K$ or there is a constant $c \in C_K$ such that $a - c$ is not invertible in $A$.*

*Proof.* Let $\overline{K}$ be the algebraic closure of $K$, $\overline{A} := A \otimes_K \overline{K}$. Let us observe that if the element $a \otimes 1 - c \otimes 1 = (a - c) \otimes 1$ is not a unit in $\overline{A}$, then the element $a - c$ will be a nonunit in $A$. Let $V_{\overline{K}}$ be the affine algebraic variety with coordinate ring $\overline{A}$. Then $a \otimes 1$ defines a $\overline{K}$-valued function $f : V_{\overline{K}} \to \overline{K}$. By Chevalley's theorem, $f(V_{\overline{K}})$ is a constructible subset of $\overline{K}$; hence it is either finite or cofinite in $\overline{K}$. If $f(V_{\overline{K}})$ is a finite set, then it is a point because $V_{\overline{K}}$ is irreducible. In this case, $f$ is a constant function; then $a \in \overline{K}$. In the second case, as the real closed field $C_K$ is infinite, there exists $c \in C_K$ such that $f(\omega) = c$, for some $\omega \in V_{\overline{K}}$. Let us observe that $f - (c \otimes 1)$ vanishes at $\omega$, and therefore $a \otimes 1 - c \otimes 1$ is not invertible in $\overline{A}$. □

## 4. Final remarks

In this section we exhibit an example due to Seidenberg which shows the necessity of our hypothesis that the field $K$ is formally real in Theorem 3.3. In the second part, we comment on the fact that, dropping the assumption that the field of constants is algebraically closed, we cannot expect to have a fundamental theorem of the Picard-Vessiot theory.

4.1. **Example of A. Seidenberg.** Let us consider the following example given by A. Seidenberg (see [13] or [8], chapter 6, Ex. 1).

Let $K$ be the differential field obtained by adjoining to $\mathbb{R}$, endowed with the trivial derivation, the general solution $a$ of the equation

$$4a^2 + a'^2 = -1, \tag{4.1}$$

such that $a' \neq 0$. It can be proved that the field of constants of $K$ is $\mathbb{R}$. Now we consider the homogeneous linear differential equation

$$y'' + y = 0 \tag{4.2}$$

defined over $K$. Seidenberg proves that for any field extension $L$ of $K$ containing a nontrivial solution of (4.2), we have $\mathbb{R} \subsetneq C_L$.

In this example the field of constants is $\mathbb{R}$, but the differential field $K = \mathbb{R}(a, a')$ is clearly not formally real. It is also interesting to observe that if we consider $\hat{K} := \mathbb{C}\langle a \rangle$, then the Picard-Vessiot extension of (4.2) over $\hat{K}$ is a quadratic extension of $\hat{K}$; hence the differential Galois group of (4.2) over $\hat{K}$ is defined over $\mathbb{R}$. It then seems that the existence of a Picard-Vessiot extension for a homogeneous linear differential equation defined over a differential field $K$ with real closed field of constants $C_K$ does not depend on the differential Galois group of the equation over the field $\hat{K} = K \otimes_{C_K} \overline{C}_K$.

4.2. **Fundamental theorem.** Picard-Vessiot theory for differential fields with no algebraically closed field of constants was described by Marvin P. Epstein (see [5] and [6]). His result states that there exists a fundamental system of solutions $(y_1, \ldots, y_n)$ for a homogeneous linear differential equation $\mathcal{L}(Y) = 0$ defined over a differential field $K$ with field of constants $C_K$ (not necessary algebraically closed), such that the field of constants of $L := K\langle y_1, \ldots, y_n \rangle$ is a normal algebraic extension of $C_K$. He defines such an $L$ to be a Picard-Vessiot extension of $K$ for the equation $\mathcal{L}(Y) = 0$. He also proves a Galois correspondence theorem for such generalized Picard-Vessiot extensions. If we do not allow new constants, the normality of the Picard-Vessiot extension can fail, as can be easily seen by considering the algebraic case. Hence obtaining Galois correspondence in this context is not possible. As stated in the introduction, we are interested in the question of whether there exists, for a homogeneous linear differential equation defined over a formally real field $K$ with real closed field of constants $C_K$, a field extension of $K$ containg a fundamental system of solutions of the equation and not adding constants.

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