THE POSITIVITY OF THE FIRST COEFFICIENTS OF NORMAL HILBERT POLYNOMIALS

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Abstract. Let $R$ be an analytically unramified local ring with maximal ideal $m$ and $d = \dim R > 0$. If $R$ is unmixed, then $e_1^I(R) \geq 0$ for every $m$-primary ideal $I$ in $R$, where $e_1^I(R)$ denotes the first coefficient of the normal Hilbert polynomial of $R$ with respect to $I$. Thus the positivity conjecture on $e_1^I(R)$ posed by Wolmer V. Vasconcelos is settled affirmatively.

1. Introduction

Throughout this paper let $R$ be a Noetherian local ring with maximal ideal $m$ and $d = \dim R > 0$. Assume that $R$ is analytically unramified, whence the $m$-adic completion $\hat{R}$ of $R$ is reduced. We fix an $m$-primary ideal $I$ in $R$ and denote by $\bar{I}^{n+1}$ (resp. $\lambda_R(R/I^{n+1})$) the integral closure of $I^{n+1}$ (resp. the length of $R/I^{n+1}$) for each $n \geq 0$. Then the normal Hilbert function

$$\lambda_R(R/I^{n+1})$$

of $R$ with respect to $I$ is of polynomial type with degree $d$, and we have integers $\{e_i^I(R)\}_{0 \leq i \leq d}$ such that the equality

$$\lambda_R(R/I^{n+1}) = e_0^I(R) \binom{n+d}{d} - e_1^I(R) \binom{n+d-1}{d-1} + \cdots + (-1)^d e_d^I(R)$$

holds true for all $n \gg 0$. We call these integers $e_i^I(R)$ the coefficients of the normal Hilbert polynomial of $R$ with respect to $I$.

In this paper we are interested in the analysis of the first coefficient $e_1^I(R)$ of the normal Hilbert polynomial. The main purpose is to study the positivity conjecture on $e_1^I(R)$ posed by Wolmer V. Vasconcelos \cite{V}, and our result is stated as follows.

Theorem 1.1. Let $R$ be an analytically unramified local ring with maximal ideal $m$ and $d = \dim R > 0$. If $R$ is unmixed, then

$$e_1^I(R) \geq 0$$

for every $m$-primary ideal $I$ in $R$. 

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Here we should note that the conjecture holds true in the case where $R$ is a Cohen-Macaulay local ring ([PUV, Theorem 2.2]). In fact, generally we have
\[ e_0^1(R) = e_0^1(R), \]
where $e_0^1(R)$ stands for the ordinary Hilbert-Samuel multiplicity of $R$ with respect to $I$. Therefore $\tau_1^1(R) \geq e_1^1(R)$, and so, if $R$ is a Cohen-Macaulay local ring, we get
\[ \tau_1^1(R) \geq e_1^1(R) \geq 0, \]
because $e_1^1(R) \geq 0$ ([N1, Corollary 1]). Mainly based on this fact, the third author, M. Mandal, along with B. Singh and J. Verma [MSV], gave several interesting answers in certain special cases, and our Theorem 1.1 now affirmatively settles the conjecture in full generality.

We shall prove Theorem 1.1 in Section 2. In Section 3 we will discuss a few results related to the positivity conjecture. We expect that the integral closure $\hat{R}$ of $R$ is a regular ring and $I_{\hat{R}}$ is normal; that is, $I_{\hat{R}}$ is integrally closed for all $n \geq 1$, once $e_1(R) = 0$ for some $m$-primary ideal $I$ in $R$. We shall give an affirmative answer in the case where $R$ is a Cohen-Macaulay ring.

Throughout this paper, unless otherwise specified, we denote by $R$ a Noetherian local ring with maximal ideal $m$ and $d = \dim R > 0$. Let $\hat{R}$ be the integral closure of $R$ in its total quotient ring. For each finitely generated $R$-module $M$, let $\mu_R(M)$ (resp. $\lambda_R(M)$) stand for the number of elements in a minimal system of generators (resp. the length) of $M$.

2. Proof of Theorem 1.1

The main purpose of this section is to prove Theorem 1.1.

Proof of Theorem 1.1. We have $\tau_1^1(R) = e_1^1(R)$, since $\hat{a}R = a\hat{R}$ for every $m$-primary ideal $a$ in $R$. Therefore, passing to the $m$-adic completion $\hat{R}$ of $R$, without loss of generality we may assume that $R$ is complete. If $d = 1$, we then have
\[ e_1^1(R) = \lambda_R(\hat{R}/R) \geq 0. \]
Suppose that $d \geq 2$ and let $S = \hat{R}$. For each $p \in \text{Ass}\,R$ we put $S(p) = \hat{R}/p$. Then $S(p)$ is a module-finite extension of $R/p$, and we get
\[ S = \prod_{p \in \text{Ass}\,R} S(p) \quad \text{and} \quad I^{n+1} = I^{n+1}S \cap R \]
for all $n \geq 0$. Hence
\[ \lambda_R(R/I^{n+1}) \leq \lambda_R(S/I^{n+1}S) = \sum_{p \in \text{Ass}\,R} \lambda_R(S(p)/I^{n+1}S(p)) \]
\[ = \sum_{p \in \text{Ass}\,R} \lambda_R(S(p)/m_{S(p)}(S(p)/I^{n+1}S(p))), \]
where \( m \) denotes the maximal ideal of \( S \). Notice that, since \( \dim S = d \) for each \( p \in \text{Ass} R \), we have
\[
\overline{e}_j^1(R) = e_j^0(R) = e_j^0(S) = \sum_{p \in \text{Ass} R} e_j^0(S(p)) = \sum_{p \in \text{Ass} R} \lambda_R(S(p)/m_{S(p)}) \cdot e_{j(S(p))}^0(S(p)) = \sum_{p \in \text{Ass} R} \lambda_R(S(p)/m_{S(p)}) \cdot e_{j(S(p))}^0(S(p)),
\]
whence
\[
0 \leq \lambda_R(S/T^{n+1}S) - \lambda_R(R/T^{n+1}) = \left[ \overline{e}_j^1(R) - \sum_{p \in \text{Ass} R} \lambda_R(S(p)/m_{S(p)}) \cdot e_{j(S(p))}^1(S(p)) \right] \left( n + d - 1 \right) \quad \frac{1}{d - 1} + \text{(terms of lower degree)},
\]
so that
\[
\overline{e}_j^1(R) \geq \sum_{p \in \text{Ass} R} \lambda_R(S(p)/m_{S(p)}) \cdot e_{j(S(p))}^1(S(p)).
\]
Thus, in order to see \( \overline{e}_j^1(R) \geq 0 \), it suffices to show that \( \overline{e}_j^1(S(p)) \geq 0 \) for each \( p \in \text{Ass} R \). If \( d = 2 \), we get
\[
\overline{e}_j^1(S(p)) \geq e_{j(S(p))}^1(S(p)) \geq 0,
\]
because \( S(p) \) is a Cohen-Macaulay local ring. Hence \( \overline{e}_j^1(R) \geq 0 \).

Suppose that \( d \geq 3 \) and that our assertion holds true for \( d - 1 \). Then thanks to the above observation, passing to the ring \( S(p) \), we may assume that \( R \) is a normal complete local ring. Let \( I = (a_1, a_2, \ldots, a_{\ell}) \) with \( a_i \in R \), where \( \ell = \mu_R(I) \). Let
\[
T = R[Z_1, Z_2, \ldots, Z_{\ell}], \quad q = mT, \quad x = \sum_{i=1}^{\ell} a_i Z_i, \quad \text{and} \quad D = T/xT,
\]
where \( Z_1, Z_2, \ldots, Z_{\ell} \) are indeterminates over \( R \). Let
\[
R' = T_q, \quad I' = I R', \quad \text{and} \quad D' = D_q.
\]
We then have \( T^{n+1}R' = T^{n+1}R' \) for all \( n \geq 0 \), so that \( \lambda_R(R'/T^{n+1}R') = \lambda_R(R/T^{n+1}) \), whence
\[
\overline{e}_j^1(R) = \overline{e}_j^1(R').
\]
Here we notice that \( \text{Ass} D' = \text{Ass} D' \), because \( R' \) is catenary and normal; hence \( D' \) is unmixed, as \( D' \) is a homomorphic image of a Cohen-Macaulay ring. The ring \( D' \) is analytically unramified. To see this, since \( D' \) is a Nagata local ring, by [Theorem 70] it suffices to show that \( D \) is reduced; that is, \( D_P = T_P/xT_P \) is an integral domain for every \( P \in \text{Ass} D \). Let \( p = P \cap R \). Then since \( h_P P = 1 \), we have \( h_P p \leq 1 \), so that \( I \not\subseteq p \), because \( h_P p \leq 1 < d = \dim R \). Without loss of generality we may assume that \( a_i \not\subseteq p \). Then, because \( x = \sum_{i=1}^{\ell} a_i Z_i \) and \( a_i \) is a unit of \( R_p \), we get
\[
T_p = R_p[Z_1, Z_2, \ldots, Z_{\ell}] = R_p[Z_1, Z_2, \ldots, Z_{\ell-1}, x],
\]
whence the ring
\[
T_p/xT_p = R_p[Z_1, Z_2, \ldots, Z_\ell]/xR_p[Z_1, Z_2, \ldots, Z_\ell] = R_p[Z_1, Z_2, \ldots, Z_{\ell-1}]
\]
is an integral domain, as it is the polynomial ring with \(\ell - 1\) indeterminates over \(R_p\). Therefore for all \(P \in \text{Ass}_T D\) the ring \(D_P = T_P/xT_P\) is an integral domain, because it is a localization of \(R_p[Z_1, Z_2, \ldots, Z_{\ell-1}]\). Thus \(D\) is reduced, whence \(D'\) is analytically unramified and unmixed.

Let us denote by \(\mathcal{A}\) the extended Rees ring of \(IT\) and by \(\overline{\mathcal{A}}\) the integral closure of \(A\) in \(T[t, t^{-1}]\), where \(t\) denotes an indeterminate. Similarly, let us denote by \(\mathcal{T}\) the extended Rees ring of \(ID\) and by \(\overline{\mathcal{T}}\) the integral closure of \(T\) in \(D[t, t^{-1}]\). We put \(N = (t^{-1}, It)\) in \(\mathcal{A}\). We look at the homomorphism
\[
\psi : T[t, t^{-1}] \to D[t, t^{-1}]
\]
of graded \(T\)-algebras such that \(\psi(t) = t\). Since \(\psi(\mathcal{A}) = \mathcal{T}\) and \(\mathcal{T}\) is a module-finite extension of \(T\), the homomorphism \(\psi\) gives rise to the finite homomorphism
\[
\varphi : \overline{\mathcal{A}}/xt\overline{\mathcal{A}} \to \overline{\mathcal{T}}
\]
of graded \(T\)-algebras. Let \(\mathcal{B}\) (resp. \(\mathcal{U}\)) denote the integral closure of \(B = A_q\) (resp. \(U = T_q\)). Then we get the homomorphism
\[
\varphi_q : \overline{\mathcal{B}}/xt\overline{\mathcal{B}} \to \overline{\mathcal{U}}
\]
of graded \(R'\)-algebras and, thanks to the proof of [HU Theorem 2.1], we furthermore have the following. Let us include a brief proof for the sake of completeness.

**Claim 1.** The homomorphism
\[
\varphi_P : \overline{\mathcal{A}}/xt\overline{\mathcal{A}}_P \to \overline{\mathcal{T}}_P
\]
is an isomorphism for all \(P \in \text{Spec}\mathcal{A} \setminus V(N)\). Hence the kernel and the cokernel of the homomorphism \(\varphi_q : \overline{\mathcal{B}}/xt\overline{\mathcal{B}} \to \overline{\mathcal{U}}\) of graded \(\mathcal{B}\)-modules are of finite length, so they are finitely graded.

**Proof.** Because \(\overline{\mathcal{A}}[t] = T[t, t^{-1}]\) and \(xt\overline{\mathcal{A}}[t] = xT[t, t^{-1}]\), the homomorphism \(\varphi_{t^{-1}}\) is an isomorphism, whence so is the homomorphism \(\varphi_P\), if \(t^{-1} \not\in P\).

Suppose now that \(It \not\subseteq P\). We may assume \(a_it \not\in P\). Notice that
\[
\overline{\mathcal{A}}/xt\overline{\mathcal{A}}_{a_it} = \left[ \overline{R}[t, t^{-1}]/[Z_1, Z_2, \ldots, Z_\ell]/xt\overline{R}[t, t^{-1}]/[Z_1, Z_2, \ldots, Z_\ell] \right]_{a_it}
\]

\[
= \left( \overline{R}[t, t^{-1}]/[1/a_it] \right)[Z_1, Z_2, \ldots, Z_\ell]/\left( \sum_{i=1}^{\ell-1} a_it Z_i/a_it + Z_\ell \right)
\]

\[
= \left( \overline{R}[t, t^{-1}]/[1/a_it] \right)[Z_1, Z_2, \ldots, Z_{\ell-1}]
\]
and that
\[ D[t, t^{-1}]_{αt} = T[t, t^{-1}, \frac{1}{αt}] / x \cdot T[t, t^{-1}, \frac{1}{αt}] \]
\[ = T[t, t^{-1}, \frac{1}{αt}] / x \cdot T[t, t^{-1}, \frac{1}{αt}] \]
\[ = R[\frac{1}{αt}, Z_1, Z_2, \ldots, Z_ℓ, t, t^{-1}] / x \cdot R[\frac{1}{αt}, Z_1, Z_2, \ldots, Z_ℓ, t, t^{-1}] \]
\[ = \left( R[\frac{1}{αt}, t, t^{-1}] \right) [Z_1, Z_2, \ldots, Z_ℓ] / \left( \sum_{i=1}^{ℓ-1} \frac{α_i Z_i}{αt} + Z_ℓ \right) \]
\[ = \left( R[t, t^{-1}] \left[ \frac{1}{αt} \right] \right) [Z_1, Z_2, \ldots, Z_{ℓ-1}] . \]

Then we get the following commutative diagram:
\[
\begin{array}{ccc}
\mathcal{A}/xt\mathcal{A} & \xrightarrow{\varphi_{αt}} & \mathcal{T}_{αt} & \xrightarrow{D[t, t^{-1}]_{αt}} \\
\end{array}
\]
\[
\begin{array}{c}
\cong \\
([R[t, t^{-1}] [\frac{1}{αt}]] \mathcal{A}, Z_1, Z_2, \ldots, Z_{ℓ-1}) & \xrightarrow{\cong} & ([R[t, t^{-1}] [\frac{1}{αt}]] \mathcal{T}, Z_1, Z_2, \ldots, Z_{ℓ-1})
\end{array}
\]
where the vertical homomorphisms are isomorphisms, so the horizontal homomorphism \( \varphi_{αt} \) is injective. Because \( \left( [R[t, t^{-1}] [\frac{1}{αt}]] \mathcal{A}, Z_1, Z_2, \ldots, Z_{ℓ-1} \right) \) is integrally closed in \( \left( [R[t, t^{-1}] [\frac{1}{αt}]] \mathcal{A}, Z_1, Z_2, \ldots, Z_{ℓ-1} \right) \) and \( \varphi_{αt} \) is finite, \( \varphi_{αt} \) is an isomorphism, whence
\[ \varphi_P : \mathcal{A}/xt\mathcal{A}_p \longrightarrow \mathcal{T}_p \]
is an isomorphism, too. This proves Claim \( \square \)

The normal ring \( \mathcal{B} \) is catenary, since it is a finitely generated \( R' \)-algebra, while we get
\[ \dim \mathcal{B}/(xt, t^{-1})\mathcal{B} = \dim \mathcal{U}/t^{-1}\mathcal{U} = d - 1 \]
by Claim \( \square \). Therefore \( t^{-1}, xt \) forms a regular sequence in the normal ring \( \mathcal{B} \). Hence \( xt \) is a non-zero divisor in the associated graded ring
\[ \mathcal{B}/t^{-1}\mathcal{B} = \bigoplus_{n \geq 0} T^n R'/T^{n+1}R' \]
of the filtration \( \{ T^n R' \}_{n \in \mathcal{Z}} \) of integrally closed ideals in \( R' \). Consequently, we have
\[ \varpi'_1(R) = \varpi'_1(R') = \varpi'(D'), \]
since \( \dim D' = \dim R' - 1 = d - 1 \geq 2 \) and since the kernel and the cokernel of the homomorphism
\[ \varpi : \mathcal{B}/(xt, t^{-1})\mathcal{B} \longrightarrow \mathcal{U}/t^{-1}\mathcal{U} \]
induced from \( \varphi_q \) are finitely graded. Thus the hypothesis of induction on \( d \) yields the assertion that \( \varpi'_1(R) \geq 0 \), which completes the proof of Theorem \( \square \)

The condition in Theorem \( \square \) that \( R \) is unmixed is not superfluous. Let us note the simplest example. See [MSV] Example 2.4 for more examples.
Example 2.1. We look at the local ring
\[ R = k[[X, Y, Z]]/a, \]
where \( k[[X, Y, Z]] \) is the formal power series ring over a field \( k \) and \( a = (X) \cap (Y, Z) \). Then \( \dim R = 2 \), \( R \) is mixed, and \( e_1^1(R) = e_2^2(R) = -1 \). Hence the famous bad example [N, p. 203, Example 2] of Nagata which is a non-regular local integral domain \((A, n)\) of dimension 2 with \( e_0^0(A) = 1 \) possess \( e_1^1(A) = e_2^2(A) = -1 \), because
\[ \hat{A} \cong k[[X, Y, Z]]/[(X) \cap (Y, Z)] \]
for some field \( k \).

Proof. We put \( T = k[[X, Y, Z]] \) and \( q = (X, Y, Z) \) in \( T \). Then \( \overline{R} = T/(X) \oplus T/(Y, Z) \), and we have the exact sequence
\[ 0 \to R \to T/(X) \oplus T/(Y, Z) \to T/q \to 0 \]
of \( T \)-modules; hence \( m\overline{R} \subseteq R \). Recall that \( m \) is a normal ideal in \( R \); that is, \( m^n = m^n \) for all \( n \geq 1 \), since the associated graded ring
\[ \text{gr}_m(R) = k[[X, Y, Z]]/[(X) \cap (Y, Z)] \]
of \( m \) is reduced. Therefore, as
\[ m^{n+1} = m^{n+1} \overline{R} \cap R = m^{n+1} \overline{R} \cap R, \]
thanks to exact sequence (E) above, we get
\[ 0 \to R/m^{n+1} \to T/[(X) + q^{n+1}] \oplus T/[(Y, Z) + q^{n+1}] \to T/q \to 0 \]
for all \( n \geq 0 \). Hence
\[ \lambda_R(R/m^{n+1}) = \binom{n+2}{2} + \binom{n+1}{1} - 1, \]
so that \( e_1^1(R) = e_2^2(R) = -1 \). \( \square \)

Let us note a consequence of Theorem 1.1.

Corollary 2.2 ([MTV, Theorem 1]). Let \( R \) be an analytically unramified unmixed local ring with maximal ideal \( m \) and \( d = \dim R > 0 \). Let \( I \) be a parameter ideal in \( R \). If \( e_1^1(R) = e_1^1(R) \), then \( R \) is a regular local ring with \( \mu_R(m/I) \leq 1 \), whence \( I \) is normal.

Proof. We get \( e_1^1(R) \geq 0 \) by Theorem 1.1 whence by [GHOPV, Theorem 2.1] \( R \) is a Cohen-Macaulay local ring with \( e_1^1(R) = 0 \). Because \( e_2^2(R) \geq e_2^2(R) \) and
\[ e_1^1(R) \geq 0 \]
([NR, Corollary 1]), we furthermore have \( e_1^1(R) = 0 \), whence \( \overline{T} \) is a parameter ideal in \( R \). Because parameter ideals contain no proper reductions ([NR], we get \( \overline{T} = I \), whence by [C, Theorem (3.1)] \( R \) is a regular local ring with \( \mu_R(m/I) \leq 1 \) and \( I \) is normal. \( \square \)

Remark 2.3. In Corollary 2.2, unless \( I \) is a parameter ideal, \( R \) is not necessarily a regular local ring, even though \( e_1^1(R = e_1^1(R) \). Let us note an example. We look at the local ring
\[ R = k[[X, Y, Z]]/(Z^2 - XY), \]
where $k[[X, Y, Z]]$ is the formal power series ring over a field $k$ of characteristic 0. Then $R$ is a rational singularity, so $\overline{e}_1(I) = e_1(R)$ for every integrally closed $m$-primary ideal $I$ in $R$.

3. A FURTHER PROBLEM

Let $R$ be an analytically unramified unmixed local ring and $I$ an $m$-primary ideal in $R$. We then expect that $\overline{R}$ is a regular ring and $IR$ is normal; that is, all the powers $I^nR$ are integrally closed, once $\overline{e}_1(I) = 0$. This is the case when $\overline{R}$ is a Cohen-Macaulay ring, as we will show in the following.

**Theorem 3.1.** Let $R$ be an analytically unramified local ring with maximal ideal $m$ and $d = \dim R > 0$. Let $S$ be an overring of $R$ and assume that $S$ is a finitely generated $R$-module with $\dim_R S/R < d$. Let $I$ be an $m$-primary ideal in $R$ such that $\overline{e}_1(I) = 0$. If $\dim_I R S = d$, then $S$ is a regular ring, $S = \overline{R}$, and $IR$ is normal.

**Proof.** We may assume that $R$ is complete. Let $Q(R)$ be the total quotient ring of $R$. We notice that $S$ is a Cohen-Macaulay $R$-module with $\dim_R S = d$; hence $R$ is unmixed. Therefore $S \subseteq Q(R)$, as $\dim_R S/R < d$, so that $S \subseteq \overline{R}$. Since $R$ is complete, we get a decomposition $S = \prod_{i=1}^\ell S_i$ of $S$, where $S_i$ is a Cohen-Macaulay local ring with $\dim S_i = d$. Consequently, for the same reason as in the proof of Theorem 1.1 we have

$$\overline{e}_1(R) = \sum_{i=1}^\ell \lambda_R(S_i/m_i) \cdot \overline{e}_1(S_i) \geq 0,$$

where $m_i$ is the maximal ideal in $S_i$; hence $\overline{e}_1(S_i) = 0$ for each $1 \leq i \leq \ell$. As $\overline{e}_1(S_i) = e_1(S_i) \geq 0$, we have $e_1(S_i/m_i) = 0$, so that $T_{S_i}$ is a parameter ideal in $S_i$. Hence $T_{S_i} = IS_i$. Therefore by [G, Theorem (3.1)] $S_i$ is a regular local ring and $IS_i$ is normal. Thus $S$ is regular and $IS$ is normal, whence $S = \overline{R}$. □

**Corollary 3.2.** Let $R$ be a two-dimensional analytically unramified unmixed local ring with maximal ideal $m$ and let $I$ be an $m$-primary ideal in $R$. If $\overline{e}_1(I) = 0$, then $\overline{R}$ is a regular ring and $IR$ is normal.

**Proof.** Notice that $\overline{R}$ is a finitely generated $R$-module and $\dim_R \overline{R} = 2$, because $R$ is analytically unramified and unmixed with $\dim R = 2$, whence the assertion follows from Theorem 3.1 taking $S = \overline{R}$. □

**Remark 3.3.** The ring $R$ itself is, however, not necessarily a regular local ring even if $\dim R = 2$. Let us note an example. We look at the local ring

$$R = k[[X, Y, Z, W]]/[(X, Y) \cap (Z, W)],$$

where $k[[X, Y, Z, W]]$ is the formal power series ring over a field $k$. We then have $\overline{e}_1(R) = 0$ and $\overline{e}_2(R) = -1$. The ring $R$ is Buchsbaum but not Cohen-Macaulay, while

$$\overline{R} = k[[X, Y]] \times k[[Z, W]]$$

is a regular ring.
References


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