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ON THE SWEEPING OUT PROPERTY FOR CONVOLUTION OPERATORS OF DISCRETE MEASURES

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ABSTRACT. Let μ_n be a sequence of discrete measures on the unit circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with $\mu_n(0) = 0$, and $\mu_n((-\delta, \delta)) \to 1$, as $n \to \infty$. We prove that the sequence of convolution operators $(f * \mu_n)(x)$ is strong sweeping out; i.e., there exists a set $E \subset \mathbb{T}$ such that

$$\limsup_{n\to\infty}(\mathbb{I}_E * \mu_n)(x) = 1, \quad \liminf_{n\to\infty}(\mathbb{I}_E * \mu_n)(x) = 0,$$

almost everywhere on $\mathbb{T}$.

1. INTRODUCTION

We consider bounded discrete measures

$$\mu = \sum_k m_k \delta_{x_k}, \quad \sum_k m_k < \infty,$$

on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, where $X = \{x_k\}$ is a finite or countable set in $\mathbb{T}$ and δ_{x_k} is Dirac measure at x_k. Denote

$$S_\mu f(x) = \int_{\mathbb{R}} f(x+t)d\mu(t).$$

Let μ_n be a sequence of discrete measures satisfying

$$\mu_n(0) = 0, \quad \mu_n((-\delta, \delta)) \to 1, \text{ as } n \to \infty, \tag{1.1}$$

for any $0 < \delta \le 1/2$. It is clear that if $f \in L^1(\mathbb{T})$ is continuous at $x \in \mathbb{T}$, then

$$S_{\mu_n} f(x) \to f(x), \tag{1.2}$$

and the convergence is uniformly if $f \in C(\mathbb{T})$. The almost everywhere convergence problem in the case of general $f \in L^1(\mathbb{T})$ is not trivial. J. Bourgain in [4] proved

Theorem 1 (J. Bourgain). *If $x_k \searrow 0$ as $k \to \infty$, and*

$$\mu_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{x_k},$$

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then there exists a function $f \in L^\infty$, such that $S_{\mu_n} f(x)$ diverges on a set of positive measure.

In fact, this theorem gave a negative answer to a problem due to A. Bellow [3], and the proof is based on a general theorem often referred to as Bourgain's entropy principle. Applying his principle, Bourgain was able to deduce an analogous theorem for Riemann sums

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(x+\frac{k}{n}\right)$$

and for the operators

$$\frac{1}{n}\sum_{k=1}^{n} f(kx).$$

We note that the first theorem was obtained earlier by W. Rudin [8] by a different technique, and the second by J. Marstrand in [7]. S. Kostyukovsky and A. Olevskii in [6], using the same entropy principle, extended Theorem 1 for general discrete sequences satisfying (1.1).

We found a new geometric proof for Theorem 1, as well as for the result from [6]. Moreover, the method allows us to obtain a stronger divergence for the operators (1.2). So in this paper we prove

Theorem 2. *If discrete measures μ_n satisfy* (1.1)*, then there exists a set $E \subset \mathbb{T}$ such that*

$$\limsup_{n\to\infty} S_{\mu_n}\mathbb{I}_E(x) = 1, \quad \liminf_{n\to\infty} S_{\mu_n}\mathbb{I}_E(x) = 0 \tag{1.3}$$

almost everywhere on $\mathbb{T}$.

The relations (1.3) for sequences of operators is called the strong sweeping out property. These kinds of operators are investigated by M. Akcoglu, A. Bellow, R. L. Jones, V. Losert, K. Reinhold-Larsson, and M. Wierdl [1] and by M. Akcoglu, M. D. Ha, and R. L. Jones [2]. In the [1] strong sweeping out property for Riemann sums operators is obtained. In [2] the authors prove a general version of Bourgain's entropy principle, which allows us to deduce sweeping out properties for some operators, but the principle is not applicable for the operators S_{μ_n}. The proof of Theorem 2 is based on Lemma 6. It will be obtained from Lemma 6 by simply applying a general result proved in [5].

2. Proof of Theorem 2

Let

$$X = \{x_i : i = 1, \dots, l\}, \quad 0 < x_1 \le x_2 \le \dots \le x_l < 1, \tag{2.1}$$

be an arbitrary sequence of reals. Suppose

$$Y = \{y_i : i = 1, \dots, \nu\}, \quad y_1 < y_2 < \dots < y_\nu = x_l$$

is a maximal independent (with respect to rational numbers) subset of X containing x_l. Then we have

$$x_k = r_1^{(k)} y_1 + \dots + r_\nu^{(k)} y_\nu, \quad k = 1, 2, \dots, l,$$

for some rational numbers $r_i^{(k)}$. Let p be the least common multiple of the denominators of $r_i^{(k)}$. Then we get

$$x_k = \frac{n_1^{(k)} y_1 + n_2^{(k)} y_2 + \ldots + n_\nu^{(k)} y_\nu}{p}, \tag{2.2}$$

for some $n_i^{(k)} \in \mathbb{Z}$. Denote

$$\tau = \max_{i,k} |n_i^{(k)}| \tag{2.3}$$

and

$$\begin{aligned} A_m = \quad &\Big\{ y = \tfrac{n_1 y_1 + n_2 y_2 + \ldots + n_\nu y_\nu}{p};\ n_i \in \mathbb{Z}, \\ &|n_i| \le m\tau,\ i = 1, 2, \ldots, \nu - 1,\ |n_\nu| \le \nu m\tau + 1 \Big\}. \end{aligned} \tag{2.4}$$

Lemma 1. *If* (2.1) *is an arbitrary sequence with* $\nu \ge 2$*, then for any interval* $I \subset (-1, 1)$ *with* $|I| \le y_\nu / p$ *we have*

$$\#\big(A_m \cap I\big) \sim \gamma m^{\nu-1} |I| \text{ as } m \to \infty, \tag{2.5}$$

where $\gamma = (2\tau)^{\nu-1} p / y_\nu$ *is a constant dependent on* X*.*

Proof. It is easy to observe that

$$\begin{aligned} A_m \cap I = \Big\{ y = \frac{n_1 y_1 + \ldots + n_\nu y_\nu}{p} :& \\ & n_1 \frac{y_1}{y_\nu} + \ldots + n_{\nu-1} \frac{y_{\nu-1}}{y_\nu} \in \frac{p}{y_\nu} \cdot I + \mathbb{Z} \cap [-(\nu m\tau + 1), (\nu m\tau + 1)], \\ & |n_i| \le m\tau,\ i = 1, 2, \ldots, \nu - 1 \Big\}. \end{aligned}$$

On the other hand if $y \in A_m \cap I$, then, by (2.4) we have

$$|n_1 \frac{y_1}{y_\nu} + \ldots + n_{\nu-1} \frac{y_{\nu-1}}{y_\nu}| \le \nu m\tau.$$

Using also the relation $|I| \le y_\nu / p$, we conclude

$$\begin{aligned} A_m \cap I = &\Big\{ y = \frac{n_1 y_1 + \ldots + n_\nu y_\nu}{p} : \\ & n_1 \frac{y_1}{y_\nu} + \ldots + n_{\nu-1} \frac{y_{\nu-1}}{y_\nu} \in \frac{p}{y_\nu} \cdot I + \mathbb{Z}, \\ & |n_i| \le m\tau,\ i = 1, 2, \ldots, \nu - 1 \Big\}. \end{aligned} \tag{2.6}$$

Since $y_1, \ldots, y_\nu$ are independent, the number

$$\theta = y_{\nu-1} / y_\nu$$

is irrational. Denoting

$$E_m = \Big\{ n_1 \frac{y_1}{y_\nu} + \ldots + n_{\nu-2} \frac{y_{\nu-2}}{y_\nu} : |n_i| \le m\tau,\ i = 1, 2, \ldots, \nu - 2 \Big\} \tag{2.7}$$

from (2.6) we get

$$\frac{p}{y_\nu} \cdot \big(A_m \cap I\big) = \big(\{n_{\nu-1}\theta : |n_{\nu-1}| \le m\tau\} + E_m\big) \cap \big(\frac{p}{y_\nu} \cdot I + \mathbb{Z}\big). \tag{2.8}$$

It is well known that $n\theta+t$, $n=1,2,\dots$ ($n=-1,-2,\dots$), is a uniformly distributed sequence. This implies that

$$\frac{\#\big(\{n_{\nu-1}\theta:\,|n_{\nu-1}|\le m\tau\}+t\big)\cap\big(\frac{p}{y_\nu}\cdot I+\mathbb{Z}\big)}{2m\tau}\to\frac{p|I|}{y_\nu},\ \text{as } m\to\infty, \tag{2.9}$$

for any $t\in\mathbb{R}$ and the convergence is uniform. Since $y_1,\dots,y_{\nu-1}$ are independent from (2.7) we obtain

$$|E_m|=(2m\tau+1)^{\nu-2}.$$

Finally, using (2.8) and (2.9), we get

$$\#\big(A_m\cap I\big)=\#\left(\frac{p}{y_\nu}\cdot\big(A_m\cap I\big)\right)\sim 2m\tau\frac{p|I|}{y_\nu}|E_m|\sim(2m\tau)^{\nu-1}\frac{p|I|}{y_\nu}.$$

□

Lemma 2. *For any set* (2.1) *we have*

$$A_m\cap(-x_l,0)+X\subset A_{m+1}\cap(-x_l,x_l),\ m=1,2,\dots, \tag{2.10}$$

where A_m is defined in (2.4).

Proof. Take an arbitrary point $x\in A_m\cap(-x_l,0)$. According to the definition of $y_1,\dots,y_\nu$ we will have

$$x=\frac{n_1y_1+n_2y_2+\ldots+n_\nu y_\nu}{p}.$$

Then suppose $x_k\in X$ has representation (2.2). Since $x\in(-x_l,0)$ and $0<x_k\le x_l$ we get

$$x+x_k\in(-x_l,x_l). \tag{2.11}$$

On the other hand,

$$x+x_k=\frac{(n_1+n_1^{(k)})y_1+(n_2+n_2^{(k)})y_2+\ldots+(n_\nu+n_\nu^{(k)})y_\nu}{p},$$

and by (2.4) and (2.3) we have

$$\begin{aligned}&|n_i+n_i^{(k)}|\le m\tau+\tau=(m+1)\tau, i=1,2,\dots,\nu-1,\\&|n_\nu+n_\nu^{(k)}|\le \nu m\tau+1+\tau<\nu(m+1)\tau.\end{aligned} \tag{2.12}$$

This means that $x+x_k\in A_{m+1}$. Combining (2.11) and (2.12) we get (2.10). □

Lemma 3. *For any numbers $\delta>0$, $0<\varepsilon<1/3$ and measure*

$$\mu=\sum_{k=1}^{l}m_k\delta_{x_k},\ m_k>0,\ 0<x_1<x_2<\ldots<x_l, \tag{2.13}$$

there exists a real number λ, with $0<\lambda\le\delta$, such that

$$S_\mu\mathbb{I}_{\{t:\,\{t/\lambda\}>\varepsilon\}}(x)=\int_{\mathbb{T}}\mathbb{I}_{\{t:\,\{t/\lambda\}>\varepsilon\}}(x+t)d\mu(t)>(1-3\varepsilon)|\mu|,\ \ as\ \{x/\lambda\}<\varepsilon. \tag{2.14}$$

Proof. Denote

$$E_t = \{\lambda > 0 : \{t/\lambda\} \in (\varepsilon, 1-\varepsilon)\}, \quad t > 0. \tag{2.15}$$

It is clear that

$$E_t = \bigcup_{k=0}^{\infty} \left(\frac{t}{k+1-\varepsilon}, \frac{t}{k+\varepsilon} \right).$$

Hence if

$$r = \min\left\{ \frac{\varepsilon x_1}{2(1-\varepsilon)}, \delta \right\}$$

and $t \geq x_1$, we obtain

$$\begin{aligned} |E_t \cap [0,r]| &> \sum_{k>t/r} \left(\frac{t}{k+\varepsilon} - \frac{t}{k+1-\varepsilon} \right) \\ &= \sum_{k>t/r} \left(\frac{(1-2\varepsilon)t}{(k+\varepsilon)(k+1-\varepsilon)} \right) > (1-2\varepsilon)t \sum_{k>t/r} \frac{1}{(k+1)^2} \\ &> \frac{(1-2\varepsilon)tr}{t+2r} > \frac{(1-2\varepsilon)x_1 r}{x_1+2r} \geq \frac{(1-2\varepsilon)x_1 r}{x_1 + \varepsilon x_1/(1-\varepsilon)} \\ &= (1-2\varepsilon)(1-\varepsilon)r > (1-3\varepsilon)r. \end{aligned} \tag{2.16}$$

Thus, denoting

$$F = \{t > 0 : \{t\} \in (\varepsilon, 1-\varepsilon)\},$$

by (2.15) we have

$$E_t = \{\lambda > 0 : t \in \lambda F\},$$

and therefore, using (2.16), we get

$$\begin{aligned} \int_0^r S_\mu \mathbb{I}_{\lambda F}(0) d\lambda &= \int_0^r \int_{\mathbb{T}} \mathbb{I}_{\lambda F}(t) d\mu(t) d\lambda \\ &= \int_{\mathbb{T}} \int_0^r \mathbb{I}_{\lambda F}(t) d\lambda d\mu(t) = \int_{\mathbb{T}} |E_t \cap [0,r]| d\mu(t) \\ &= \sum_{i=1}^{l} m_i |E_{x_i} \cap [0,r]| \geq (1-3\varepsilon) r |\mu|. \end{aligned} \tag{2.17}$$

This implies that

$$S_\mu \mathbb{I}_{\lambda F}(0) > (1-3\varepsilon)|\mu| \tag{2.18}$$

for some $0 < \lambda \leq r \leq \delta$. From (2.18) it follows that

$$S_\mu \mathbb{I}_{\lambda F + x}(x) > (1-3\varepsilon)|\mu|, \quad x \in \mathbb{R}. \tag{2.19}$$

It is clear that

$$\bigcup_{x:\, \{x/\lambda\} < \varepsilon} (\lambda F + x) = \{t : \{t/\lambda\} > \varepsilon\}. \tag{2.20}$$

Thus, using (2.19) and (2.20), for any x, $\{x/\lambda\} < \varepsilon$, we obtain

$$S_\mu \mathbb{I}_{\{t:\, \{t/\lambda\} > \varepsilon\}}(x) \geq S_\mu \mathbb{I}_{\lambda F + x}(x) > (1-3\varepsilon)|\mu|.$$

This implies (2.14), and the lemma is proved. □

Lemma 4. *For any measure* (2.13) *and number* $0<\varepsilon<1/3$ *there exist finite sets* $E,G\subset(-x_l,x_l)$ *such that*

$$E\cap G=\varnothing,\quad \#E>\frac{\varepsilon\#G}{4}, \tag{2.21}$$

$$S_\mu\mathbb{I}_G(x)>(1-3\varepsilon)|\mu|,\quad x\in E. \tag{2.22}$$

Proof. Denote

$$U_\lambda=\{t\in(-x_l,0):\ \{t/\lambda\}<\varepsilon\},\quad V_\lambda=\{t\in(-x_l,x_l):\ \{t/\lambda\}>\varepsilon\}. \tag{2.23}$$

It is clear that $|U_\lambda|\to\varepsilon x_l$ and $|V_\lambda|\to 2(1-\varepsilon)x_l$ as $\lambda\to0$. On the other hand, by Lemma 3, for λ small enough we have (2.14). So we can fix λ satisfying (2.14) and the conditions

$$0<\lambda<x_1,\quad |V_\lambda|<2x_l,\quad |U_\lambda|>\frac{\varepsilon x_l}{2}. \tag{2.24}$$

Denote

$$E_m=A_m\cap U_\lambda,\quad G_m=A_{m+1}\cap V_\lambda. \tag{2.25}$$

Since the sets U_λ and V_λ are a finite union of intervals in $(-1,1)$, according to Lemma 1 we have

$$\#E_m\sim\gamma m^{\mu-1}|U_\lambda|,\quad \#G_m\sim\gamma m^{\mu-1}|V_\lambda|$$

as $m\to\infty$. Hence for an integer m large enough, denoting

$$E=E_m,\quad G=G_m$$

and taking into account (2.24), we will have

$$\#E>\frac{\varepsilon\#G}{4}. \tag{2.26}$$

Besides, since $U_\lambda\cap V_\lambda=\varnothing$ we have $E\cap G=\varnothing$ and so (2.21). To show (2.22) we take an arbitrary $x\in E$. Because of (2.23) and (2.25) we will have

$$x\in A_m\cap(-x_l,0),\quad \{x/\lambda\}<\varepsilon.$$

From Lemma 2 we get $x+X\in A_{m+1}\cap(-x_l,x_l)$. Thus we get

$$S_\mu\mathbb{I}_G(x)=S_\mu\mathbb{I}_{V_\lambda}(x)=S_\mu\mathbb{I}_{\{t:\,\{\lambda t\}>\varepsilon\}}(x),$$

and therefore, since we have $\{x/\lambda\}<\varepsilon$, from Lemma 3 we obtain (2.22). □

For an arbitrary nonempty finite set $A\subset\mathbb{R}\setminus\{0\}$ we define

$$(A)=\begin{cases}\min\{|x-y|:\ x,y\in A,\ x\neq y\}, & \text{if } \#A\geq 2,\\ |x|, & \text{if } A=\{x\}.\end{cases}$$

Lemma 5. *Let* $A_k\subset\mathbb{R}\setminus\{0\}$, $k=1,2,\ldots$, *be a sequence of nonempty finite sets such that*

$$\max A_{k+1}\leq\frac{1}{4}\cdot(A_k),\quad k=1,2,\ldots. \tag{2.27}$$

Then the equality

$$x_1+x_2+\ldots+x_n=y_1+y_2+\ldots+y_n,\quad x_i,y_i\in A_i,\ i=1,2,\ldots,n \tag{2.28}$$

implies that $x_i=y_i$, $i=1,2,\ldots,n$.

Proof. Suppose to the contrary in (2.28) that we have $x_i = y_i$, $i < k$, and $x_k \neq y_k$. Hence we get

$$x_k + \ldots + x_n = y_k + \ldots + y_n. \tag{2.29}$$

From (2.27) and the relation

$$\max A_i \leq \frac{1}{4} \cdot (A_{i-1}) \leq \frac{1}{2} \max A_{i-1}$$

it follows that

$$\begin{aligned} |x_i|, |y_i| \leq \max A_i \leq \frac{1}{2} \max A_{i-1} \leq \ldots \\ \leq \frac{1}{2^{i-k-1}} \max A_{k+1} \leq \frac{(A_k)}{2^{i-k+1}} \leq \frac{|x_k - y_k|}{2^{i-k+1}} \end{aligned} \tag{2.30}$$

for any $i = k+1, k+2, \ldots, n$. Thus, using (2.29) and (2.30), we get

$$|x_k - y_k| \leq |x_{k+1}| + |y_{k+1}| + \ldots + |x_n| + |y_n| < 2|x_k - y_k| \sum_{i=1}^{\infty} \frac{1}{2^{i+1}} = |x_k - y_k|,$$

which is a contradiction, and so $x_i = y_i$ for all $i = 1, 2, \ldots, n$. □

Lemma 6. *Let μ_n be a sequence of measures satisfying the condition* (1.1). *Then for any numbers $\Delta > 0$ and $0 < \delta < 1$ there exists a measurable set $A \subset \mathbb{T}$, $|A| > 0$, such that*

$$|\{x \in \mathbb{T} : \sup_{n \in \mathbb{N}} S_{\mu_n} \mathbb{I}_A(x) > \delta\}| > \Delta \cdot |A|. \tag{2.31}$$

Proof. It is easy to observe that each $\operatorname{supp} \mu_n$ is a finite set and moreover

$$\mu_n = \sum_{i=l(n-1)+1}^{l(n)} m_i \delta_{x_i}, \quad n = 1, 2, \ldots,$$

where $0 = l(0) < l(1) < l(2) < \ldots$ are integers, $1 > x_i \searrow 0$ and $m_i > 0$, $i = 1, 2, \ldots$. Applying Lemma 4 with $\varepsilon = (1-\delta)/3$ we define finite sets E_n and G_n with

$$E_n, G_n \subset (-x_{l(n)}, x_{l(n)}), \quad E_n \cap G_n = \varnothing, \tag{2.32}$$

$$\#(E_n) > \frac{(1-\delta)\#(G_n)}{12}, \tag{2.33}$$

$$S_{\mu_n} \mathbb{I}_{G_n}(x) > \delta, \quad x \in E_n. \tag{2.34}$$

Clearly we can choose a sequence of integers n_k, $k = 1, 2, \ldots$, satisfying

$$\max(E_{n_{k+1}} \cap G_{n_{k+1}}) < \frac{(E_{n_k} \cap G_{n_k})}{4}, \quad k = 1, 2, \ldots. \tag{2.35}$$

So the sequence of sets $A_k = E_{n_k} \cup G_{n_k}$ satisfies the condition (2.27). Fix an integer

$$m > \frac{12\Delta}{1-\delta} \tag{2.36}$$

and denote

$$G = G_{n_1} + G_{n_2} + \ldots + G_{n_m}, \tag{2.37}$$

$$F_k = \sum_{i \neq k} G_{n_i} + E_{n_k}, \quad E = \bigcup_{i=1}^{n} F_i. \tag{2.38}$$

Notice that the sets F_k are mutually disjoint. Indeed, suppose to the contrary that $F_p \cap F_q \neq \varnothing$, $p \neq q$, and $x \in F_p \cap F_q$. We then have

$$x = x_1 + \ldots + x_m = y_1 + \ldots + y_m, \text{ where}$$
$$x_i, y_i \in A_i, \quad x_p \in E_{n_p}, \, y_p \in G_{n_p}.$$

Since $G_{n_p} \cap E_{n_p} = \varnothing$ (see (2.32)), we have $x_{n_p} \neq y_{n_p}$. On the other hand, because $x_i, y_i \in A_i$ and the family A_i satisfies the hypothesis of Lemma 5, we get $x_i = y_i$ for all $i = 1, 2, \ldots, m$. This is a contradiction and so the F_k are mutually disjoint. Similarly we can prove that any point $x \in G$ has the unique representation

$$x = x_1 + \ldots + x_m, \quad x_i \in G_{n_i}, \, i = 1, 2, \ldots, m.$$

This implies that

$$\#G = \prod_{i=1}^{m} \#(G_{n_i}).$$

By the same argument, using (2.33), we get

$$\#F_k = \prod_{i \neq k} \#(G_{n_i}) \cdot \#(E_{n_k}) \geq \prod_{i \neq k} \#(G_{n_i}) \cdot \frac{(1-\delta)\#(G_{n_k})}{12} = \frac{(1-\delta)\#G}{12}.$$

Combining this and (2.35) we conclude that

$$\#E = \sum_{k=1}^{m} \#F_k > \frac{m(1-\delta)\#G}{12} > \Delta \cdot \#G. \tag{2.39}$$

To prove (2.31), we take an arbitrary $x \in E$. We have $x \in F_k$ for some $1 \leq k \leq m$, and so

$$x = x_1 + \ldots + x_m, \quad x_i \in G_{n_i}, \, i \neq k, \, x_k \in E_{n_k}.$$

From (2.37) it follows that $G_{n_k} \subset G - \sum_{i \neq k} x_i$. Therefore, by (2.34), we get

$$S_{\mu_{n_k}} \mathbb{I}_G(x) = S_{\mu_{n_k}} \mathbb{I}_{G - \sum_{i \neq k} x_i}(x_k) \geq S_{\mu_{n_k}} \mathbb{I}_{G_{n_k}}(x_k) > \delta.$$

Hence we have

$$\sup_k S_{\mu_{n_k}} \mathbb{I}_G(x) > \delta, \quad x \in E. \tag{2.40}$$

Finally we let $\varepsilon = (G \cup E)/2$ and denote

$$A = G + (-\varepsilon, \varepsilon), \quad B = E + (-\varepsilon, \varepsilon).$$

It is clear that the intervals $t + (-\varepsilon, \varepsilon)$, $t \in G \cup E$, are pairwise disjoint. Hence

$$|A| = 2\varepsilon\#G, \quad |B| = 2\varepsilon\#E,$$

and so, by (2.39), we conclude

$$|B| > \Delta|A|. \tag{2.41}$$

Then for an arbitrary $x \in B$ we have $x = t + y$, where $t \in E$ and $|y| < \varepsilon$. Hence, using (2.40), we get

$$\sup_k S_{\mu_{n_k}} \mathbb{I}_A(x) \geq \sup_k S_{\mu_{n_k}} \mathbb{I}_{G+y}(x) = \sup_k S_{\mu_{n_k}} \mathbb{I}_G(t) > \delta, \quad x \in B. \tag{2.42}$$

Collecting (2.41) and (2.42) we obtain (2.31). The lemma is proved. $\square$

Definition. A sequence of linear operators

$$U_n : L^1(\mathbb{T}) \to \{\text{measurable functions on } \mathbb{T}\}$$

is said to be strong sweeping out if given $\varepsilon > 0$ there is a set E with $mE < \varepsilon$ such that $\limsup_{n\to\infty} U_n \mathbb{I}_E(x) = 1$ and $\liminf_{n\to\infty} U_n \mathbb{I}_E(x) = 0$ a.e.

To prove the theorem we need to show that the sequence S_{μ_n} is strong sweeping out. The following theorem gives a sufficient condition for a sequence of operators to be strong sweeping out.

Theorem 3 ([5, §7, Theorem 6]). *If the sequence of positive translation-invariant operators U_n satisfies the conditions leftmargin.2inrightargin.5in*

(a) $U_n(\mathbb{I}_\mathbb{T}) \to 1$ *as* $n \to \infty$,

(b) *for any* $\varepsilon > 0$ *and* $n \in \mathbb{N}$ *there exists a number* $\delta = \delta(\varepsilon, n) > 0$ *such that if* $G \subset \mathbb{T}$ *and* $m(G) < \delta$, *then*

$$m\{x \in \mathbb{T} : U_n \mathbb{I}_G(x) > \varepsilon\} < \varepsilon, \tag{2.43}$$

(c) *for any* $0 < \delta < 1$ *we have*

$$\sup_{G\subset\mathbb{T},\, |G|>0} \frac{|\{x \in X : \sup_{n\in\mathbb{N}} U_n \mathbb{I}_G(x) \geq \delta\}|}{|G|} = \infty,$$

then it is strong sweeping out.

Observe that each S_{μ_n} is positive translation invariant. Condition (**a**) follows from (1.1). To show (**b**) we simply note that

$$\int_\mathbb{T} S_{\mu_n} \mathbb{I}_G(x)dx = \int_\mathbb{T}\int_\mathbb{T} \mathbb{I}_G(x+t)dtdx = |\mu_n| \cdot |G|,$$

and therefore, by the Chebishev inequality, we will have (2.43) provided $|G| < \delta = |\mu_n|/\varepsilon$. Condition (**c**) immediately follows from Lemma 6. The theorem is proved.

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