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HOMEOMORPHISMS OF TWO-POINT SETS

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ABSTRACT. Given a cardinal $\kappa \leq \mathfrak{c}$, a subset of the plane is said to be a κ-point set if and only if it meets every line in precisely κ many points. In response to a question of Cobb, we show that for all $2 \leq \kappa, \lambda < \mathfrak{c}$ there exists a κ-point set which is homeomorphic to a λ-point set, and further, we also show that it is consistent with ZFC that for all $2 \leq \kappa < \mathfrak{c}$, there exists a κ-point set X such that for all $2 \leq \lambda < \mathfrak{c}$, X is homeomorphic to a λ-point set. On the other hand, we prove that it is consistent with ZFC that for all $2 \leq \kappa, \lambda < \mathfrak{c}$, there exists a κ-point set, such that for all homeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$, if $f(X)$ is a λ-point set, then $\lambda = \kappa$.

1. INTRODUCTION

Given a cardinal $\kappa \leq \mathfrak{c}$, a subset of the plane is a κ-point set[1] if and only if it meets every line in precisely κ many points, and is said to be a partial κ-point set if and only if it meets every line in at most κ many points. By considering infinite families of concentric circles, it is easily seen that κ-point sets exist for $\aleph_0 \leq \kappa \leq \mathfrak{c}$, and it is obvious that one-point sets do not exist. However, to demonstrate the existence of n-point sets for $2 \leq n < \aleph_0$, it seems apparent that we must resort to transfinite techniques. The standard approach, which we take in this paper, is essentially due to Mazurkiewicz[2] [13] and is based on the existence of a well-ordering of the real line, but we note that Chad et al. [3] describe an alternative construction of two-point sets which is consistent with ZF and requires only that some suitable fragment of the real line be well-ordered.

It is arguable that problems concerning two-point sets were first widely advertised amongst topologists by Mauldin [10] in his article of problems for "Open Problems in Topology" [14]. Mauldin gave three problems concerning two-point sets, and to this day, the only remaining problem is to determine if a two-point set can be chosen to be a Borel subset of the plane.[3] This problem is apparently very

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[1]We also allow ourselves to refer to (partial) two-point sets, (partial) three-point sets, and so on.

[2]A French translation [12] of Mazurkiewicz's paper is available.

[3]Mauldin says in [11] that he "believes" he first heard of the problem from Erdős, who in turn said that it had been around since he (Erdős) was a "baby."

deep, and it is likely that if we are to make any progress on it, then we will need to further our knowledge about the structure of two-point sets.

There are interesting results that are known about the structure of κ-point sets which demonstrate that differing values of κ give rise to related but distinct classes of topological objects. For example, it is known that n-point sets are not F_σ subsets of the plane for $2 \leq n < \aleph_0$, the case of $n = 2$ originally having been shown by Larman [9], with corrections later supplied by Baston and Bostock [1], and the more general case having been shown by Bouhjar et al. [2]. On the other hand, Larman [9] showed that two-point sets cannot contain arcs, and whilst Bouhjar et al. [2] showed that three-point sets are also required to have this property, they further showed that four-point sets are not. As continuing evidence of our claim, Kulesza [8] showed that two-point sets must be zero-dimensional, Fearnley et al. [7] showed that three-point sets must also be zero-dimensional, but the work of Bouhjar et al. [2] leads to the result that four-point sets may be either zero-dimensional or one-dimensional.

This paper investigates some general relationships which hold between the classes of κ-point sets. Our first main result can be seen to be saying that the classes of κ-point sets are pairwise overlapping (up to homeomorphism), and our second main result shows that it is consistent with ZFC that the intersection of these classes over all $2 \leq \kappa < \mathfrak{c}$ is non-empty (up to homeomorphism). Our final main result shows that it is consistent with ZFC that for all $2 \leq \kappa < \mathfrak{c}$ there exists a κ-point set X such that if $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism, then $f(X)$ is either a κ-point set or it is disjoint from some line.

Our analysis has two motivations. First, given his result that there exists an n-point set which is homeomorphic to a function from $\mathbb{R}$ to $\mathbb{R}$ for $2 \leq n < \aleph_0$, Cobb [6] asks if there exists an n-point set which is homeomorphic to an m-point set for some distinct $2 \leq n, m < \aleph_0$. A corollary to our first main result gives an affirmative answer to Cobb's question.

The second motivation for our work is a desire to better understand the structure of two-point sets and to follow a line of research started by Chad and Suabedissen [4, 5]. These papers have studied autohomeomorphims of two-point sets, and their main results include the facts that two-point sets may be chosen to be rigid or homogeneous or to have isometry group isomorphic to any subgroup of S^1 of cardinality less than $\mathfrak{c}$. These results examine a two-point set by looking for similarity within itself; our results will examine a two-point set by looking for similarity with other distinct types of geometric objects.

Throughout, we let $\mathcal{L}$ denote the collection of all lines in the plane. Also, if $2 \leq \kappa < \mathfrak{c}$, if P is a partial κ-point set, and if f is a homeomorphism of the plane, then we let

$$\mathcal{L}(P,\kappa) = \{L \in \mathcal{L}\colon\ |P \cap L| = \kappa\} \quad \text{and} \quad \mathcal{L}_f(P,\kappa) = \{A \in f(\mathcal{L})\colon\ |P \cap A| = \kappa\}\ .$$

2. Homeomorphisms of κ-point sets

We begin by answering Cobb's question.

Lemma 2.1. *There exists a family $\{f_t\colon t \in [0,1]\}$ of distinct homeomorphisms of $\mathbb{R}^2$ such that:*

(1) *f_0 is the identity function;*
(2) *$f_s(\mathcal{L}) \cap f_t(\mathcal{L}) = \emptyset$ for all distinct $s, t \in [0,1]$;*

(3) $|A \cap B| \leq \aleph_0$ *for all distinct* $A, B \in \bigcup_{t\in[0,1]} f_t(\mathcal{L})$.

Proof. We find it convenient throughout to identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way.

For each $r \geq 0$, let $\lfloor r \rfloor$ denote the integer part of r and let $r' = r - \lfloor r \rfloor$, so that r' denotes the fractional part of r. Let $g\colon [0,1] \to [0,1]$ be defined by $g(x) = 1 - |2x-1|$, and for each $t \in [0,1]$, let $f_t\colon \mathbb{C} \to \mathbb{C}$ be the homeomorphism defined by $f_t(re^{i\theta}) = re^{i(\theta + tg(r')\pi)}$. We note then that f_0 is the identity function. For each $r \geq 0$, let C_r denote the (possibly degenerate) circle centered at the origin of radius r. Then each f_t rotates each C_r by a factor of $tg(r')\pi$ and so leaves it invariant.

Since each f_s and f_t are homeomorphisms such that $f_s^{-1} \circ f_t = f_{t-s}$, it suffices to show that for all $t \in (0,1]$, $\mathcal{L} \cap f_t(\mathcal{L}) = \emptyset$ and that for all distinct $A, B \in \mathcal{L} \cup f_t(\mathcal{L})$, $|A \cap B| \leq \aleph_0$.

Let $t \in (0,1]$. Noting that $g^{-1}(\{0\}) = \{0,1\}$, we see that f_t fixes every point of C_r precisely when $r \in \mathbb{Z}$ and moves every point of C_r precisely when $r \notin \mathbb{Z}$, and so $\mathcal{L} \cap f_t(\mathcal{L}) = \emptyset$.

Let $A, B \in \mathcal{L} \cup f_t(\mathcal{L})$ be distinct. If $A, B \in \mathcal{L}$, then $|A \cap B| \leq 1$, as must also be the case if $A, B \in f_t(\mathcal{L})$. To complete the proof, let $A = L$ and $B = f_t(K)$ for some $L, K \in \mathcal{L}$. To see that $|A \cap B| \leq \aleph_0$, it is enough to note that if $r \geq 0$ and if $S \subseteq K$ is a line segment contained in the connected region enclosed by C_r and C_{r+1}, then $f_t(S)$ is a path which meets every line in at most four points. □

Theorem 2.2. *Let* $2 \leq \kappa, \lambda < \mathfrak{c}$. *Then there exists a* κ*-point set which is homeomorphic to a* λ*-point set.*

Proof. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism such that $\mathcal{L} \cap f(\mathcal{L}) = \emptyset$ and distinct members of $\mathcal{L} \cup f(\mathcal{L})$ meet in at most countably many points (for example, take f to be the homeomorphism f_1 furnished by Lemma 2.1), and let $\langle A_\alpha\colon \alpha < \mathfrak{c}\rangle$ be an enumeration of $\mathcal{L} \cup f(\mathcal{L})$.

We will construct an increasing sequence $\langle X_\alpha\colon \alpha < \mathfrak{c}\rangle$ of partial κ-point sets such that for all $\alpha < \mathfrak{c}$:

(1) $X_\alpha \setminus \bigcup_{\beta<\alpha} X_\beta \subseteq A_\alpha \setminus \left(\bigcup \mathcal{L}\left(\bigcup_{\beta<\alpha} X_\beta, \kappa\right) \cup \bigcup \mathcal{L}_f\left(\bigcup_{\beta<\alpha} X_\beta, \lambda\right)\right)$;
(2) X_α meets each member of $\mathcal{L}$ in at most κ many points and each member of $\mathcal{L} \cap \{A_\beta\colon \beta \leq \alpha\}$ in precisely κ many points;
(3) X_α meets each member of $f(\mathcal{L})$ in at most λ many points and each member of $f(\mathcal{L}) \cap \{A_\beta\colon \beta \leq \alpha\}$ in precisely λ many points.

Suppose then that for some $\alpha < \mathfrak{c}$ we have already chosen the partial sequence $\langle X_\beta\colon \beta < \alpha\rangle$. Let $P = \bigcup_{\beta<\alpha} X_\beta$. Then our hypotheses imply that $|P| < \mathfrak{c}$. Let $\mu < \mathfrak{c}$ be the unique cardinal number such that

$$|P \cap A_\alpha| + \mu = \begin{cases} \kappa & \text{if } A_\alpha \in \mathcal{L}, \\ \lambda & \text{if } A_\alpha \in f(\mathcal{L}). \end{cases}$$

If $\mu = 0$, then let $X_\alpha = P$. Otherwise, we will select X_α in a recursion of length μ. Let the sequence $\langle x_\delta\colon \delta < \mu\rangle$ be chosen such that

$$x_\delta \in A_\alpha \setminus \left(\bigcup \mathcal{L}(P \cup \{x_\gamma : \gamma < \delta\}, \kappa) \cup \bigcup \mathcal{L}_f(P \cup \{x_\gamma : \gamma < \delta\}, \lambda) \cup \{x_\gamma : \gamma < \delta\}\right).$$

To confirm that such a sequence exists, we note that each member of $\mathcal{L}$ is uniquely defined by two points on it, that

$$|\mathcal{L}(P \cup \{x_\gamma : \gamma < \delta\}, \kappa)| < \mathfrak{c} \quad \text{and} \quad |\mathcal{L}_f(P \cup \{x_\gamma : \gamma < \delta\}, \lambda)| < \mathfrak{c}\,,$$

and that A_α cannot be covered by fewer than $\mathfrak{c}$ many members of $(\mathcal{L} \cup f(\mathcal{L})) \setminus \{A_\alpha\}$. We then set $X_\alpha = P \cup \{x_\delta : \delta < \mu\}$, and since $\langle x_\delta \colon \delta < \mu \rangle$ is injective, it follows that $|X_\alpha \cap A_\alpha|$ is the required cardinal taken from $\{\kappa, \lambda\}$.

Let the X_α now be defined for all $\alpha < \mathfrak{c}$ and let $X = \bigcup_{\alpha < \mathfrak{c}} X_\alpha$. Conditions (2) and (3) in our recursion were chosen so that X meets each member of $\mathcal{L}$ in at least κ many points and each member of $f(\mathcal{L})$ in at least λ many points. In the case that κ is finite, condition (2) is sufficient to ensure that X meets each member of $\mathcal{L}$ in at most κ many points; however, in the case that κ is infinite, we must appeal to conditions (1) and (2) to guarantee this. Similarly, it can be argued that X meets each member of $f(\mathcal{L})$ in at most λ many points. Then $f^{-1}|X \colon X \to f^{-1}(X)$ is a homeomorphism between the κ-point set X and the λ-point set $f^{-1}(X)$. $\square$

The following corollary answers Cobb's question.

Corollary 2.3. *Let $2 \leq n, m < \aleph_0$. Then there exists an n-point set which is homeomorphic to an m-point set.*

We remark that there exists a homeomorphism $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(X)$ is not a κ-point set for all two-point sets X and all $\aleph_0 \leq \kappa < \mathfrak{c}$. Let H denote the upper half plane (including the real axis) and let f be defined by

$$f(x, y) = \begin{cases} (x, y) & \text{if } 0 \leq y, \\ (x + y, y) & \text{if } y \leq 0. \end{cases}$$

Then f has the properties that:

(1) f is the identity on H (and so maps line segments in H to line segments in H);
(2) $f(\mathbb{R}^2 \setminus H) = \mathbb{R}^2 \setminus H$ and maps line segments in $\mathbb{R}^2 \setminus H$ to line segments in $\mathbb{R}^2 \setminus H$;
(3) if L is a line, then there exist lines A and B such that $L \subseteq f(A) \cup f(B)$;
(4) $f(X)$ is a partial $2n$-point set, and hence not a κ-point set, for all $2 \leq n < \aleph_0$, all $\aleph_0 \leq \kappa < \mathfrak{c}$, and all n-point sets X.

3. The existence of universal two-point sets

It turns out that we can strengthen Theorem 2.2, provided that we assume that $\mathfrak{c}$ is regular and that there are fewer than $\mathfrak{c}$ many cardinals less than $\mathfrak{c}$. These assumptions are consistent with ZFC, for they hold in models of ZFC + CH.

Definition 3.1. Let $2 \leq \kappa < \mathfrak{c}$. Then a κ-point set is said to be *universal* if it is homeomorphic to a λ-point set for all $2 \leq \lambda < \mathfrak{c}$.

To prove the existence of universal κ-point sets, it will be sufficient to demonstrate the existence of a universal two-point set.

Theorem 3.2 ($\mathfrak{c}$ is regular and $\mathfrak{c} < \aleph_{\mathfrak{c}}$)**.** *There exists a universal two-point set.*

Proof. Let $\mu < \mathfrak{c}$ be a cardinal admitting an enumeration $\langle \kappa_\gamma \colon \gamma < \mu \rangle$ of the set of all cardinals $2 < \kappa < \mathfrak{c}$, let $\langle t_\gamma \colon \gamma < \mu \rangle$ be an injective sequence on $(0, 1]$, let $\{f_t \colon t \in [0, 1]\}$ be a family of functions given by Lemma 2.1, let f_γ denote f_{t_γ}, and let $\{A_\alpha : \alpha < \mathfrak{c}\}$ enumerate $\mathcal{L} \cup \bigcup_{\gamma < \mu} f_\gamma(\mathcal{L})$.

We will construct an increasing sequence $\langle X_\alpha \colon \alpha < \mathfrak{c} \rangle$ of subsets of the plane such that for all $\alpha < \mathfrak{c}$:

(1) $X_\alpha \setminus \bigcup_{\beta<\alpha} X_\beta \subseteq A_\alpha \setminus \Big(\bigcup \mathcal{L}\Big(\bigcup_{\beta<\alpha} X_\beta, 2\Big) \cup \bigcup_{\gamma<\mu} \bigcup \mathcal{L}_{f_\gamma}\Big(\bigcup_{\beta<\alpha} X_\beta, \kappa_\gamma\Big)\Big)$;
(2) X_α meets each member of $\mathcal{L}$ in at most two points and each member of $\mathcal{L} \cap \{A_\beta : \beta \le \alpha\}$ in precisely two points;
(3) for each $\gamma < \mu$, X_α meets each member of $f_\gamma(\mathcal{L})$ in at most κ_γ many points and each member of $f_\gamma(\mathcal{L}) \cap \{A_\beta : \beta \le \alpha\}$ in precisely κ_γ many points.

Suppose that for some $\alpha < \mathfrak{c}$ we have already chosen the partial sequence $\langle X_\beta : \beta < \alpha\rangle$. Let $P = \bigcup_{\beta<\alpha} X_\beta$. Then $\Big|X_\beta \setminus \bigcup_{\gamma<\beta} X_\gamma\Big| < \mathfrak{c}$ for all $\beta < \alpha$, and since $\mathfrak{c}$ is regular, it follows that $|P| < \mathfrak{c}$. We choose X_α more or less as we did in the proof of Theorem 2.2: if $A_\alpha \in \mathcal{L}$, then we choose X_α so that $|X_\alpha \setminus P| \le 2$ and X_α meets A_α in precisely two points; otherwise, $A_\alpha \in f_\gamma(\mathcal{L})$ for some $\gamma < \mu$, and we choose X_α so that $|X_\alpha \setminus P| \le \kappa_\gamma$ and X_α meets A_α in precisely κ_γ many points.

Let $X = \bigcup_{\alpha<\mathfrak{c}} X_\alpha$. Then X is a two-point set and for any $\kappa < \mathfrak{c}$, $\kappa = \kappa_\gamma$ for some $\gamma < \mu$, and $f_\gamma^{-1}(X)$ is a κ-point set homeomorphic to X. □

Corollary 3.3 ($\mathfrak{c}$ is regular and $\mathfrak{c} < \aleph_{\mathfrak{c}}$)**.** *Let $2 \le \kappa \le \aleph_0$. Then there exists a universal κ-point set.*

4. The existence of delicate κ-point sets

We will now consider how κ-point sets are embedded in the plane and obtain a result which is complementary to those we have previously discussed. We will make use of the axiom that "$\mathbb{R}$ cannot be covered by fewer than $\mathfrak{c}$ many of its nowhere dense subsets", and we will refer to this axiom, which is implied by the Continuum Hypothesis or Martin's Axiom, by $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$.

Definition 4.1. Let $2 \le \kappa < \mathfrak{c}$. Then a κ-point set X is said to be *delicate* if for every homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$, if f is not affine, then $f(X)$ is disjoint from some line.

Lemma 4.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a non-affine homeomorphism. Then there exists a collection $\mathcal{K}$ of lines such that:*
(1) *$f^{-1}(K)$ is not a line for each $K \in \mathcal{K}$;*
(2) *the members of $\mathcal{K}$ are pairwise parallel;*
(3) *$|\mathcal{K}| = \mathfrak{c}$.*

Proof. Note that a bijection $\mathbb{R}^2 \to \mathbb{R}^2$ is affine if and only if its inverse is affine. Let $x, y, z \in \mathbb{R}^2$ be collinear and such that $f^{-1}(x)$, $f^{-1}(y)$ and $f^{-1}(z)$ form the vertices of a non-degenerate triangle T . Let $\varepsilon_1 > 0$ be the length of the altitude of T joining $f^{-1}(z)$ to the line segment joining $f^{-1}(x)$ and $f^{-1}(y)$, let $\varepsilon_2 > 0$ be the distance between $f^{-1}(x)$ and $f^{-1}(y)$, and let $\varepsilon = \min\{\varepsilon_1/2, \varepsilon_2/2\}$. Then letting $B(v, r)$ denote the ball of radius $r > 0$ about $v \in \mathbb{R}^2$, it is easily seen that $B(f^{-1}(x), \varepsilon)$, $B(f^{-1}(y), \varepsilon)$ and $B(f^{-1}(z), \varepsilon)$ are pairwise disjoint. Further, it can be seen that for all $a \in B(f^{-1}(x), \varepsilon)$, all $b \in B(f^{-1}(y), \varepsilon)$ and all $c \in B(f^{-1}(z), \varepsilon)$, the points a, b and c form the vertices of a non-degenerate triangle.

Given that f^{-1} is continuous, let $\delta > 0$ be such that $f^{-1}(B(x, \delta)) \subseteq B(f^{-1}(x), \varepsilon)$ and $f^{-1}(B(y, \delta)) \subseteq B(f^{-1}(y), \varepsilon)$ and $f^{-1}(B(z, \delta)) \subseteq B(f^{-1}(z), \varepsilon)$. Let L be the line spanned by x and y, let $v \in \mathbb{R}^2$ be of unit length and perpendicular to L, and let $\mathcal{K} = \{L + rv : r \in [0, \delta)\}$. Then it is clear that the members of $\mathcal{K}$ are pairwise parallel and $|K| = \mathfrak{c}$. Further, for each $r \in [0, \delta)$, the points $f^{-1}(x+rv)$, $f^{-1}(y+rv)$ and $f^{-1}(z + rv)$ witness that $f^{-1}(L + rv)$ is not a line. □

Theorem 4.3 ($\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$)**.** *Let $2 \leq \kappa < \mathfrak{c}$. Then there exists a delicate κ-point set.*

Proof. Let $2 \leq \kappa < \mathfrak{c}$, let $\langle L_\alpha \colon \alpha < \mathfrak{c}\rangle$ enumerate the collection of all lines, and let $\langle f_\alpha \colon \alpha < \mathfrak{c}\rangle$ enumerate all non-affine autohomeomorphims of $\mathbb{R}^2$. By the previous lemma, let $\langle \mathcal{K}_\alpha \colon \alpha < \mathfrak{c}\rangle$ be a sequence of collections of lines such that for each $\alpha < \mathfrak{c}$:

(a) for each $K \in \mathcal{K}_\alpha$, $f_\alpha^{-1}(K)$ is not a line;
(b) the members of $\mathcal{K}_\alpha$ are pairwise parallel;
(c) $|\mathcal{K}_\alpha| = \mathfrak{c}$.

We note then that the members of each $\mathcal{K}_\alpha$ are pairwise disjoint, as are the members of each $f_\alpha^{-1}(\mathcal{K}_\alpha)$.

We will construct an increasing sequence $\langle X_\alpha \colon \alpha < \mathfrak{c}\rangle$ of partial κ-point sets and a sequence $\langle K_\alpha \colon \alpha < \mathfrak{c}\rangle$ of lines such that for all $\alpha < \mathfrak{c}$:

(1) $\left|X_\alpha \setminus \bigcup_{\beta<\alpha} X_\beta\right| \leq \kappa + \aleph_0$ and $\left(X_\alpha \setminus \bigcup_{\beta<\alpha} X_\beta\right) \cap \bigcup \mathcal{L}\left(\bigcup_{\beta<\alpha} X_\beta, \kappa\right) = \emptyset$;
(2) X_α meets each member of $\mathcal{L}$ in at most κ many points and each member of $\{L_\beta : \beta \leq \alpha\}$ in precisely κ many points;
(3) $K_\alpha \in \mathcal{K}_\alpha$;
(4) X_α and $\bigcup_{\beta\leq\alpha} f_\beta^{-1}(K_\beta)$ are disjoint;
(5) for each $\beta < \alpha$ and for each $\gamma < \mathfrak{c}$, if $|X_\beta \cap L_\gamma| < \kappa$, then $f_\beta^{-1}(K_\beta) \cap L_\gamma$ is nowhere dense in the relative topology on L_γ.

Suppose then that for some $\alpha < \mathfrak{c}$ we have already chosen the partial sequences $\langle X_\beta \colon \beta < \alpha\rangle$ and $\langle K_\beta \colon \beta < \alpha\rangle$. Let $P = \bigcup_{\beta<\alpha} X_\beta$ and let $F = \bigcup_{\beta<\alpha} f_\beta^{-1}(K_\beta)$. We will choose X_α in two steps. First, we will extend P to Q so as to satisfy conditions (1) - (4), and then we will extend Q to X_α so as to preserve these properties and to additionally satisfy condition (5).

If $|P \cap L_\alpha| = \kappa$, then let $Q = P$. Otherwise it follows from the inductive hypothesis and $\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$ that $|L_\alpha \setminus F| = \mathfrak{c}$, and so by choosing suitable points in

$$L_\alpha \setminus \left(\bigcup \mathcal{L}(P, \kappa) \cup F\right),$$

let Q be a partial κ-point set such that $P \subseteq Q$ and $Q \setminus P \subseteq L_\alpha$ and $|L_\alpha \cap Q| = \kappa$.

It is easily argued from (1) that $|Q| < \mathfrak{c}$, and so we may choose $K_\alpha \in \mathcal{K}_\alpha$ to be such that $Q \cap f_\alpha^{-1}(K_\alpha) = \emptyset$. Let

$$\mathcal{A} = \left\{L \in \mathcal{L} \colon |Q \cap L| < \kappa \text{ and } f_\alpha^{-1}(K_\alpha) \cap L \text{ is somewhere dense in } L\right\}.$$

We will now argue that $\mathcal{A}$ is countable. Towards this, let

$$\mathcal{S} = \left\{S \subseteq f_\alpha^{-1}(K_\alpha) \colon S \text{ is a maximal non-degenerate line segment}\right\},$$

and so noting that $L \in \mathcal{A}$ if and only if $L \in \mathcal{L}$ and L contains a member of S, it will suffice to show that $\mathcal{S}$ is countable. This is obvious once we note that a maximal non-degenerate line segment contained in $f_\alpha^{-1}(K_\alpha)$ is the image under f_α^{-1} of some interval contained in K_α.

It now remains to describe how to extend Q to X_α so that X_α meets each member of $\mathcal{A}$ in precisely κ many points. For each $L \in \mathcal{A}$ it is a routine exercise to show that $L \setminus f_\alpha^{-1}(K_\alpha) \neq \emptyset$, and so we can choose $x_L \in L \setminus f_\alpha^{-1}(K_\alpha)$. Since $\mathbb{R}^2 \setminus f_\alpha^{-1}(K_\alpha)$ is an open neighbourhood of x_L, it follows that there exists an interval $I_L \subseteq L \setminus f_\alpha^{-1}(K_\alpha)$.

By choosing suitable points in

$$\left(\bigcup_{L\in\mathcal{A}} I_L\right) \setminus \left(\bigcup \mathcal{L}(Q,\kappa) \cup F \cup f_\alpha^{-1}(K_\alpha)\right) ,$$

we select X_α to be a partial κ-point set which extends Q and meets each $L \in \mathcal{A}$ in κ many points. Then X_α and K_α satisfy our inductive hypothesis.

Let $X = \bigcup_{\alpha<\mathfrak{c}} X_\alpha$. Then X is a κ-point set. To see that X is delicate, it suffices to note that if $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a non-affine homeomorphism, then $f = f_\alpha$ for some $\alpha < \mathfrak{c}$, whence $f(X) = f_\alpha(X)$ is disjoint from the line K_α. □

Corollary 4.4 ($\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$)**.** *There exists a delicate two-point set.*

Note that if $2 \leq \kappa, \lambda < \mathfrak{c}$ and X is a delicate κ-point set and $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism, then whenever $f(X)$ is a λ-point set we must have that $\lambda = \kappa$.

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