

## ON THE ASYMPTOTIC BEHAVIOR OF WEAKLY LACUNARY SERIES

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ABSTRACT. Let  $f$  be a measurable function satisfying

$$f(x+1) = f(x), \quad \int_0^1 f(x) dx = 0, \quad \text{Var}_{[0,1]} f < +\infty,$$

and let  $(n_k)_{k \geq 1}$  be a sequence of integers satisfying  $n_{k+1}/n_k \geq q > 1$  ( $k = 1, 2, \dots$ ). By the classical theory of lacunary series, under suitable Diophantine conditions on  $n_k$ ,  $(f(n_k x))_{k \geq 1}$  satisfies the central limit theorem and the law of the iterated logarithm. These results extend for a class of subexponentially growing sequences  $(n_k)_{k \geq 1}$  as well, but as Fukuyama showed, the behavior of  $f(n_k x)$  is generally not permutation-invariant; e.g. a rearrangement of the sequence can ruin the CLT and LIL. In this paper we construct an infinite order Diophantine condition implying the permutation-invariant CLT and LIL without any growth conditions on  $(n_k)_{k \geq 1}$  and show that the known finite order Diophantine conditions in the theory do not imply permutation-invariance even if  $f(x) = \sin 2\pi x$  and  $(n_k)_{k \geq 1}$  grows almost exponentially. Finally, we prove that in a suitable statistical sense, for almost all sequences  $(n_k)_{k \geq 1}$  growing faster than polynomially,  $(f(n_k x))_{k \geq 1}$  has permutation-invariant behavior.

### 1. INTRODUCTION

Let  $f$  be a measurable function satisfying

$$(1) \quad f(x+1) = f(x), \quad \int_0^1 f(x) dx = 0, \quad \text{Var}_{[0,1]} f < +\infty$$

and let  $(n_k)_{k \geq 1}$  be a sequence of positive integers satisfying the Hadamard gap condition

$$(2) \quad n_{k+1}/n_k \geq q > 1 \quad (k = 1, 2, \dots).$$

In the case  $n_k = 2^k$ , Kac [14] proved that  $f(n_k x)$  satisfies the central limit theorem

$$(3) \quad N^{-1/2} \sum_{k=1}^N f(n_k x) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$$

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with respect to the probability space  $[0, 1]$  equipped with the Lebesgue measure, where

$$\sigma^2 = \int_0^1 f^2(x) dx + 2 \sum_{k=1}^{\infty} \int_0^1 f(x)f(2^k x) dx.$$

Gaposhkin [11] extended (3) to the case when the fractions  $n_{k+1}/n_k$  are all integers or if  $n_{k+1}/n_k \rightarrow \alpha$ , where  $\alpha^r$  is irrational for  $r = 1, 2, \dots$ . On the other hand, an example of Erdős and Fortet (see [15], p. 646) shows that the CLT (3) fails if  $n_k = 2^k - 1$ . Gaposhkin also showed (see [12]) that the asymptotic behavior of  $\sum_{k=1}^N f(n_k x)$  is intimately connected with the number of solutions of the Diophantine equation

$$an_k + bn_l = c, \quad 1 \leq k, l \leq N.$$

Improving these results, Aistleitner and Berkes [1] gave a necessary and sufficient condition for the CLT (3). For related laws of the iterated logarithm, see [5], [11], [13], and [17].

The previous results show that for arithmetically “nice” sequences  $(n_k)_{k \geq 1}$ , the system  $f(n_k x)$  behaves like a sequence of independent random variables. However, as an example of Fukuyama [9] shows, this result is not permutation-invariant: a rearrangement of  $(n_k)_{k \geq 1}$  can change the variance of the limiting Gaussian law or ruin the CLT altogether. A complete characterization of the permutation-invariant CLT and LIL for  $f(n_k x)$  under the Hadamard gap condition (2) is given in our forthcoming paper [3]. In particular, it is shown there that in the harmonic case  $f(x) = \cos 2\pi x$ ,  $f(x) = \sin 2\pi x$  the CLT and LIL for  $f(n_k x)$  hold after any permutation of  $(n_k)_{k \geq 1}$ .

For subexponentially growing  $(n_k)_{k \geq 1}$  the situation changes radically. Note that in the case  $f(x) = \cos 2\pi x$ ,  $f(x) = \sin 2\pi x$ , the unpermuted CLT and LIL remain valid under the weaker gap condition

$$n_{k+1}/n_k \geq 1 + ck^{-\alpha}, \quad 0 < \alpha < 1/2;$$

see Erdős [8], Takahashi [18], [19]. However, as the following theorem shows, the slightest weakening of the Hadamard gap condition (2) can ruin the permutation-invariant CLT and LIL.

**Theorem 1.** *For any positive sequence  $(\varepsilon_k)_{k \geq 1}$  tending to 0, there exists a sequence  $(n_k)_{k \geq 1}$  of positive integers satisfying*

$$(4) \quad n_{k+1}/n_k \geq 1 + \varepsilon_k, \quad k \geq k_0$$

and a permutation  $\sigma : \mathbb{N} \rightarrow \mathbb{N}$  of the positive integers such that

$$(5) \quad N^{-1/2} \sum_{k=1}^N \cos 2\pi n_{\sigma(k)} x - b_N \xrightarrow{\mathcal{D}} G,$$

where  $G$  is a non-Gaussian distribution with characteristic function given by (14)-(16) and  $(b_N)_{N \geq 1}$  is a numerical sequence with  $b_N = O(1)$ . Moreover, there exists a permutation  $\sigma : \mathbb{N} \rightarrow \mathbb{N}$  of the positive integers such that

$$(6) \quad \limsup_{N \rightarrow \infty} \frac{\sum_{k=1}^N \cos 2\pi n_{\sigma(k)} x}{\sqrt{2N \log \log N}} = +\infty \quad \text{a.e.}$$

As we will see, with a slight change of the norming sequence  $N^{-1/2}$  in (5) the limit distribution  $G$  can also be chosen as the Cauchy distribution with density  $\pi^{-1}(1+x^2)^{-1}$ . The proof of Theorem 1 will also show that (6) can be improved to

$$(7) \quad \limsup_{N \rightarrow \infty} \frac{\sum_{k=1}^N \cos 2\pi n_{\sigma(k)} x}{\sqrt{N \log N}} > 0 \quad \text{a.e.}$$

It should be noted that the subexponential gap condition (4) does not imply the permutation-invariant behavior of  $f(n_k x)$  even for arithmetically “nice” sequences  $(n_k)_{k \geq 1}$ . Indeed, the sequence  $(n_k)_{k \geq 1}$  in Theorem 1 can be chosen so that it satisfies conditions **B**, **C**, **G** in our paper [7] implying very strong independence properties of  $\cos 2\pi n_k x$ ,  $\sin 2\pi n_k x$ , including the CLT and LIL. In fact, it is not easy to construct subexponential sequences  $(n_k)_{k \geq 1}$  satisfying the permutation-invariant CLT and LIL: the only known example (see [2]) is the Hardy-Littlewood-Pólya sequence, i.e. the sequence generated by finitely many primes and arranged in increasing order; the proof uses deep number-theoretic tools. The purpose of this paper is to introduce a new, infinite order Diophantine condition  $\mathbf{A}_\omega$  which implies the permutation-invariant CLT and LIL for  $f(n_k x)$  and then to show that, in a suitable statistical sense, almost all sequences  $(n_k)_{k \geq 1}$  growing faster than polynomially satisfy  $\mathbf{A}_\omega$ . Thus, despite the difficulties to construct explicit examples, the permutation-invariant CLT and LIL are rather the rule than the exception.

Given a nondecreasing sequence  $\omega = (\omega_1, \omega_2, \dots)$  of positive numbers tending to  $+\infty$ , let us say that a sequence  $(n_k)_{k \geq 1}$  of different positive integers satisfies

**Condition  $\mathbf{A}_\omega$**  if for any  $N \geq N_0$  the Diophantine equation

$$(8) \quad a_1 n_{k_1} + \dots + a_r n_{k_r} = 0, \quad 2 \leq r \leq \omega_N, \quad 0 < |a_1|, \dots, |a_r| \leq N^{\omega_N}$$

with different indices  $k_j$  and nonzero integer coefficients  $a_j$  has only such solutions, where all  $n_{k_j}$  belong to the smallest  $N$  elements of the sequence  $(n_k)_{k \geq 1}$ .

Clearly, this property is permutation-invariant and it implies that for any fixed nonzero integer coefficients  $a_j$  the number of solutions of (8) with different indices  $k_j$  is at most  $N^r$ .

**Theorem 2.** *Let  $\omega = (\omega_1, \omega_2, \dots)$  be a nondecreasing sequence tending to  $+\infty$  and let  $(n_k)_{k \geq 1}$  be a sequence of different positive integers satisfying condition  $\mathbf{A}_\omega$ . Then for any  $f$  satisfying (1) we have*

$$(9) \quad N^{-1/2} \sum_{k=1}^N f(n_k x) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \|f\|^2),$$

where  $\|f\|$  denotes the  $L_2(0, 1)$  norm of  $f$ . If  $\omega_k \geq (\log k)^\alpha$  for some  $\alpha > 0$  and  $k \geq k_0$ , then we also have

$$(10) \quad \limsup_{N \rightarrow \infty} \frac{\sum_{k=1}^N f(n_k x)}{(2N \log \log N)^{1/2}} = \|f\| \quad \text{a.e.}$$

Condition  $\mathbf{A}_\omega$  is different from the usual Diophantine conditions in lacunarity theory, which typically involve 4 or fewer terms. In contrast,  $\mathbf{A}_\omega$  is an “infinite order” condition; namely, it involves equations with arbitrary large order. As noted, the usual Diophantine conditions do not suffice in Theorem 2. Given any  $\omega_k \uparrow \infty$ , it is not hard to see that any sufficiently rapidly growing sequence  $(n_k)_{k \geq 1}$  satisfies  $\mathbf{A}_\omega$ ; on the other hand, we do not have any “concrete” subexponential examples for

$\mathbf{A}_\omega$ . However, we will show that, in a suitable statistical sense, almost all sequences growing faster than polynomially satisfy condition  $\mathbf{A}_\omega$  for some appropriate  $\omega$ . To make this precise requires defining a probability measure over the set of such sequences, or, equivalently, a natural random procedure to generate such sequences. A simple procedure is to choose  $n_k$  independently and uniformly from the integers in the interval

$$(11) \quad I_k = [a(k-1)^{\omega_{k-1}}, ak^{\omega_k}), \quad k = 1, 2, \dots$$

Note that the length of  $I_k$  is at least  $a\omega_k(k-1)^{\omega_{k-1}} \geq a\omega_1$  for  $k = 2, 3, \dots$  and equals  $a$  for  $k = 1$ , and thus choosing  $a$  large enough, each  $I_k$  contains at least one integer. Let  $\mu_\omega$  be the distribution of the random sequence  $(n_k)_{k \geq 1}$  in the product space  $I_1 \times I_2 \times \dots$ .

**Theorem 3.** *Let  $\omega_k \uparrow \infty$  and let  $f$  be a function satisfying (1). Then with probability one with respect to  $\mu_\omega$  the sequence  $(f(n_k x))_{k \geq 1}$  satisfies the CLT (9) after any permutation of its terms, and if  $\omega_k \geq (\log k)^\alpha$  for some  $\alpha > 0$  and  $k \geq k_0$ ,  $(f(n_k x))_{k \geq 1}$  also satisfies the LIL (10) after any permutation of its terms.*

The sequences  $(n_k)_{k \geq 1}$  provided by  $\mu_\omega$  satisfy  $n_k = O(k^{\omega_k})$ ; for slowly increasing  $\omega_k$ , the so-obtained sequences grow much slower than exponentially, in fact they grow barely faster than polynomial speed. If  $\omega_k$  grows so slowly that  $\omega_k - \omega_{k-1} = o((\log k)^{-1})$ , then the so-obtained sequence  $(n_k)_{k \geq 1}$  has the precise speed  $n_k \sim k^{\omega_k}$ . We do not know if there exist polynomially growing sequences  $(n_k)_{k \geq 1}$  satisfying the permutation-invariant CLT or LIL. The proof of Theorem 3 will also show that with probability 1, the sequences provided by  $\mu_\omega$  satisfy  $\mathbf{A}_{\omega^*}$  with  $\omega^* = (c\omega_1^{1/2}, c\omega_2^{1/2}, \dots)$ .

## 2. PROOFS

**2.1. Proof of Theorem 1.** We begin with the CLT part. Let  $(\varepsilon_k)_{k \geq 1}$  be a positive sequence tending to 0. Let  $m_1 < m_2 < \dots$  be positive integers such that  $m_{k+1}/m_k \geq 2^{k^2}$ ,  $k = 1, 2, \dots$  and all the  $m_k$  are powers of 2; let  $r_1 \leq r_2 \leq \dots$  be positive integers satisfying  $1 \leq r_k \leq k^2$ . Put  $I_k = \{m_k, 2m_k, \dots, r_k m_k\}$ ; clearly the sets  $I_k$ ,  $k = 1, 2, \dots$  are disjoint. Define the sequence  $(n_k)_{k \geq 1}$  by

$$(12) \quad (n_k)_{k \geq 1} = \bigcup_{j=1}^{\infty} I_j.$$

Clearly, if  $n_k, n_{k+1} \in I_j$ , then  $n_{k+1}/n_k \geq 1 + 1/r_j$  and thus if  $r_j$  grows sufficiently slowly, the sequence  $(n_k)_{k \geq 1}$  satisfies the gap condition (4). Also, if  $r_j$  grows sufficiently slowly, there exists a subsequence  $(n_{k_\ell})_{\ell \geq 1}$  of  $(n_k)_{k \geq 1}$  which has exactly the same structure as the sequence in (12), just with  $r_k \sim k$ . By the proof of Theorem 1 in [4],  $(\cos 2\pi n_{k_\ell} x)_{\ell \geq 1}$  satisfies

$$(13) \quad \frac{1}{\sqrt{N}} \sum_{\ell=1}^N \cos 2\pi n_{k_\ell} x - b_N \xrightarrow{D} G,$$

where  $(b_N)_{N \geq 1}$  is a numerical sequence with  $b_N = O(1)$  and  $G$  is a non-Gaussian infinitely divisible distribution with characteristic function

$$(14) \quad \exp \left\{ \int_{\mathbb{R} \setminus \{0\}} \left( e^{itx} - 1 - \frac{itx}{1+x^2} \right) dL(x) \right\},$$

where

$$(15) \quad L(x) = \begin{cases} -\frac{1}{\pi} \int_x^1 \frac{F(t)}{t} dt & \text{if } 0 < x \leq 1, \\ \frac{1}{\pi} \int_{-x}^1 \frac{G(t)}{t} dt & \text{if } -1 \leq x < 0, \\ 0 & \text{if } |x| > 1 \end{cases}$$

and

$$(16) \quad F(t) = \lambda\{x > 0 : \sin x/x \geq t\}, \quad G(t) = \lambda\{x > 0 : \sin x/x \leq -t\} \quad (t > 0),$$

where  $\lambda$  is the Lebesgue measure. Define a permutation  $\sigma$  in the following way:

- for  $k \notin \{1, 2, 4, \dots, 2^m, \dots\}$ ,  $\sigma(k)$  takes the values of the set  $\{k_1, k_2, \dots\}$  in consecutive order;
- for  $k \in \{1, 2, 4, \dots, 2^m, \dots\}$ ,  $\sigma(k)$  takes the values of the set  $\mathbb{N} \setminus \{k_1, k_2, \dots\}$  in consecutive order.

Then  $\sigma$  is a permutation of  $\mathbb{N}$  and the sums

$$\sum_{k=1}^N \cos 2\pi n_{\sigma(k)} x \quad \text{and} \quad \sum_{l=1}^N \cos 2\pi n_{k_l} x$$

differ at most in  $2 \log_2 N$  terms. Therefore, (13) implies (5), proving the first part of Theorem 1.

The proof of the LIL part of Theorem 1 is modeled after the proof of Theorem 1 in Berkes and Philipp [6]. Similarly as above, we construct a sequence  $(n_k)_{k \geq 1}$  satisfying (4) that contains a subsequence  $(\mu_k)_{k \geq 1}$  of the form (12) with  $I_k = \{m_k, 2m_k, \dots, r_k m_k\}$ , where  $r_k \sim k \log k$  and  $(m_k)_{k \geq 1}$  is growing fast; specifically we choose  $m_k$  in such a way that it is a power of 2 and  $m_{k+1} \geq r_k 2^{2k} m_k$ . Let  $\mathcal{F}_i$  denote the  $\sigma$ -field generated of the dyadic intervals

$$[\nu 2^{-(\log_2 m_i)-i}, (\nu + 1) 2^{-(\log_2 m_i)-i}), \quad 0 \leq \nu < 2^{(\log_2 m_i)+i}.$$

Write

$$X_i = \cos 2\pi m_i x + \dots + \cos 2\pi r_i m_i x$$

and

$$Z_i = \mathbb{E}(X_i | \mathcal{F}_i).$$

Then for all  $x \in (0, 1)$ ,

$$|X_i(x) - Z_i(x)| \ll r_i^2 m_i 2^{-(\log_2 m_i)-i}$$

and thus

$$\sum_{i \geq 1} |X_i(x) - Z_i(x)| < \infty \quad \text{for all } x \in (0, 1).$$

As in [6, Lemma 2.1], the random variables  $Z_1, Z_2, \dots$  are independent, and as in [6, Lemma 2.2], for almost every  $x \in (0, 1)$  we have

$$\limsup_{i \rightarrow \infty} X_i/r_i \geq 2/\pi.$$

Assume that for a fixed  $x$  and some  $i \geq 1$  we have  $X_i/r_i \geq 1/\pi$ . Then either

$$|X_1 + \dots + X_{i-1}| \geq r_i/2\pi$$

or

$$|X_1 + \dots + X_i| \geq r_i/2\pi.$$

Since the total number of summands in  $X_1, \dots, X_i$  is  $\ll i^2 \log i$ , and since

$$r_i \gg (i^2 \log i)^{1/2} (\log(i^2 \log i))^{1/2},$$

we have

$$\limsup_{N \rightarrow \infty} \frac{\left| \sum_{k=1}^N \cos 2\pi \mu_k x \right|}{\sqrt{N \log N}} > 0 \quad \text{a.e.,}$$

and, in particular,

$$\limsup_{N \rightarrow \infty} \frac{\left| \sum_{k=1}^N \cos 2\pi \mu_k x \right|}{\sqrt{2N \log \log N}} = +\infty \quad \text{a.e.}$$

Thus we constructed a subsequence of  $(n_k)_{k \geq 1}$  failing the LIL and similarly as above, we can construct a permutation  $(n_{\sigma(k)})_{k \geq 1}$  of  $(n_k)$  failing the LIL as well.

Note that the just completed proof of Theorem 1 provides the stronger relation (7) instead of (6). Also, if relation  $r_k \sim k$  just preceding relation (13) is replaced by  $r_k \sim k \log \log k$ , then the proof of Theorem 2 in [4] shows that (5) will be replaced by

$$a_N^{-1} \sum_{k=1}^N \cos 2\pi n_{\sigma(k)} x - b_N \xrightarrow{\mathcal{D}} G,$$

where  $G$  is the Cauchy distribution with density  $\pi^{-1}(1+x^2)^{-1}$  and  $a_N \sim c\sqrt{N/\log \log N}$  with some  $c > 0$  and  $b_N = O(1)$ .

### 2.2. Proof of Theorem 2.

**Lemma 1.** *Let  $\omega_k \uparrow \infty$  and let  $(n_k)_{k \geq 1}$  be a sequence of different positive integers satisfying condition  $\mathbf{A}_\omega$ . Let  $f$  satisfy (1) and put  $S_N = \sum_{k=1}^N f(n_k x)$ ,  $\sigma_N = (\mathbb{E}S_N^2)^{1/2}$ . Then for any  $p \geq 3$  we have*

$$\mathbb{E}S_N^p = \begin{cases} \frac{p!}{(p/2)!} 2^{-p/2} \sigma_N^p + O(T_N) & \text{if } p \text{ is even,} \\ O(T_N) & \text{if } p \text{ is odd,} \end{cases}$$

where

$$T_N = \exp(p^2) N^{(p-1)/2} (\log N)^p$$

and the constants implied by the  $O$  are absolute.

*Proof.* Fix  $p \geq 2$  and choose the integer  $N$  so large that  $\omega_{[N^{1/4}]} \geq 8p$ . Without loss of generality we may assume that  $f$  is an even function and that  $\|f\|_\infty \leq 1$ ,  $\text{Var}_{[0,1]} f \leq 1$ ; the proof in the general case is similar. Let

$$(17) \quad f \sim \sum_{j=1}^{\infty} a_j \cos 2\pi j x$$

be the Fourier series of  $f$ .  $\text{Var}_{[0,1]} f \leq 1$  implies that

$$(18) \quad |a_j| \leq j^{-1}$$

(see Zygmund [20, p. 48]), and writing

$$g(x) = \sum_{j=1}^{N^p} a_j \cos 2\pi jx, \quad r(x) = f(x) - g(x),$$

we have

$$\|g\|_\infty \leq \text{Var}_{[0,1]} f + \|f\|_\infty \leq 2, \quad \|r\|_\infty \leq \|f\|_\infty + \|g\|_\infty \leq 3$$

by (4.12) of Chapter II and (1.25) and (3.5) of Chapter III of Zygmund [20]. Letting  $\|\cdot\|$  and  $\|\cdot\|_p$  denote the  $L_2(0, 1)$ , resp.  $L_p(0, 1)$ , norms, (18) yields for any positive integer  $n$ ,

$$(19) \quad \|r(nx)\|^2 = \|r(x)\|^2 = \frac{1}{2} \sum_{j=N^p+1}^\infty a_j^2 \leq N^{-p}.$$

By Minkowski's inequality,

$$\|S_N\|_p \leq \left\| \sum_{k=1}^N g(n_k x) \right\|_p + \left\| \sum_{k=1}^N r(n_k x) \right\|_p$$

and

$$(20) \quad \left\| \sum_{k=1}^N r(n_k x) \right\|_p \leq 3 \sum_{k=1}^N \|r(n_k x)/3\|_p \leq 3 \sum_{k=1}^N \|r(n_k x)/3\|^{2/p} \leq 3 \sum_{k=1}^N N^{-1} \leq 3.$$

Similarly,

$$\left| \|S_N\| - \left\| \sum_{k=1}^N g(n_k x) \right\| \right| \leq \left\| \sum_{k=1}^N r(n_k x) \right\| \leq N^{-\frac{p}{2}+1},$$

and therefore

$$\begin{aligned} & \left| \|S_N\|^p - \left\| \sum_{k=1}^N g(n_k x) \right\|^p \right| \\ & \leq p \max \left( \|S_N\|^{p-1}, \left\| \sum_{k=1}^N g(n_k x) \right\|^{p-1} \right) \cdot \left| \|S_N\| - \left\| \sum_{k=1}^N g(n_k x) \right\| \right| \\ & \ll p (N(\log \log N)^2)^{\frac{p-1}{2}} N^{-\frac{p}{2}+1} \\ (21) \quad & \ll p(\log \log N)^{p-1} N^{1/2} \end{aligned}$$

since by a result of Gál [10] and Koksma [16],

$$(22) \quad \|S_N\|^2 \ll N(\log \log N)^2 \quad \text{and} \quad \left\| \sum_{k=1}^N g(n_k x) \right\|^2 \ll N(\log \log N)^2,$$

where the implied constants are absolute.

By expanding and using elementary properties of the trigonometric functions we get

$$\begin{aligned} & \mathbb{E} \left( \sum_{k=1}^N g(n_k x) \right)^p \\ (23) \quad & = 2^{-p} \sum_{1 \leq j_1, \dots, j_p \leq N^p} a_{j_1} \cdots a_{j_p} \sum_{1 \leq k_1, \dots, k_p \leq N} \mathbb{I}\{\pm j_1 n_{k_1} \pm \dots \pm j_p n_{k_p} = 0\}, \end{aligned}$$

with all possibilities of the signs  $\pm$  within the indicator function. Assume that  $j_1, \dots, j_p$  and the signs  $\pm$  are fixed, and consider a solution of  $\pm j_1 n_{k_1} \pm \dots \pm j_p n_{k_p} = 0$ . Then the set  $\{1, 2, \dots, p\}$  can be split into disjoint sets  $A_1, \dots, A_l$  such that for each such set  $A$  we have  $\sum_{i \in A} \pm j_i n_{k_i} = 0$  and no further subsums of these sums are equal to 0. Group the terms of  $\sum_{i \in A} \pm j_i n_{k_i}$  with equal  $k_i$ . If after grouping there are at least two terms, then by the restriction on subsums, the sum of the coefficients  $j_i$  in each group will be different from 0 and will not exceed

$$pN^p \leq \omega_{[N^{1/4}]} N^{\frac{1}{8}\omega_{[N^{1/4}]}} \leq 2^{\frac{1}{8}\omega_{[N^{1/4}]}} N^{\frac{1}{8}\omega_{[N^{1/4}]}} \leq N^{\frac{1}{4}\omega_{[N^{1/4}]}} \quad (N \geq N_0).$$

Also the number of terms after grouping will be at most  $p \leq \omega_{[N^{1/4}]}$  and thus applying condition  $\mathbf{A}_\omega$  with the index  $[N^{1/4}]$  shows that within a block  $A$  the  $n_{k_i}$  belong to the smallest  $[N^{1/4}]$  terms of the sequence. Thus letting  $|A| = m$ , the number of solutions of  $\sum_{i \in A} \pm j_i n_{k_i} = 0$  is at most  $N^{m/4}$ . If after grouping there is only one term, then all the  $k_i$  are equal and thus the number of solutions of  $\sum_{i \in A} \pm j_i n_{k_i} = 0$  is at most  $N$ . Thus if  $m \geq 3$ , then the number of solutions of  $\sum_{i \in A} \pm j_i n_{k_i} = 0$  in a block is at most  $N^{m/3}$ . If  $m = 2$ , then the number of solutions is clearly at most  $N$ . Thus if  $s_i = |A_i|$  ( $1 \leq i \leq l$ ) denotes the cardinality of  $A_i$ , the number of solutions of  $\pm j_1 n_{k_1} \pm \dots \pm j_p n_{k_p} = 0$  admitting such a decomposition with fixed  $A_1, \dots, A_l$  is at most

$$\begin{aligned} & \prod_{\{i:s_i \geq 3\}} N^{s_i/3} \prod_{\{i:s_i=2\}} N = N^{\frac{1}{3} \sum_{\{i:s_i \geq 3\}} s_i + \sum_{\{i:s_i=2\}} 1} \\ & = N^{\frac{1}{3} \sum_{\{i:s_i \geq 3\}} s_i + \frac{1}{2} \sum_{\{i:s_i=2\}} s_i} = N^{\frac{1}{3} \sum_{\{i:s_i \geq 3\}} s_i + \frac{1}{2}(p - \sum_{\{i:s_i \geq 3\}} s_i)} \\ & = N^{\frac{p}{2} - \frac{1}{6} \sum_{\{i:s_i \geq 3\}} s_i}. \end{aligned}$$

If there is at least one  $i$  with  $s_i \geq 3$ , then the last exponent is at most  $(p - 1)/2$  and since the number of partitions of the set  $\{1, \dots, p\}$  into disjoint subsets is at most  $p! 2^p$ , we see that the number of solutions of  $\pm j_1 n_{k_1} \pm \dots \pm j_p n_{k_p} = 0$ , where at least one of the sets  $A_i$  has cardinality  $\geq 3$ , is at most  $p! 2^p N^{(p-1)/2}$ . If  $p$  is odd, there are no other solutions and thus using (18) the inner sum in (23) is at most  $p! 2^p N^{(p-1)/2}$  and consequently, taking into account the  $2^p$  choices for the signs  $\pm 1$ ,

$$\begin{aligned} & \left| \mathbb{E} \left( \sum_{k \leq N} g(n_k x) \right)^p \right| \\ & \leq p! 2^p N^{(p-1)/2} \sum_{1 \leq j_1, \dots, j_p \leq N^p} |a_{j_1} \cdots a_{j_p}| \ll \exp(p^2) N^{(p-1)/2} (\log N)^p. \end{aligned}$$



If  $p$  is even, there are also solutions where each  $A$  has cardinality 2. Clearly, the contribution of the terms in (23), where  $A_1 = \{1, 2\}, A_2 = \{3, 4\}, \dots$ , is

$$\begin{aligned} & \left( \frac{1}{4} \sum_{1 \leq i, j \leq N^{2p}} \sum_{1 \leq k, \ell \leq N} a_i a_j \mathbb{I}\{\pm i n_k \pm j n_\ell = 0\} \right)^{p/2} = \left( \mathbb{E} \left( \sum_{k \leq N} g(n_k x) \right)^2 \right)^{p/2} \\ & = \left\| \sum_{k \leq N} g(n_k x) \right\|^p \\ & = \|S_N\|^p + \mathcal{O} \left( p(\log \log N)^{p-1} N^{1/2} \right) \end{aligned}$$

by (21).

Since the splitting of  $\{1, 2, \dots, p\}$  into pairs can be done in  $\frac{p!}{(p/2)!} 2^{-p/2}$  different ways, we proved that

$$(24) \quad \mathbb{E} \left( \sum_{k \leq N} g(n_k x) \right)^p = \begin{cases} \frac{p!}{(p/2)!} 2^{-\frac{p}{2}} \sigma_N^p + O(T_N), \\ O(T_N) \end{cases}$$

according as  $p$  is even or odd; here

$$T_N = \exp(p^2) N^{(p-1)/2} (\log N)^p.$$

Now, letting  $G_N = \sum_{k \leq N} g(n_k x)$  we get, using (20), (22) and (24),

$$\begin{aligned} & |\mathbb{E}S_N^p - \mathbb{E}G_N^p| \\ & \leq p \max(\|S_N\|_p^{p-1}, \|G_N\|_p^{p-1}) \cdot \|\|S_N\|_p - \|G_N\|_p\| \\ & \ll p \left( \frac{p!}{(p/2)!} 2^{-\frac{p}{2}} \sigma_N^p \right)^{\frac{p-1}{p}} \\ & \ll T_N, \end{aligned}$$

where in the last step we used the fact that  $\sigma_N \ll \sqrt{N} \log \log N$  by (22). This completes the proof of Lemma 1.  $\square$

**Lemma 2.** *Let  $\omega_k \uparrow \infty$  and let  $(n_k)_{k \geq 1}$  be a sequence of different positive integers satisfying condition  $\mathbf{A}_\omega$ . Then for any  $f$  satisfying (1) we have*

$$(25) \quad \int_0^1 \left( \sum_{k=1}^N f(n_k x) \right)^2 dx \sim \|f\|^2 N \quad \text{as } N \rightarrow \infty.$$

*Proof.* Clearly,  $\omega_{[N^{1/4}]} \geq 4$  for sufficiently large  $N$  and thus applying Condition  $\mathbf{A}_\omega$  for the index  $[N^{1/4}]$  it follows that for  $N \geq N_0$  the Diophantine equation

$$(26) \quad j_1 n_{i_1} + j_2 n_{i_2} = 0, \quad i_1 \neq i_2, \quad 0 < |j_1|, |j_2| \leq N$$

has only such solutions, where  $n_{i_1}, n_{i_2}$  belong to the set  $J_N$  of  $[N^{1/4}]$  smallest elements of the sequence  $(n_k)_{k \geq 1}$ . Write  $p_N(x)$  for the  $N$ -th partial sum of the Fourier series of  $f$ , and  $r_N$  for the  $N$ -th remainder term. Then we have for any  $f$

satisfying (1),

(27)

$$\left\| \sum_{k=1}^N f(n_k x) \right\| \geq \left\| \sum_{k \in [1, N] \setminus J_N} p_N(n_k x) \right\| - \left\| \sum_{k \in J_N} f(n_k x) \right\| - \left\| \sum_{k \in [1, N] \setminus J_N} r_N(n_k x) \right\|.$$

Using the previous remark on the number of solutions of (26) we get, as in (23),

$$\left\| \sum_{k \in [1, N] \setminus J_N} p_N(n_k x) \right\| = (N - [N^{1/4}])^{1/2} \|p_N\| \sim N^{1/2} \|f\|,$$

since  $\|p_N\| \rightarrow \|f\|$ . Further,  $\|r_N\| \ll N^{-1/2}$  by (18) and thus using Minkowski's inequality and the results of Gál and Koksma mentioned in (22), we get

$$\left\| \sum_{k \in J_N} f(n_k x) \right\| \ll N^{1/4}, \quad \left\| \sum_{k \in [1, N] \setminus J_N} r_N(n_k x) \right\| \ll \sqrt{N} \log \log N \|r_N\| \ll \log \log N.$$

These estimates, together with (27), prove Lemma 2.

Lemma 1 and Lemma 2 imply that for any fixed  $p \geq 2$ , the  $p$ -th moment of  $S_N/\sigma_N$  converges to  $\frac{p!}{(p/2)!} 2^{-p/2}$  if  $p$  is even and to 0 if  $p$  is odd; in other words, the moments of  $S_N/\sigma_N$  converge to the moments of the standard normal distribution. By  $\sigma_N \sim \|f\| \sqrt{N}$  and a well-known result in probability theory, this proves the CLT part of Theorem 2. The proof of the LIL part of Theorem 2 is more involved, and we will give just a sketch of the proof. The details can be modeled after the proof of [2, Theorem 1]. The crucial ingredient is Lemma 3 below, which yields the LIL part of Theorem 2, just as [2, Theorem 1] follows from [2, Lemma 3].

Let  $\theta > 1$  and define  $\Delta'_M = \{k \in \mathbb{N} : \theta^M < k \leq \theta^{M+1}\}$  and  $T'_M = \sum_{k \in \Delta'_M} f(n_k x)$ . By the standard method of proof of the LIL, we need precise bounds for the tails of  $T'_M$  and also, a near independence relation for the  $T'_M$  for the application of the Borel-Cantelli lemma in the lower half of the LIL. From the set  $\Delta'_M$  we remove its  $[\theta^{M/4}]$  elements with the smallest value of  $n_k$  (recall that the sequence  $(n_k)_{k \geq 1}$  is not assumed to be increasing) and denote the remaining set by  $\Delta_M$ . Since the number of removed elements is  $\ll |\Delta'_M|^{1/4}$ , this operation does not influence the partial sum asymptotics of  $T'_M$ . As in the proof of Lemma 1, we assume that we have a representation of  $f$  in the form (17) and that (18) holds. Define

$$g_M(x) = \sum_{j=1}^{[\theta^M]^2} a_j \cos 2\pi j x, \quad \sigma_M^2 = \int_0^1 \left( \sum_{k \in \Delta_M} g_M(n_k x) \right)^2 dx$$

and

$$T_M = \sum_{k \in \Delta_M} g_M(n_k x), \quad Z_M = T_M/\sigma_M.$$

From Lemma 2 it follows easily that

$$(28) \quad \sigma_M \gg |\Delta_M|^{1/2}.$$

Assume that  $(n_k)_{k \geq 1}$  satisfies Condition  $\mathbf{A}_\omega$  for a sequence  $(\omega_k)_{k \geq 1}$  with  $\omega_k \geq (\log k)^\alpha$  for some  $\alpha > 0$ ,  $k \geq k_0$ . Without loss of generality we may assume

$0 < \alpha < 1/2$ . Choose  $\delta > 0$  so small that for sufficiently large  $r$ ,

$$(29) \quad \left(\log \theta^{\sqrt{r}/4}\right)^\alpha > 4(\log \theta^r)^\delta. \quad \square$$

**Lemma 3.** For sufficiently large  $M, N$  satisfying  $N^{1-\alpha/2} \leq M \leq N$ , and for positive integers  $p, q$  satisfying  $p + q \leq (\log \theta^N)^\delta$  we have

$$\mathbb{E}Z_M^p Z_N^q = \begin{cases} \frac{p!}{(p/2)!2^{p/2}} \frac{q!}{(q/2)!2^{q/2}} + \mathcal{O}(R_{M,N}) & \text{if } p, q \text{ are even,} \\ \mathcal{O}(R_{M,N}) & \text{otherwise,} \end{cases}$$

where

$$R_{M,N} = 2^{p+q}(p+q)! (\log M)^{p+q} |\Delta_M|^{-1/2}.$$

*Proof.* Note that  $N \leq M^{1+3\alpha/4}$  and thus, setting  $L = [\theta^{M/4}]$ , relation  $p + q \leq (\log \theta^N)^\delta$  and (29) imply

$$p + q \leq \frac{1}{4}(\log \theta^{\sqrt{N}/4})^\alpha \leq \frac{1}{4}(\log \theta^{M/4})^\alpha \leq \omega_L$$

and by a simple calculation,

$$(p+q)(\theta^N)^2 \leq [\theta^{M/4}]^{(\log[\theta^{M/4}])^\alpha} \leq L^{\omega_L}$$

provided  $N$  is large enough. Applying condition  $\mathbf{A}_\omega$  we get that for all solutions of the equation

$$(30) \quad \pm j_1 n_{k_1} \pm \dots \pm j_p n_{k_p} \pm j_{p+1} n_{k_{p+1}} \pm \dots \pm j_{p+q} n_{k_{p+q}} = 0$$

with different indices  $k_1, \dots, k_{p+q}$ , where

$$(31) \quad 1 \leq j_i \leq (p+q)(\theta^N)^2,$$

the  $n_{k_j}$  belong to the  $[\theta^{M/4}]$  smallest elements of  $(n_k)_{k \geq 1}$ . By construction not a single one of these elements is contained in  $\Delta_M$  or  $\Delta_N$ . Thus the equation (30) subject to (31) has no solution  $(k_1, \dots, k_{p+q})$ , where  $k_1, \dots, k_{p+q}$  are different and satisfy

$$k_1, \dots, k_p \in \Delta_M, \quad k_{p+1}, \dots, k_{p+q} \in \Delta_N.$$

Now

$$(32) \quad \mathbb{E}Z_M^p Z_N^q = \frac{2^{-p-q}}{\sigma_M^p \sigma_N^q} \sum_{\substack{1 \leq j_1, \dots, j_p \leq [\theta^{M/4}]^2, \\ 1 \leq j_{p+1}, \dots, j_{p+q} \leq [\theta^{N/4}]^2}} \sum_{\substack{k_1, \dots, k_p \in \Delta_M, \\ k_{p+1}, \dots, k_{p+q} \in \Delta_N}} a_{j_1} \dots a_{j_{p+q}} \mathbf{1}\{\pm j_1 n_{k_1} \pm \dots \pm j_{p+q} n_{k_{p+q}} = 0\}.$$

If for some  $k_1, \dots, k_{p+q}$  we have (note that in (32) these indices need not be different)

$$(33) \quad \pm j_1 n_{k_1} \pm \dots \pm j_{p+q} n_{k_{p+q}} = 0,$$

then grouping the terms of the equation according to identical indices, we get a new equation of the form

$$j'_1 n_{l_1} + \dots + j'_s n_{l_s} = 0, \quad l_1 < \dots < l_s, \quad s \leq p+q, \quad j'_i \leq (p+q)(\theta^N)^2$$

and using the above observation, all the coefficients  $j'_1, \dots, j'_s$  must be equal to 0. In other words, in any solution of (33) the terms can be divided into groups such that in each group the  $n_{k_j}$  are equal and the sum of the coefficients is 0. Consider

first the solutions where all groups have cardinality 2. This can happen only if both  $p$  and  $q$  are even, and similarly to the proof of Lemma 1, the contribution of such solutions in (32) is

$$\frac{p!}{(p/2)!2^{p/2}} \frac{q!}{(q/2)!2^{q/2}}.$$

Consider now the solutions of (33), where at least one group has cardinality  $\geq 3$ . Clearly the sets  $\{k_1, \dots, k_p\}$  and  $\{k_{p+1}, \dots, k_{p+q}\}$  are disjoint; let us denote the number of groups within these two sets by  $R$  and  $S$ , respectively. Evidently  $R \leq p/2$ ,  $S \leq q/2$ , and at least one of the inequalities is strict. Fixing  $j_1, \dots, j_{p+q}$  and the number of groups, the number of such solutions cannot exceed

$$|\Delta_M|^R |\Delta_N|^S \leq |\Delta_M|^{p/2} |\Delta_N|^{q/2} |\Delta_M|^{-1/2} \ll \sigma_M^p \sigma_N^q |\Delta_M|^{-1/2},$$

where we used (28) and the fact that  $|\Delta_M| \leq |\Delta_N|$ . Since the number of partitions of the set  $\{1, 2, \dots, p + q\}$  into disjoint subsets is at most  $(p + q)!2^{p+q}$  and since the number of choices for the signs  $\pm$  in (33) is at most  $2^{p+q}$ , we see, after summing over all possible values of  $j_1, \dots, j_{p+q}$ , that the contribution of the solutions containing at least one group with cardinality  $\geq 3$  in (32) is at most  $2^{p+q}(p + q)!|\Delta_M|^{-1/2}(\log[\theta^N])^{p+q}$ . This completes the proof of Lemma 3.  $\square$

The rest of the proof of the LIL part of Theorem 2 can be modeled following the lines of Lemma 4, Lemma 5, Lemma 6 and the proof of Theorem 1 in [2].

**2.3. Proof of Theorem 3.** Let  $\omega_k \uparrow \infty$  and set  $\eta_k = \frac{1}{2}\omega_k^{1/2}$ ,  $\boldsymbol{\eta} = (\eta_1, \eta_2, \dots)$ . Clearly

$$(34) \quad (2k)^{\eta_k^2 + 2\eta_k} \leq (2k)^{\omega_k/2} \leq k^{-2}|I_k| \quad \text{for } k \geq k_0$$

since, as we noted,  $|I_k| \geq a\omega_k(k - 1)^{\omega_k - 1} \geq (k/2)^{\omega_k - 1}$  for large  $k$ . We choose  $n_k$ ,  $k = 1, 2, \dots$ , independently and uniformly from the integers of the intervals  $I_k$  in (11). We claim that, with probability 1, the sequence  $(n_k)_{k \geq 1}$  is increasing and satisfies condition  $\mathbf{A}_\eta$ . To see this, let  $k \geq 1$  and consider the numbers of the form

$$(35) \quad (a_1 n_{i_1} + \dots + a_s n_{i_s})/d,$$

where  $1 \leq s \leq \eta_k$ ,  $1 \leq i_1, \dots, i_s \leq k - 1$ ,  $a_1, \dots, a_s, d$  are nonzero integers with  $|a_1|, \dots, |a_s|, |d| \leq k^{\eta_k}$ . Since the number of values in (35) is at most  $(2k)^{\eta_k^2 + 2\eta_k}$ , and by (34), the probability that  $n_k$  equals any of these numbers is at most  $k^{-2}$ . Thus by the Borel-Cantelli lemma, with probability 1 for  $k \geq k_1$ ,  $n_k$  will be different from all the numbers in (35) and thus the equation

$$a_1 n_{i_1} + \dots + a_s n_{i_s} + a_{s+1} n_k = 0$$

has no solution with  $1 \leq s \leq \eta_k$ ,  $1 \leq i_1 < \dots < i_s \leq k - 1$ ,  $0 < |a_1|, \dots, |a_{s+1}| \leq k^{\eta_k}$ . By monotonicity, the equation

$$a_1 n_{i_1} + \dots + a_s n_{i_s} = 0$$

has no solutions provided the indices  $i_\nu$  are all different, the maximal index is at least  $k$ , the number of terms is at most  $\eta_k$ , and  $0 < |a_1|, \dots, |a_s| \leq k^{\eta_k}$ . In other words,  $(n_k)$  satisfies condition  $\mathbf{A}_\eta$ . Now using Theorem 2, we get Theorem 3.

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