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# BUSER'S ISOPERIMETRIC INEQUALITIES WITH INTEGRAL NORMS OF RICCI CURVATURE

SEONG-HUN PAENG

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ABSTRACT. We generalize Buser's isoperimetric inequality with integral norms of Ricci curvature.

## 1. INTRODUCTION

The isoperimetric inequality is one of the most important topics in geometry. In Riemannain geometry, several isoperimetric inequalities have been proved under curvature pinching conditions. Buser's isoperimetric inequality is as follows [B], [Ch]: Let $M$ be an $n$-dimensional complete Riemannian manifold and $B(x,r)$ be the $r$-ball in $M$ centered at $x$ for $r > 0$. We denote the $(n-1)$-volume of a hypersurface $\Gamma$ by $A(\Gamma)$ and $\min\{\mathrm{vol}(D_1), \mathrm{vol}(D_2)\}$ by $V(D_1, D_2)$ for disjoint open sets $D_1, D_2$.

**Theorem 1.1.** *If the Ricci curvature satisfies that* $\mathrm{Ric}_M \geq (n-1)\lambda$ *for* $\lambda \leq 0$*, then there exists a positive constant* $c(n,\lambda,r)$ *depending on* $n, \lambda, r$ *such that for any* $x \in M$*, a dividing smooth hypersurface* $\Gamma$ *in* $B(x,r)$ *with* $\bar{\Gamma}$ *embedded in* $\overline{B(x,r)}$ *and* $B(x,r) \setminus \Gamma = D_1 \cup D_2$*, we have*

$$A(\Gamma) \geq c(n,\lambda,r)V(D_1,D_2).$$

Recently, there have been many attempts to replace curvature bounds with integral norms of curvature. In [Ga], Gallot obtained an isoperimetric inequality with integral norms of Ricci curvature. He obtain a lower bound of $A(\partial\Omega)$ with $V(\Omega, M \setminus \Omega)$ and a condition on the integral norm of Ricci curvature. Note that $\Gamma$ does not need to be $\partial\Omega$ in the case of Buser's isoperimetric inequality.

In [PW], Petersen and Wei generalized the Bishop-Gromov volume comparison estimate with integral norms of Ricci curvature. In this paper, we generalize Buser's isoperimetric inequalities with integral norms of Ricci curvature by using the volume comparison in [PW].

First, we define the following notation for the integral norm of Ricci curvature. Let $g(x)$ be the smallest eigenvalue of the Ricci tensor at $x \in M$ and $u_+ = \max(0,u)$

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be the positive part of $u$. For $2p > n$, $\lambda \le 0$, we define $k_x(\lambda, p, R)$ as follows:

$$k_x(\lambda, p, R) = \int_{B(x,R)} ((-g(x) + (n-1)\lambda)_+)^p dv \text{ for } x \in M. \tag{1.1}$$

If $\mathrm{Ric}_M \ge \lambda$, then $k_x(\lambda, p, R) = 0$.

We will prove the following isoperimetric inequalities:

**Theorem 1.2.** *If $B(x,r)$ is a convex $r$-ball and $k_x(\lambda, p, r) \le K$ for $x \in M$, then there exist positive constants $c_1(n, \lambda, r)$ depending on $n, \lambda, r$ and $c_2(n, p, \lambda, r, K)$ depending on $n, p, \lambda, r, K$ such that for a dividing smooth hypersurface $\Gamma$ in $B(x,r)$ with $\bar{\Gamma}$ embedded in $\overline{B(x,r)}$ and $B(x,r) \setminus \Gamma = D_1 \cup D_2$, we have*

$$A(\Gamma) \ge c_1(n, \lambda, r) V(D_1, D_2) - c_2(n, p, \lambda, r, K),$$

*where $c_2(n, p, \lambda, r, K) \to 0$ as $K \to 0$.*

We can obtain $c_1, c_2$ explicitly. In [Ch], Theorem 1.1 is proved for a more general domain, i.e. a star-shaped domain $D$ with $B(x, r/2) \subset D \subset B(x,r)$. We can also prove Theorem 1.2 for a convex domain $D$ with $B(x, r/2) \subset D \subset B(x,r)$.

The boundary rigidity problem is *to what extent a Riemannian metric on a compact manifold with boundary is determined from the distances between boundary points* [C], [S]. Linearizing the boundary rigidity problem, the integral geometry problem is *to what extent is a symmetric tensor field determined by the set of integrals along geodesics connecting boundary points* [S]. In [Pa], an upper bound of the volume entropy and the simplicial volume with integral norms of Ricci curvature over closed geodesics are obtained. It is also interesting to consider Buser's isoperimetric problem with integral norms of curvature over all geodesics connecting boundary points.

For a geodesic segment $\gamma$, define $k_\gamma$ for $p > 1$ as follows:

$$k_\gamma(p) = \int_\gamma \max\{0, -\mathrm{Ric}(\gamma', \gamma')\}^p. \tag{1.2}$$

Let $G(x,r)$ be the set of geodesic segments in $B(x,r)$ connecting two points in $\partial B(x,r)$.

**Theorem 1.3.** *If $B(x,r)$ is a convex $r$-ball and $k_\gamma(p) \le k$ for any $\gamma \in G(x,r)$, then there exists a positive constant $c_3(n,p,k,r)$ depending on $n, p, k, r$ such that for a dividing smooth hypersurface $\Gamma$ in $B(x,r)$ with $\bar{\Gamma}$ embedded in $\overline{B(x,r)}$ and $B(x,r) \setminus \Gamma = D_1 \cup D_2$, we have*

$$A(\Gamma) \ge c_3(n,p,k,r) V(D_1, D_2).$$

Note that in this theorem, there are no additional terms such as $c_2(n, p, \lambda, r, K)$ in Theorem 1.2. Also $c_3$ can be obtained explicitly.

## 2. Proof of Theorem 1.2

We will follow the proof of Buser's isoperimetric inequality in [Ch]. Around $x$, use exponential polar coordinates and write the volume element as $d\mathrm{vol} = \omega dt \wedge d\theta$ inside the cut locus, where $d\theta$ is the standard volume element on the unit sphere $S^{n-1} \cong U_x M$, where $U_x M$ is the unit tangent space of $x$. Sometimes we abbreviate $\omega(t, \theta)$ to $\omega(t)$. Outside the cut locus, $\omega = 0$. Let $\omega_\lambda$ be the $\omega$ for the $n$-dimensional space form $M^n_\lambda$ with constant curvature $\lambda$. Let $V_\lambda(s) = \int_0^s \omega_\lambda(t) dt$. We know that

$\omega' = h\omega$, where $h$ is the mean curvature of the distance spheres around $x$ and $h$, satisfies

$$h' + \frac{h^2}{n-1} \leq -\mathrm{Ric}(\partial_t, \partial_t), \tag{2.1}$$

where $\partial_t$ is the unit gradient of the distance function $d(\cdot, x)$.

The Bishop-Gromov volume comparison theorem implies that $\frac{d}{ds}\frac{\omega(s,\theta)}{\omega_\lambda(s)} \leq 0$ if $\mathrm{Ric}_M \geq (n-1)\lambda$. Then we have

$$\frac{\omega(l,\theta)}{\omega_\lambda(l)} \geq \frac{1}{V_\lambda(s) - V_\lambda(l)} \int_l^s \omega(t,\theta)dt \tag{2.2}$$

for $l \leq s$ and

$$\frac{1}{V_\lambda(r_1) - V_\lambda(r_0)} \int_{r_0}^{r_1} \omega(s,\theta)ds \geq \frac{1}{V_\lambda(r_2) - V_\lambda(r_1)} \int_{r_1}^{r_2} \omega(s,\theta)ds \tag{2.3}$$

for $r_0 \leq r_1 \leq r_2 \leq r$.

With integral norms of Ricci curvature, we obtain the following comparisons. Let $\omega'_\lambda = h_\lambda \omega$ and define $\psi = \psi(s,\cdot) = \max\{0, h(s,\cdot) - h_\lambda(s,\cdot)\}$. Then we have

$$\frac{d}{ds}\frac{\omega}{\omega_\lambda} \leq \psi \frac{\omega}{\omega_\lambda}.$$

Integrating the above, for $r_1 \leq s$, we obtain that

$$\frac{\omega(s)}{\omega_\lambda(s)} - \frac{\omega(r_1)}{\omega_\lambda(r_1)} \leq \int_{r_1}^s \psi \frac{\omega}{\omega_\lambda} \leq \frac{1}{\omega_\lambda(r_1)} \int_{r_1}^s \psi\omega. \tag{2.4}$$

Then we have for $r_2 > r_1$,

$$\begin{aligned} \int_{r_1}^{r_2} \omega(s)ds &\leq \Big(\frac{\omega(r_1)}{\omega_\lambda(r_1)} + \frac{1}{\omega_\lambda(r_1)} \int_{r_1}^{r_2} \psi\omega\Big) \int_{r_1}^{r_2} \omega_\lambda(s)ds \\ &\leq \frac{V_\lambda(r_2) - V_\lambda(r_1)}{\omega_\lambda(r_1)} (\omega(r_1) + \int_{r_1}^{r_2} \psi\omega), \end{aligned} \tag{2.5}$$

so

$$\frac{\int_{r_1}^{r_2} \omega(s)ds}{V_\lambda(r_2) - V_\lambda(r_1)} \leq \frac{\omega(r_1)}{\omega_\lambda(r_1)} + \frac{1}{\omega_\lambda(r_1)} \int_{r_1}^{r_2} \psi\omega. \tag{2.6}$$

For $r_0 \leq l \leq r_1 \leq r_2 \leq r$,

$$\begin{aligned} \frac{\int_{r_1}^{r_2} \omega(s)ds}{V_\lambda(r_2) - V_\lambda(r_1)} &\leq \frac{\omega(r_1)}{\omega_\lambda(r_1)} + \frac{1}{\omega_\lambda(r_1)} \int_{r_1}^{r_2} \psi\omega \\ &\leq \frac{\omega(l)}{\omega_\lambda(l)} + \frac{1}{\omega_\lambda(l)} \int_l^{r_1} \psi\omega + \frac{1}{\omega_\lambda(r_1)} \int_{r_1}^{r_2} \psi\omega \\ &\leq \frac{\omega(l)}{\omega_\lambda(l)} + \frac{2}{\omega_\lambda(l)} \int_0^r \psi\omega. \end{aligned} \tag{2.7}$$

Then we have

$$\frac{\int_{r_1}^{r_2} \omega(s)ds}{V_\lambda(r_2) - V_\lambda(r_1)} \leq \frac{\int_{r_0}^{r_1} \omega(s)ds}{V_\lambda(r_1) - V_\lambda(r_0)} + \frac{2r}{V_\lambda(r_1) - V_\lambda(r_0)} \int_0^r \psi\omega. \tag{2.8}$$

From now on, we follow the proof of Theorem 1.1 in [Ch] with (2.6) and (2.8) instead of (2.2) and (2.3).

Fix $t \in (0, r/2)$. We set $B_s = B(x,s)$, $V_s = \mathrm{vol}(B(x,s))$. We may assume that $\mathrm{vol}(D_1 \cap B_{r/2}) \le V_{r/2}/2$. Fix any $\alpha \in (0,1)$, $t < r/2$. We consider two cases. The first is $\mathrm{vol}(D_1 \cap B_{r/2}) \le \alpha \mathrm{vol}(D_1)$ and the second is $\mathrm{vol}(D_1 \cap B_{r/2}) \ge \alpha \mathrm{vol}(D_1)$.

*Case 1.* Let $C(x)$ be the cut locus of $x$. For $p \in D_1 \setminus C(x)$, let $p^*$ be the first point on the geodesic segment $px$ from $p$ to $x$ where $px$ intersects $\Gamma$. If $px \subset D_1$, then $p^* = x$. We define the subsets $A_1, A_2, A_3$ of $D_1$ as follows:

$$\begin{aligned} A_1 &= \{p \in D_1 \setminus (C(x) \cup \overline{B_{r/2}}) \mid p^* \notin \overline{B_t}\}, \\ A_2 &= \{p \in D_1 \setminus (C(x) \cup \overline{B_{r/2}}) \mid p^* \in \overline{B_t}\}, \\ A_3 &= (B_{r/2} \setminus B_t) \cap \bigcup_{p \in A_2} \{\exp_x \tau\theta \mid t \le \tau \le s, \text{ where } p = \exp_x s\theta\}. \end{aligned} \tag{2.9}$$

It will be helpful to refer to Figure 6.1 in [Ch].

We define $\rho = \rho(t,\cdot) = \max\{0, (n-1)\lambda - \mathrm{Ric}(\partial_t, \partial_t)\}$. From Lemma 2.2 in [PW], for a subset $S$ of $U_xM$, we have

$$\begin{aligned} \int_S \int_0^r \psi\omega &\le \Big(\int_{S^{n-1}} \int_0^r \psi^{2p}\omega\Big)^{\frac{1}{2p}} \Big(\int_{S^{n-1}} \int_0^r \omega\Big)^{1-\frac{1}{2p}} \\ &\le c_1(n,p)\mathrm{vol}(B(x,r))^{1-\frac{1}{2p}} \Big(\int_{S^{n-1}} \int_0^r \rho^p\omega\Big)^{\frac{1}{2p}} \\ &\le c_1(n,p)\mathrm{vol}(B(x,r))^{1-\frac{1}{2p}} k_x(\lambda,p,r)^{\frac{1}{2p}} \end{aligned} \tag{2.10}$$

for some constant $c_1(n,p) > 0$. By (2.8) and (2.10), we obtain that

$$\begin{aligned} \mathrm{vol}(A_2) \le {} & \frac{V_\lambda(r) - V_\lambda(\frac{r}{2})}{V_\lambda(\frac{r}{2}) - V_\lambda(t)} \mathrm{vol}(A_3) \\ & + \frac{2rc_1(n,p)\mathrm{vol}(B(x,r))^{1-\frac{1}{2p}} K^{\frac{1}{2p}} (V_\lambda(r) - V_\lambda(\frac{r}{2}))}{V_\lambda(\frac{r}{2}) - V_\lambda(t)}. \end{aligned} \tag{2.11}$$

By [PW], we have $\mathrm{vol}(B(x,r)) \le v(n,p,\lambda,r,K)$ for some constant $v(n,p,\lambda,r,K)$ depending on $n,p,\lambda,r,K$. Let

$$\begin{aligned} \gamma_1 &= \frac{V_\lambda(r) - V_\lambda(\frac{r}{2})}{V_\lambda(\frac{r}{2}) - V_\lambda(t)}, \\ \gamma_2 &= \frac{2c_1(n,p)v(n,p,\lambda,r,K)^{1-\frac{1}{2p}} K^{\frac{1}{2p}} (V_\lambda(r) - V_\lambda(\frac{r}{2}))}{V_\lambda(\frac{r}{2}) - V_\lambda(t)} \\ &= 2c_1(n,p)v(n,p,\lambda,r,K)^{1-\frac{1}{2p}} K^{\frac{1}{2p}} \gamma_1. \end{aligned} \tag{2.12}$$

Then we obtain that

$$\mathrm{vol}(A_2) \le \gamma_1 \mathrm{vol}(A_3) + r\gamma_2.$$

Since $(1-\alpha)\mathrm{vol}(D_1) \le \mathrm{vol}(D_1 \setminus B_{r/2}) = \mathrm{vol}(A_1) + \mathrm{vol}(A_2)$ and $\mathrm{vol}(A_3) \le \mathrm{vol}(D_1 \cap B_{r/2}) \le \alpha \mathrm{vol}(D_1)$, we have

$$(1 - \alpha(1+\gamma_1))\mathrm{vol}(D_1) \le \mathrm{vol}(A_1) + r\gamma_2. \tag{2.13}$$

Now we only need to compute an upper bound of $\mathrm{vol}(A_1)$. Let $\{\exp_x(t\theta)\} \cap A_1 = \bigcup_{j_\theta} \exp_x[\alpha'_{j_\theta}, \beta_{j_\theta}]$. We set $\alpha_{j_\theta} = \alpha'_{j_\theta}$ if $\alpha'_{j_\theta} > r/2$ and $\alpha_{j_\theta} = |\exp_x^{-1}(\exp_x(r/2)\theta)^*|$ if $\alpha'_{j_\theta} = \frac{r}{2}$, i.e. $\exp_x \alpha_{j_\theta} \in \Gamma$.

With (2.6), for $t < r/2 \le \alpha_{j_\theta} \le s \le \beta_{j_\theta} \le r$, we obtain that

$$\frac{\int_{\alpha_{j_\theta}}^{\beta_{j_\theta}} \omega(s)ds}{V_\lambda(\beta_{j_\theta}) - V_\lambda(\alpha_{j_\theta})} \le \frac{\omega(\alpha_{j_\theta})}{\omega_\lambda(\alpha_{j_\theta})} + \frac{1}{\omega_\lambda(\alpha_{j_\theta})} \int_{\alpha_{j_\theta}}^{\beta_{j_\theta}} \psi\omega. \tag{2.14}$$

Let $\nu$ be the projection to $U_xM$ such that $\nu(\exp_x s\theta) = \theta$ and let $S$ be the subset $\nu(A_1)$ of $U_xM$. By (2.14),

$$\begin{aligned}\int_S \sum_{j_\theta} \int_{\alpha_{j_\theta}}^{\beta_{j_\theta}} \omega(s)dsd\theta \le \frac{V_\lambda(r) - V_\lambda(t)}{\omega_\lambda(t)} \Big( \int_S \sum_{j_\theta} \omega(\alpha_{j_\theta})d\theta \\ + \int_S \sum_{j_\theta} \int_{\alpha_{j_\theta}}^{\beta_{j_\theta}} \psi\omega d\theta \Big).\end{aligned} \tag{2.15}$$

We have

$$\begin{aligned}\int_S \sum_{j_\theta} \int_{\alpha_{j_\theta}}^{\beta_{j_\theta}} \psi\omega d\theta &\le \Big( \int_S \sum_{j_\theta} \int_{\alpha_{j_\theta}}^{\beta_{j_\theta}} \psi^{2p}\omega d\theta dt \Big)^{\frac{1}{2p}} \Big( \int_S \sum_{j_\theta} \int_{\alpha_{j_\theta}}^{\beta_{j_\theta}} \omega d\theta dt \Big)^{1-\frac{1}{2p}} \\ &\le \Big( \int_S \int_0^r \psi^{2p}\omega d\theta dt \Big)^{\frac{1}{2p}} \Big( \int_S \int_0^r \omega d\theta dt \Big)^{1-\frac{1}{2p}}.\end{aligned} \tag{2.16}$$

From the definition of $p^*$, we have that

$$\Big( \int_S \int_0^r \omega d\theta dt \Big)^{1-\frac{1}{2p}} \le \mathrm{vol}(B(x,r))^{1-\frac{1}{2p}} \le v(n,p,\lambda,r,K)^{1-\frac{1}{2p}}. \tag{2.17}$$

As in (2.10), we have

$$\int_S \sum_{j_\theta} \int_{\alpha_{j_\theta}}^{\beta_{j_\theta}} \psi\omega d\theta \le c_1(n,p) K^{\frac{1}{2p}} v(n,p,\lambda,r,K)^{1-\frac{1}{2p}}. \tag{2.18}$$

Then we obtain that

$$\begin{aligned}\mathrm{vol}(A_1) &= \int_S \sum_{j_\theta} \int_{\alpha_{j_\theta}}^{\beta_{j_\theta}} \omega(s)dsd\theta \\ &\le \frac{V_\lambda(r) - V_\lambda(t)}{\omega_\lambda(t)} \Big( \int_S \sum_{j_\theta} \omega(\alpha_{j_\theta})d\theta + c_1(n,p) K^{\frac{1}{2p}} v(n,p,\lambda,r,K)^{1-\frac{1}{2p}} \Big) \\ &\le \frac{V_\lambda(r) - V_\lambda(t)}{\omega_\lambda(t)} \Big( A(\Gamma) + c_1(n,p) K^{\frac{1}{2p}} v(n,p,\lambda,r,K)^{1-\frac{1}{2p}} \Big).\end{aligned} \tag{2.19}$$

So we obtain that

$$\begin{aligned}A(\Gamma) &\ge \frac{\omega_\lambda(t)}{V_\lambda(r) - V_\lambda(t)} \mathrm{vol}(A_1) - c_1(n,p) K^{\frac{1}{2p}} v(n,p,\lambda,r,K)^{1-\frac{1}{2p}} \\ &\ge \frac{\omega_\lambda(t)}{V_\lambda(r) - V_\lambda(t)} \mathrm{vol}(A_1) - c_1(n,p) K^{\frac{1}{2p}} v(n,p,\lambda,r,K)^{1-\frac{1}{2p}}.\end{aligned} \tag{2.20}$$

From the definition (2.12), we have

$$c_1(n,p) K^{\frac{1}{2p}} v(n,p,\lambda,r,K)^{1-\frac{1}{2p}} = \frac{\gamma_2}{2\gamma_1}. \tag{2.21}$$

Since $\mathrm{vol}(A_1) \geq (1-\alpha(1+\gamma_1))\mathrm{vol}(D_1) - r\gamma_2$ and $v(n,p,\lambda,r,K) \geq \mathrm{vol}(B(x,r))$, we obtain from (2.21) that

$$\begin{aligned} A(\Gamma) &\geq (1-\alpha(1+\gamma_1))\frac{\omega_\lambda(t)}{V_\lambda(r)-V_\lambda(t)}\mathrm{vol}(D_1) - \frac{r\gamma_2\omega_\lambda(t)}{V_\lambda(r)-V_\lambda(t)} \\ &\quad - c_1(n,p)K^{\frac{1}{2p}}v(n,p,\lambda,r,K)^{1-\frac{1}{2p}} \\ &\geq (1-\alpha(1+\gamma_1))\frac{\omega_\lambda(t)}{V_\lambda(r)-V_\lambda(t)}\mathrm{vol}(D_1) - \frac{r\gamma_2\omega_\lambda(t)}{V_\lambda(r)-V_\lambda(t)} - \frac{\gamma_2}{2\gamma_1}. \end{aligned} \tag{2.22}$$

*Case 2.* In Case 2, we consider another exponential map $\exp_{w_0}$ centered at $w_0$ instead of $\exp_x$ to compute the volume, where $w_0$ is determined by the following lemma:

**Lemma 2.1.** *Set either $W_0 = D_1 \cap B_{r/2}$, $W_1 = D_2 \cap B_{r/2}$ or $W_0 = D_2 \cap B_{r/2}$, $W_1 = D_1 \cap B_{r/2}$. Then for at least one of two choices of the pair $\{W_0, W_1\}$, we have the existence of a point $w_0 \in W_0$ and $\mathcal{W}_1 \subset W_1$ such that*

(i) $\mathrm{vol}(\mathcal{W}_1) \geq \mathrm{vol}(W_1)/2$*;*
(ii) *for $q \in \mathcal{W}_1$, every minimizing directed geodesic segment $qw_0$ from $q$ to $w_0$ intersects $\Gamma$ in a first point $q^*$ such that $d(q,q^*) \leq d(q^*,w_0)$.*

For Case 2, $\omega(t,\theta)$ is the volume element of $\partial B(w_0,t)$ for $\exp_{w_0}$. Let $\nu_1 : B(x,\frac{r}{2}) \setminus C(w_0) \to U_{w_0}M$ be a projection as described before for $x$. To each $\theta \in \nu_1(\mathcal{W}_1 \setminus C(w_0))$, we determine a collection of disjoint intervals $\{(\alpha_{j_\theta}, \beta_{j_\theta})\}$ as follows: For $q = \exp_{w_0} t_0\theta \in \mathcal{W}_1$, $\alpha_q = d(w_0,q^*)$ and $\beta_q = \sup\{t > \alpha_q \mid \exp_{w_0} t\theta \in \mathcal{W}_1,\ \{\exp_{w_0} s\theta \mid s \in (\alpha_q, t)\} \subset B(x,r) \setminus \Gamma\}$. Then $\mathcal{W}_1 \cap \{\exp_{w_0} t\theta\} \subset \bigcup_{j_\theta}\{\exp_{w_0} s\theta \mid s \in (\alpha_{j_\theta}, \beta_{j_\theta})\}$ and

$$\beta_{j_\theta} \leq 2\alpha_{j_\theta} < r. \tag{2.23}$$

Similarly as in (2.19), we have by (2.23),

$$\begin{aligned} \mathrm{vol}(\mathcal{W}_1) &\leq \int_{\nu_1(\mathcal{W}_1\setminus C(w_0))} d\theta \sum_{j_\theta} \int_{\alpha_{j_\theta}}^{\beta_{j_\theta}} \omega(s)ds \\ &\leq \frac{V_\lambda(r)-V_\lambda(\frac{r}{2})}{\omega_\lambda(\frac{r}{2})}\Big(A(\Gamma) + c_1(n,p)K^{\frac{1}{2p}}v(n,p,\lambda,r,K)^{1-\frac{1}{2p}}\Big). \end{aligned} \tag{2.24}$$

By (2.21) and $\alpha\mathrm{vol}(D_1) \leq 2\mathrm{vol}(\mathcal{W}_1)$,

$$\begin{aligned} A(\Gamma) &\geq \frac{\omega_\lambda(\frac{r}{2})}{V_\lambda(r)-V_\lambda(\frac{r}{2})}\mathrm{vol}(\mathcal{W}_1) - c_1(n,p)K^{\frac{1}{2p}}v(n,p,\lambda,r,K)^{1-\frac{1}{2p}} \\ &\geq \frac{\omega_\lambda(\frac{r}{2})}{V_\lambda(r)-V_\lambda(\frac{r}{2})}\mathrm{vol}(\mathcal{W}_1) - \frac{\gamma_2}{2\gamma_1} \\ &\geq \frac{\alpha}{2}\frac{\omega_\lambda(\frac{r}{2})}{V_\lambda(r)-V_\lambda(\frac{r}{2})}\mathrm{vol}(D_1) - \frac{\gamma_2}{2\gamma_1}. \end{aligned} \tag{2.25}$$

Simply if $t = \frac{r}{4}$ and $\alpha = \frac{1}{2(1+\gamma_1)}$ for $t = \frac{r}{4}$, then we obtain that

$$c_1(n,\lambda,r) = \min\Big(\frac{1}{4(1+\gamma_1)}\frac{\omega_\lambda(\frac{r}{2})}{V_\lambda(r)-V_\lambda(\frac{r}{2})}, \frac{1}{2}\frac{\omega_\lambda(\frac{r}{4})}{V_\lambda(r)-V_\lambda(\frac{r}{4})}\Big) \tag{2.26}$$

and

$$c_2(n,p,\lambda,r,K) = \frac{\gamma_2}{2\gamma_1} + \frac{r\gamma_2\omega_\lambda(\frac{r}{4})}{V_\lambda(r) - V_\lambda(\frac{r}{4})}, \tag{2.27}$$

although these coefficients are not optimized. Since $\gamma_2 \to 0$ as $K \to 0$, we obtain that $c_2(n,p,\lambda,r,K) \to 0$ as $K \to 0$. Since $c_1(n,p)$ and $v(n,p,\lambda,r,K)$ are obtained explicitly in [PW], we can obtain $c_1, c_2$ explicitly, too.

## 3. Proof of Theorem 1.3

We denote $\frac{n-1}{t}$ by $h_0(t)$. Then $h_0' + \frac{h_0^2}{n-1} = 0$. Let $\psi(t,\theta) = \max\{0, h(t,\theta) - h_0(t)\}$. Since $\frac{\omega'}{\omega} = h$, integrating this equation, we obtain that

$$\log\Big(\frac{\omega(r_2,\theta)}{\omega(r_1,\theta)}\Big) = \int_{r_1}^{r_2} h dt \le \int_{r_1}^{r_2} h_0 dt + \int_{r_1}^{r_2} \psi dt. \tag{3.1}$$

So we have

$$\begin{aligned} \omega(r_2,\theta) &\le e^{\int_{r_1}^{r_2}\psi dt} e^{\int_{r_1}^{r_2} h_0 dt}\omega(r_1,\theta) \\ &= \Big(\frac{r_2}{r_1}\Big)^{n-1} e^{\int_{r_1}^{r_2}\psi dt}\omega(r_1,\theta). \end{aligned} \tag{3.2}$$

By [Pa], for any $p > 1$, we have

$$\int_0^s \psi^{2p} dt \le (n-1)^p \int_0^s \rho^p dt,$$

so for $0 < r_1 \le r_2 \le r$,

$$\begin{aligned} \int_{r_1}^{r_2} \psi dt \le \int_0^r \psi dt &\le \sqrt{n-1}\, r^{1-\frac{1}{2p}} \Big(\int_0^r \rho^p dt\Big)^{\frac{1}{2p}} \\ &\le \sqrt{n-1}\, r^{1-\frac{1}{2p}} k^{\frac{1}{2p}}. \end{aligned} \tag{3.3}$$

Then for $r_1 < r_2$,

$$\omega(r_2) \le C_1(n,p,k,r)\Big(\frac{r_2}{r_1}\Big)^{n-1}\omega(r_1) \tag{3.4}$$

for some constant $C(n,p,k,r)$. So, for $r_0 \le l \le r_1 < r_2 < r$,

$$\begin{aligned} \int_{r_1}^{r_2} \omega(s) ds &\le C_1(n,p,k,r)(r_2 - r_1)\Big(\frac{r_2}{r_1}\Big)^{n-1}\omega(r_1) \\ &\le C_1(n,p,k,r)^2 (r - r_1)\Big(\frac{r}{r_0}\Big)^{2(n-1)}\omega(l) \end{aligned} \tag{3.5}$$

and

$$\int_{r_1}^{r_2} \omega(s) ds \le C_1(n,p,k,r)^2 \frac{r - r_1}{r_1 - r_0}\Big(\frac{r}{r_0}\Big)^{2(n-1)} \int_{r_0}^{r_1} \omega(s) ds. \tag{3.6}$$

Instead of (2.2), (2.3), we use (3.5), (3.6) and follow the proof of [Ch]. For Case 1, if we denote $C_1(n,p,k,r)^2 \frac{r/2}{r/2-t}(\frac{r}{t})^{2(n-1)}$ by $C_2(n,p,k,r,t)$, we have

$$\frac{\mathrm{vol}(A_2)}{\mathrm{vol}(A_3)} \le C_2(n,p,k,r,t), \tag{3.7}$$

and by (3.5),

$$
\begin{aligned}
\mathrm{vol}(A_1) &= \int_S \sum_{j_\theta} \int_{\alpha_{j_\theta}}^{\beta_{j_\theta}} \omega(s) ds d\theta \\
&\le C_1(n,p,k,r) r \Big(\frac{r}{t}\Big)^{n-1} \int_S \sum_{j_\theta} \omega(\alpha_{j_\theta}) d\theta \qquad (3.8)\\
&\le C_1(n,p,k,r) r \Big)\frac{r}{t}\Big)^{n-1} A(\Gamma).
\end{aligned}
$$

For Case 2, we have $\beta_{j_\theta} \le 2\alpha_{j_\theta}$. With the first inequality in (3.5), we have

$$
\begin{aligned}
\int_{\alpha_{j_\theta}}^{\beta_{j_\theta}} \omega(s) ds &\le C_1(n,p,k,r) r \Big(\frac{\beta_{j_\theta}}{\alpha_{j_\theta}}\Big)^{n-1} \omega(\alpha_{j_\theta}) \qquad (3.9)\\
&\le 2^{n-1} r C_1(n,p,k,r) \omega(\alpha_{j_\theta}).
\end{aligned}
$$

Then

$$
\mathrm{vol}(\mathcal{W}_1) \le 2^{n-1} r C_1(n,p,k,r) A(\Gamma). \qquad (3.10)
$$

Then we can prove Theorem 1.3 by the same arguments as [Ch].

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Department of Mathematics, Konkuk University, 1 Hwayang-dong, Gwangjin-gu, Seoul 143-701, Republic of Korea

*E-mail address*: shpaeng@konkuk.ac.kr