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A GENERALIZED POINCARÉ INEQUALITY FOR A CLASS OF CONSTANT COEFFICIENT DIFFERENTIAL OPERATORS

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Abstract. We study first order differential operators $\mathcal{P} = \mathcal{P}(D)$ with constant coefficients. The main question is under what conditions the following full gradient L^p estimate holds:

$$\|D(f - f_0)\|_{L^p} \leq C\|\mathcal{P}f\|_{L^p}, \text{ for some } f_0 \in \ker \mathcal{P}.$$

We show that the constant rank condition is sufficient. The concept of the Moore-Penrose generalized inverse of a matrix comes into play.

1. Introduction

The aim of this paper is to investigate a class of generalized Poincaré inequalities. We begin by recalling the classical Poincaré Inequality.

Theorem 1.1. *For each $f \in \mathcal{D}'(\mathbb{R}^n)$ such that $\nabla f \in L^p(\mathbb{R}^n)$ and each ball $B = B(x_0, r) \subset \mathbb{R}^n$, there exists a constant f_B such that*

$$\int_B |f - f_B|^p \leq C(n,p) r^p \int_B |\nabla f|^p .$$

We view f_B as an element in $\mathcal{D}'(\mathbb{R}^n)$ with $\nabla f_B = 0$, that is, $f_B \in \ker \nabla$.

Our investigation of ways to generalize this to other differential operators ended up relying on full gradient estimates. This leads to our main question:

Question 1.2. For what homogeneous first order partial differential operators $\mathcal{P} = \sum_{i=1}^n A_i \frac{\partial}{\partial x_i}$ is it true that for every $f \in \mathcal{D}'(\mathbb{R}^n, \mathbb{U})$ such that $\mathcal{P}f \in L^p(\mathbb{R}^n, \mathbb{V})$, there exists $f_0 \in \mathcal{D}'(\mathbb{R}^n, \mathbb{U})$ such that $\mathcal{P}f_0 = 0$ and

$$\sum_{i=1}^n \left\| \frac{\partial}{\partial x_i} (f - f_0) \right\|_p \leq C_p(\mathcal{P}) \|\mathcal{P}f\|_p ? \tag{1.1}$$

Notice that with the change from ∇ to $\mathcal{P}$ we also had to change some other details. First, there is no need for the ball that appears in the classical Poincaré inequality; our methods have been able to achieve global estimates. These are based on Calderón-Zygmund estimates while the classical Poincaré inequality is based on the fundamental theorem of calculus. Our estimates are on the first order partial

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derivatives of f, not f itself. The local L^p estimates of $f - f_0$ will follow from (1.1) by the usual Poincaré inequality. We will confine our investigations to the case where $\mathcal{P}$ has constant coefficients.

In section 2 we review elliptic complexes and provide a previously known result, see [4] for example, that derives a generalized Poincaré inequality using elliptic complexes. In section 3 we review the notion of a generalized inverse of a matrix, and use this to prove a new generalized Poincaré inequality. In section 4 we prove a structure theorem for elliptic complexes that allows us to see the relationship between these two generalized Poincaré inequalities. This paper is based upon results contained in the author's dissertation [6].

2. Elliptic complexes

Let $\mathbb{U}$, $\mathbb{V}$, and $\mathbb{W}$ be finite dimensional inner product spaces, whose inner products are denoted by $\langle\ ,\ \rangle_{\mathbb{U}}$, $\langle\ ,\ \rangle_{\mathbb{V}}$, and $\langle\ ,\ \rangle_{\mathbb{W}}$, respectively, or just $\langle\ ,\ \rangle$ when the space is clear. Let $\mathcal{P}$ and $\mathcal{Q}$ be the first order differential operators with constant coefficients

$$\mathcal{P} = \sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}, \quad \mathcal{Q} = \sum_{i=1}^{n} B_i \frac{\partial}{\partial x_i},$$

where the A_i are linear operators from $\mathbb{U}$ to $\mathbb{V}$ and the B_i are linear operators from $\mathbb{V}$ to $\mathbb{W}$. We will use

$$\mathcal{P}(\xi) = \sum_{i=1}^{n} \xi_i A_i \quad \text{and} \quad \mathcal{Q}(\xi) = \sum_{i=1}^{n} \xi_i B_i$$

to denote the symbols of $\mathcal{P}$ and $\mathcal{Q}$, respectively. We denote by $\mathcal{D}'(\mathbb{R}^n, \mathbb{V})$ the space of distributions valued in $\mathbb{V}$. We define a *short elliptic complex of order* 1 *over* $\mathbb{R}^n$ to be

$$\mathcal{D}'(\mathbb{R}^n, \mathbb{U}) \xrightarrow{\ \mathcal{P}\ } \mathcal{D}'(\mathbb{R}^n, \mathbb{V}) \xrightarrow{\ \mathcal{Q}\ } \mathcal{D}'(\mathbb{R}^n, \mathbb{W})$$

such that the *symbol complex*

$$\mathbb{U} \xrightarrow{\ \mathcal{P}(\xi)\ } \mathbb{V} \xrightarrow{\ \mathcal{Q}(\xi)\ } \mathbb{W}$$

is exact for all $\xi \neq 0 \in \mathbb{R}^n$.

From an elliptic complex, we form the adjoint complex

$$\mathcal{D}'(\mathbb{R}^n, \mathbb{W}) \xrightarrow{\ \mathcal{Q}^*\ } \mathcal{D}'(\mathbb{R}^n, \mathbb{V}) \xrightarrow{\ \mathcal{P}^*\ } \mathcal{D}'(\mathbb{R}^n, \mathbb{U}).$$

Here $\mathcal{P}^*$ is the formal adjoint defined by

$$\int_{\mathbb{R}^n} \langle \mathcal{P}^* f, g \rangle_{\mathbb{U}} = \int_{\mathbb{R}^n} \langle f, \mathcal{P} g \rangle_{\mathbb{V}}$$

for $f \in C_0^\infty(\mathbb{R}^n, \mathbb{V})$ and $g \in C_0^\infty(\mathbb{R}^n, \mathbb{U})$. So, we have

$$\mathcal{P}^* = -\sum_{i=1}^{n} A_i^* \frac{\partial}{\partial x_i}, \tag{2.1}$$

and similarly for $\mathcal{Q}^*$. Here, we have identified $\mathbb{U}^*$, $\mathbb{V}^*$, and $\mathbb{W}^*$ with $\mathbb{U}$, $\mathbb{V}$ and $\mathbb{W}$, respectively, by use of their inner products. Note that the adjoint complex is elliptic if and only if the original complex is.

From this, we define an associated second order *Laplace-Beltrami operator* by

$$\triangle = \triangle_{\mathbb{V}} = -\mathcal{P}\mathcal{P}^* - \mathcal{Q}^*\mathcal{Q} : \mathcal{D}'(\mathbb{R}^n, \mathbb{V}) \to \mathcal{D}'(\mathbb{R}^n, \mathbb{V}),$$

with symbol denoted by $\triangle(\xi) : \mathbb{V} \to \mathbb{V}$. Linear algebra shows that for every $v \in \mathbb{V}$, $\langle -\triangle(\xi)v, v\rangle = |\mathcal{P}^*(\xi)v|^2 + |\mathcal{Q}(\xi)v|^2 \geq 0$. That equality only occurs when $\xi = 0$ follows from the definition of an elliptic complex. Thus, the linear operator $\triangle(\xi) : \mathbb{V} \to \mathbb{V}$ is invertible for $\xi \neq 0$. We also have that as a function in ξ, $\triangle(\xi)$ is homogeneous of degree 2. So, letting

$$c = \max_{|\xi|=1} \left\|\triangle^{-1}(\xi) : \mathbb{V} \to \mathbb{V}\right\|,$$

we get the estimate

$$\left\|\triangle^{-1}(\xi)\right\| \leq c\,|\xi|^{-2}.$$

So, solving the Poisson equation

$$\triangle\varphi = F$$

with $F \in C_0^\infty(\mathbb{R}^n, \mathbb{V})$, we find the second derivatives of φ by noting that

$$\widehat{\frac{\partial^2\varphi}{\partial x_i \partial x_j}}(\xi) = \xi_i\xi_j\triangle^{-1}(\xi)\widehat{F}(\xi).$$

Since $\xi_i\xi_j\triangle^{-1}(\xi) : \mathbb{V} \to \mathbb{V}$ is bounded, this gives rise to a Calderón-Zygmund type singular integral operator, $R_{ij}F = \frac{\partial^2}{\partial x_i \partial x_j}\varphi$ which is bounded on L^p for $1 < p < \infty$. We will refer to these as the second order Riesz type transforms, due to the similarities with the classical Riesz transforms. A detailed discussion of Calderón-Zygmund singular integral operators and, in particular, the classical Riesz transforms, can be found in [8].

We refer the reader to [2], [4], [9], and [10] for further reading on elliptic complexes.

We now present a previously known generalized Poincaré inequality; see [4] for example.

Theorem 2.1. *Let $1 < p < \infty$, and let*

$$\mathcal{D}'(\mathbb{R}^n, \mathbb{X}) \xrightarrow{\mathcal{R}} \mathcal{D}'(\mathbb{R}^n, \mathbb{U}) \xrightarrow{\mathcal{P}} \mathcal{D}'(\mathbb{R}^n, \mathbb{V}) \xrightarrow{\mathcal{Q}} \mathcal{D}'(\mathbb{R}^n, \mathbb{W})$$

be an elliptic complex of order 1, and let $f \in \mathcal{D}'(\mathbb{R}^n, \mathbb{U})$ such that $\mathcal{P}f \in L^p(\mathbb{R}^n, \mathbb{V})$. Then there exists $f_0 \in \mathcal{D}'(\mathbb{R}^n, \mathbb{U}) \cap \ker\mathcal{P}$ with

$$\sum_j \left\|\frac{\partial}{\partial x_j}(f - f_0)\right\|_p \leq C\,\|\mathcal{P}f\|_p.$$

Proof. Here we shall need not only the Laplace-Beltrami operator for functions valued in $\mathbb{V}$, but also the Laplace-Beltrami operator for functions valued in $\mathbb{U}$, $-\triangle_{\mathbb{U}} = \mathcal{R}\mathcal{R}^* + \mathcal{P}^*\mathcal{P}$. There exists $\varphi \in \mathcal{D}'(\mathbb{R}^n, \mathbb{U})$ such that $\triangle_{\mathbb{U}}\varphi = f$. Note that because of the exactness of the elliptic complex, we have the identity

$$\triangle_{\mathbb{V}}\mathcal{P}\varphi = -\mathcal{P}\mathcal{P}^*\mathcal{P}\varphi - \mathcal{Q}^*\mathcal{Q}\mathcal{P}\varphi = -\mathcal{P}\mathcal{P}^*\mathcal{P}\varphi - \mathcal{P}\mathcal{R}\mathcal{R}^*\varphi = \mathcal{P}\triangle_{\mathbb{U}}\varphi = \mathcal{P}f.$$

Let $f_0 = f + \mathcal{P}^*\mathcal{P}\varphi$. Now it simply remains to verify that f_0 satisfies the conclusions of the theorem. First,

$$\mathcal{P}f_0 = \mathcal{P}f + \mathcal{P}\mathcal{P}^*\mathcal{P}\varphi = \mathcal{P}f + \mathcal{P}\mathcal{P}^*\mathcal{P}\varphi + \mathcal{P}\mathcal{R}\mathcal{R}^*\varphi = \mathcal{P}f - \mathcal{P}\triangle_{\mathbb{U}}\varphi = 0.$$

Also,

$$\begin{aligned}\sum_j \left\| \frac{\partial}{\partial x_j}(f-f_0)\right\|_p &= \sum_j \left\| -\frac{\partial}{\partial x_j}\mathcal{P}^*\mathcal{P}\varphi\right\|_p \le \sum_{i,j}\left\| A_i^*\frac{\partial^2}{\partial x_i \partial x_j}\mathcal{P}\varphi\right\|_p \\ &\le \sum_{i,j}\|A_i^* R_{ij}\mathcal{P}f\|_p \le \sum_{i,j}\|A_i^*\|\, C_{i,j}\,\|\mathcal{P}f\|_p \\ &\le C\,\|\mathcal{P}f\|_p .\end{aligned}$$

□

3. Generalized inverses

Before we are able to present the second theorem, we need to look at the theory of generalized inverses.

Proposition 3.1. *For $A \in \mathrm{Hom}(\mathbb{U},\mathbb{V})$, there exists a unique $A^\dagger \in \mathrm{Hom}(\mathbb{V},\mathbb{U})$, called the Moore-Penrose generalized inverse,with the following properties:*

(1) $AA^\dagger A = A : \mathbb{U} \to \mathbb{V}$,
(2) $A^\dagger A A^\dagger = A^\dagger : \mathbb{V} \to \mathbb{U}$,
(3) $(AA^\dagger)^* = AA^\dagger : \mathbb{V} \to \mathbb{V}$,
(4) $(A^\dagger A)^* = A^\dagger A : \mathbb{U} \to \mathbb{U}$.

The linear map $A^\dagger$ has properties similar to inverse matrices that make it valuable as a tool.

Proposition 3.2. *For $\lambda \neq 0$, $(\lambda A)^\dagger = \lambda^{-1}A^\dagger$.*

Proposition 3.3. *For a continuous matrix-valued function $P = P(\xi)$, the function $P^\dagger = P^\dagger(\xi)$ is continuous at ξ if and only if there is a neighborhood of ξ on which P has constant rank.*

Proposition 3.4. *$AA^\dagger$ is the orthogonal projection onto the image of A. $A^\dagger A$ is the orthogonal projection onto the orthogonal complement of the kernel of A.*

For a more detailed discussion of generalized inverses and the proofs of these results, consult [1] and the references cited there. Generalized inverses are the additional tools we need for the following theorem.

Theorem 3.5. *Let $\mathcal{P} : \mathcal{D}'(\mathbb{R}^n,\mathbb{U}) \to \mathcal{D}'(\mathbb{R}^n,\mathbb{V})$ be a differential operator of order 1 with constant coefficients and symbol $\mathcal{P}(\xi)$ which is of constant rank for $\xi \neq 0$, and let $f \in \mathcal{D}'(\mathbb{R}^n,\mathbb{U})$ such that $\mathcal{P}f \in L^p(\mathbb{R}^n,\mathbb{V})$, $1 < p < \infty$. Then there exists $f_0 \in \mathcal{D}'(\mathbb{R}^n,\mathbb{U})$ such that $\mathcal{P}f_0 = 0$ and*

$$\sum_j \left\| \frac{\partial}{\partial x_j}(f-f_0)\right\|_p \le C\,\|\mathcal{P}f\|_p .$$

Note that this is the same constant rank condition investigated in [3] in relation to quasiconvexity of variational integrals.

Proof. From the symbol $\mathcal{P}(\xi) : \mathbb{U} \to \mathbb{V}$, we have its generalized inverse $\mathcal{P}^\dagger(\xi) : \mathbb{V} \to \mathbb{U}$. We use this to define pseudodifferential operators R_j, which we will refer to as the first order Riesz type transforms. For $h \in C_0^\infty(\mathbb{R}^n,\mathbb{V})$, we define $R_j h(x) = (2\pi)^{-n/2}\int i e^{ix\cdot\xi}\xi_j \mathcal{P}^\dagger(i\xi)\widehat{h}(\xi)d\xi$. Note that since $\mathcal{P}(\xi)$ is homogeneous of degree 1, we get that for $\lambda \neq 0$,

$$(\lambda\xi_j)\mathcal{P}^\dagger(i\lambda\xi) = \lambda\xi_j\,(\lambda\mathcal{P}(i\xi))^\dagger = \xi_j\mathcal{P}(i\xi).$$

So, $\xi_j \mathcal{P}^\dagger(i\xi)$ is homogeneous of degree 0. Since $\mathcal{P}(\xi)$ is a polynomial, $\mathcal{P}^\dagger(i\xi)$ is infinitely differentiable on $|\xi| = 1$ by Theorem 4.3 of [5], which gives a formula for the derivative of $\mathcal{P}^\dagger(\xi)$ in terms of $\mathcal{P}^\dagger(\xi)$, $\mathcal{P}(\xi)$, and the derivative of $\mathcal{P}(\xi)$. Thus, R_j extends continuously to a Calderón-Zygmund singular integral operator from $L^p(\mathbb{R}^n, \mathbb{V})$ to $L^p(\mathbb{R}^n, \mathbb{U})$. Recalling the definition of the operator $\mathcal{P} = \sum_j A_j \frac{\partial}{\partial x_j}$, we note that

$$\sum_j A_j R_j h = (2\pi)^{-n/2} \int e^{ix\cdot\xi} \mathcal{P}(i\xi)\mathcal{P}^\dagger(i\xi)\widehat{h}(\xi)d\xi.$$

So, if $h = \mathcal{P}g$, then

$$\begin{aligned} \sum_j A_j R_j h &= (2\pi)^{-n/2} \int e^{ix\cdot\xi} \mathcal{P}(i\xi)\mathcal{P}^\dagger(i\xi)\mathcal{P}(i\xi)\widehat{g}(\xi)d\xi \\ &= (2\pi)^{-n/2} \int e^{ix\cdot\xi} \mathcal{P}(i\xi)\widehat{g}(\xi)d\xi = \mathcal{P}g = h. \end{aligned}$$

Also, since this is defined by a Calderón-Zygmund singular integral operator, the identity

$$\sum_j A_j R_j = Id$$

extends to the L^p closure of the set $\{\mathcal{P}g : g \in C_0^\infty\}$, which is sufficient for the subsequent computations.

The reader may wish to notice that

$$\left(\frac{\partial}{\partial x_j} R_k h\right)^\wedge = (i\xi_j)\left(i\xi_k \mathcal{P}^\dagger(i\xi)\widehat{h}(\xi)\right) = (i\xi_k)\left(i\xi_j \mathcal{P}^\dagger(i\xi)\widehat{h}(\xi)\right) = \left(\frac{\partial}{\partial x_k} R_j h\right)^\wedge,$$

which means that

$$\frac{\partial}{\partial x_j} R_k h = \frac{\partial}{\partial x_k} R_j h.$$

Thus,

$$\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_k} f - R_k \mathcal{P} f\right) = \frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial x_j} f - R_j \mathcal{P} f\right).$$

This can be viewed as saying that for any basis vector v_l of $\mathbb{V}$,

$$\operatorname{curl}\begin{bmatrix} \left\langle \frac{\partial}{\partial x_1} f - R_1 \mathcal{P} f, v_l \right\rangle_{\mathbb{V}} \\ \vdots \\ \left\langle \frac{\partial}{\partial x_n} f - R_n \mathcal{P} f, v_l \right\rangle_{\mathbb{V}} \end{bmatrix} = 0.$$

Thus, assuming that we can find a distribution f_l such that

$$\frac{\partial}{\partial x_j} f_l = \left\langle \frac{\partial}{\partial x_1} f - R_1 \mathcal{P} f, v_l \right\rangle_{\mathbb{V}}, \tag{3.1}$$

we can define f_0 by

$$f_0 = \sum_l f_l v_l,$$

which means that $\frac{\partial}{\partial x_j} f_0 = \frac{\partial}{\partial x_j} f - R_j \mathcal{P} f$, for $j = 1, \ldots, n$. Then, we have

$$\mathcal{P} f_0 = \mathcal{P} f - \sum_j A_j R_j \mathcal{P} f = \mathcal{P} f - \mathcal{P} f = 0.$$

Also,

$$\sum_j \left\| \frac{\partial}{\partial x_j} (f - f_0) \right\|_p = \sum_j \|R_j \mathcal{P} f\|_p \leq C \|\mathcal{P} f\|_p ,$$

where the constant depends on the norms of the Riesz type transforms. Thus, it only remains to prove Lemma 3.6, a distributional Poincaré lemma, which is used in (3.1). □

Lemma 3.6 (Distributional Poincaré Lemma)**.** *Let $\omega \in \mathcal{D}'\left(\mathbb{R}^n, \Lambda^l(\mathbb{R}^n)\right)$ with $1 \leq l \leq n$, where $\Lambda^l(\mathbb{R}^n)$ denotes the space of l-covectors on $\mathbb{R}^n$, such that $d\omega = 0$. Then there exists $\nu \in \mathcal{D}'\left(\mathbb{R}^n, \Lambda^{l-1}(\mathbb{R}^n)\right)$ such that $\omega = d\nu$.*

Note that this was used in the sense that we can view a vector field $F \in \mathcal{D}'(\mathbb{R}^n, \mathbb{R}^n)$ as a differential form in $\mathcal{D}'\left(\mathbb{R}^n, \Lambda^1(\mathbb{R}^n)\right)$ and that the conditions $\operatorname{curl} F = 0$ and $\nabla f = F$ become $dF = 0$ and $df = F$ from this point of view.

Proof. We will use the notation dx_I to denote $dx_{i_1} \wedge \cdots \wedge dx_{i_l}$, where $\{x_{i_1}, \ldots, x_{i_l}\}$ is any collection of l basis elements and $i_1 < i_2 < \cdots < i_l$. Let $\omega = \sum \omega_I dx_I$, where $\omega_I \in \mathcal{D}'(\mathbb{R}^n)$. It is well known that ω_I can be written as $\omega_I = \triangle \eta_I = (d\, d^* + d^*\, d)\, \eta_I$ for $\eta_I \in \mathcal{D}'(\mathbb{R}^n)$. See, for example, Corollary 3.6.2 in [7]. Thus, $\omega = \triangle \eta = (d\, d^* + d^*\, d)\, \eta$, where $\eta = \sum \eta_I dx_I$. Notice that

$$\triangle d\eta = d\triangle \eta = d\omega = 0.$$

Thus, by Weyl's Lemma, we get that $d\eta$ is infinitely differentiable, which gives us that $d^*\, d\eta \in C^\infty\left(\mathbb{R}^n, \Lambda^l(\mathbb{R}^n)\right)$. Also, we have that

$$d\,(d^*\, d\eta) = d\omega - d\, d\, d^* \eta = 0.$$

So, since the De Rham cohomology of $\mathbb{R}^n$ is trivial except in dimension 0, we get that $d^*\, d\eta = d\kappa$ for some $\kappa \in C^\infty\left(\mathbb{R}^n, \Lambda^{l-1}(\mathbb{R}^n)\right)$. Letting $\nu = \kappa + d^*\eta$ we get that

$$d\nu = d\kappa + d\, d^*\eta = d^*\, d\eta + d\, d^*\eta = \omega,$$

as desired. □

4. Sufficiency of generalized inverses

At this point we have proved Theorem 2.1 and Theorem 3.5 in an attempt to answer our question about when a generalized Poincaré inequality is true. What is unclear is if these two results are related in any way.

Since this next result is true for a broader class of elliptic complexes than what we have previously defined, we will take a moment for definitions so that we may state our result in this broader sense. For the differential operators

$$\mathcal{P} = \sum_{|\alpha| \leq m} A_\alpha(x) D^\alpha$$

and

$$\mathcal{Q} = \sum_{|\alpha| \leq m} B_\alpha(x) D^\alpha$$

of order m with variable coefficients, then

$$\mathcal{D}'(\mathbb{R}^n, \mathbb{U}) \xrightarrow{\mathcal{P}} \mathcal{D}'(\mathbb{R}^n, \mathbb{V}) \xrightarrow{\mathcal{Q}} \mathcal{D}'(\mathbb{R}^n, \mathbb{W})$$

is an elliptic complex of order m if $\mathcal{Q}\mathcal{P} = 0$ and the symbol complex

$$\mathbb{U} \xrightarrow{\mathcal{P}_m(x,\xi)} \mathbb{V} \xrightarrow{\mathcal{Q}_m(x,\xi)} \mathbb{W}$$

is exact for every x and every $\xi \neq 0$. Here, $\mathcal{P}_m$ denotes the principal symbol of $\mathcal{P}$, that is, $\sum_{|\alpha|=m} A_\alpha(x)\xi^\alpha$, and similarly for $\mathcal{Q}_m$.

Theorem 4.1. *A sequence*

$$\mathcal{D}'(\mathbb{R}^n, \mathbb{U}) \xrightarrow{\mathcal{P}} \mathcal{D}'(\mathbb{R}^n, \mathbb{V}) \xrightarrow{\mathcal{Q}} \mathcal{D}'(\mathbb{R}^n, \mathbb{W})$$

with continuous coefficients is an elliptic complex if and only if all of the following hold:

(i) $\mathcal{Q}\mathcal{P} = 0$.

(ii) *The sequence*

$$\mathbb{U} \xrightarrow{\mathcal{P}_m(y,\zeta)} \mathbb{V} \xrightarrow{\mathcal{Q}_m(y,\zeta)} \mathbb{W}$$

is exact for some y and some $\zeta \neq 0$.

(iii) *For each multi-index γ of length $2m$,*

$$\sum_{\substack{\alpha+\beta=\gamma,\\ |\alpha|=|\beta|=m}} B_\beta(x)A_\alpha(x) = 0$$

as operators from $\mathbb{U}$ to $\mathbb{V}$.

(iv) *The matrix $\mathcal{P}_m(x,\xi)$ has constant rank for all x and all $\xi \neq 0$.*

(v) *The matrix $\mathcal{Q}_m(x,\xi)$ has constant rank for all x and all $\xi \neq 0$.*

Proof. We will begin by showing that an elliptic complex has the stated properties. Note that (i) and (ii) follow trivially from the definition. Since $\mathcal{Q}_m(x,\xi)\mathcal{P}_m(x,\xi) = 0$ as functions of ξ, we get (iii) by equating coefficients of ξ^γ. Since the A_α and B_β are continuous, we get that $\operatorname{rank}\mathcal{P}_m$ and $\operatorname{rank}\mathcal{Q}_m$ are lower semicontinuous. By the Rank-Nullity Theorem, the fact that the symbol complex is exact, and the lower semicontinuity of $\operatorname{rank}\mathcal{P}_m$, we get that $\operatorname{rank}\mathcal{Q}_m$ is upper semicontinuous. Therefore $\operatorname{rank}\mathcal{Q}_m$ is continuous, and, since it is valued in a discrete set, we get (v). Then, (iv) follows by the Rank-Nullity Theorem.

Now, we will assume that properties (i) through (iv) hold and show that the complex is elliptic. Property (iii) gives us that the composition $\mathcal{Q}_m\mathcal{P}_m$ is identically 0, which means that $\operatorname{image}\mathcal{P}_m \subseteq \ker\mathcal{Q}_m$. Now, the Rank-Nullity Theorem, and properties (ii), (iv), and (v) give us that $\operatorname{rank}\mathcal{P}_m = \operatorname{Null}\mathcal{Q}_m(x,\xi)$, proving ellipticity. □

This result shows that Theorem 2.1 follows from Theorem 3.5. The following example shows that Theorem 3.5 is a strictly stronger result.

Example 4.2. Consider the differential operator $\mathcal{P} : \mathcal{D}'(\mathbb{R}^2, \mathbb{R}^2) \to \mathcal{D}'(\mathbb{R}^2, \mathbb{R}^3)$ given by

$$\mathcal{P} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial y} \end{bmatrix}.$$

Clearly, the symbol of $\mathcal{P}$ has constant rank away from $\xi = 0$.

We will show that there is no first order constant coefficient differential operator $\mathcal{Q} = B_1\frac{\partial}{\partial x} + B_2\frac{\partial}{\partial y}$ such that

$$\mathcal{D}'(\mathbb{R}^2, \mathbb{R}^2) \xrightarrow{\mathcal{P}} \mathcal{D}'(\mathbb{R}^2, \mathbb{R}^3) \xrightarrow{\mathcal{Q}} \mathcal{D}'(\mathbb{R}^2, \mathbb{W})$$

is an elliptic complex. Since the cokernel of A_i has dimension 1 for each i, the largest that $\mathbb{W}$ need be is $\mathbb{R}^2$, corresponding to the possibility that the images of

the B_i only share 0. Now, solving the equation $\mathcal{Q}(\xi)\mathcal{P}(\xi) = 0$ with $\mathbb{W} = \mathbb{R}^2$, we see that $\mathcal{Q}$ must be the zero operator, but the image of $\mathcal{P}$ is not all of $\mathbb{R}^3$. Thus, Theorem 3.5 applies to $\mathcal{P}$, but Theorem 2.1 does not.

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