THE GL-L.U.ST. CONSTANT AND ASYMMETRY OF THE KALTON-PECK TWISTED SUM IN FINITE DIMENSIONS

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(Communicated by Nigel J. Kalton)

Dedicated to the memory of Nigel J. Kalton

Abstract. We prove that the Kalton-Peck twisted sum $Z^n_2$ of $n$-dimensional Hilbert spaces has a GL-l.u.st. constant of order $\log n$ and bounded GL constant. This is the first concrete example which shows different explicit orders of growth in the GL and GL-l.u.st. constants. We also discuss the asymmetry constants of $Z^n_2$.

1. Introduction

Local unconditional structure, or l.u.s.t., is an important notion in the study of the geometry of Banach spaces (see for instance [GL1], [KT], [PW]). The variant of l.u.s.t. that we investigate here was introduced by Gordon and Lewis [GL1] and is frequently referred to as GL-l.u.s.t.; another, formally more restrictive, notion of l.u.s.t. was introduced in [FJT].

This notion was not, however, fully studied in the finite-dimensional case, where the asymptotic values of the various constants that it involves are not yet fully understood. We show in this paper that in the finite-dimensional setting of $n$-dimensional normed spaces, the GL constant and the l.u.s.t. constant can be of significantly different orders of magnitude, by considering the $2n$-dimensional Kalton-Peck twisted sum of $n$-dimensional Hilbert spaces.

Let us now recall some definitions ([GL1]). A basis $B = (b_i)_{i \in I}$ for a Banach space $E$ is called unconditional if there is a constant $C > 0$ such that for every $x \in E$, $x = \sum_{i \in I} \xi_i b_i$ and every choice of signs $\varepsilon_i = \pm 1$, with $\varepsilon_i = 1$ for all but a finite number of $i \in I$, one has

$$\left\| \sum_{i \in I} \varepsilon_i \xi_i b_i \right\| \leq C \| x \|.$$

The smallest $C$ satisfying this is called the unconditional constant of $B$ and denoted by $\chi(B)$. The unconditional constant of $E$ is

$$\chi(E) := \inf \{ \chi(B); \ B \text{ is a basis of } E \}.$$

2010 Mathematics Subject Classification. Primary 46B20; Secondary 46B07.

Key words and phrases. Banach spaces, local unconditional structure, asymmetry.

The first, third and fourth authors were supported in part by the France-Israel Research Network Program in Mathematics contract #3-4301.

The second author was supported in part by NSF grant DMS-0901457.

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More generally, define the local unconditional structure constant of \(E\) (see [GL1]), \(\chi_u(E)\), by

\[
\chi_u(E) := \sup_{F \subseteq E} \left( \inf_{A, B, U} \|B\| \|\chi(U)\| \|A\| \right),
\]

where the sup is taken over all finite-dimensional subspaces \(F\) of \(E\) and the infimum over all Banach spaces \(U\) and all continuous operators \(A : F \rightarrow U\) and \(B : U \rightarrow E\) such that \(BA = I_E\), the identity on \(F\). Clearly \(\chi_u(E) \leq \chi(E)\). By [GL1], \(\chi_u(E) = \chi_u(E')\) and \(\chi_u(E)\) is finite if \(E''\) is isomorphic to a complemented subspace of a Banach lattice.

Given a subset \(\{x_i; i \in I\}\) of \(E\), we denote

\[
\varepsilon_1(\{x_i; i \in I\}) := \sup \left\{ \sum_{i \in I} \|x'(x_i)\|; x' \in E', \|x'\| \leq 1 \right\}.
\]

Let \(E\) and \(F\) be Banach spaces. The \textit{weakly nuclear norm} of a linear operator \(u : E \rightarrow F\), which has a representation \(u = \sum_{i \in I} x'_i \otimes y_i\), which converges unconditionally in \(L(E, F)\), is defined by

\[
\eta(u) = \inf \varepsilon_1(\{x'_i \otimes y_i; i \in I\}),
\]

where the infimum is taken over all such representations of \(u\).

By Proposition 1.2. of [GL2], one has for all \(u \in L(E, F)\)

\[
\eta(u) = \inf \|A\| \|\chi(U)\| \|B\|,
\]

where the infimum is taken over all factorizations \(u = BA\), with \(A : E \rightarrow U\) compact, \(B : U \rightarrow F\), and \(U\) is a Banach space with an unconditional basis. Hence \(\eta(I_E) = \chi_u(E)\). It is easy to show, using e.g. techniques from [R], that when \(E = F\) is finite dimensional and \(u = I_E\), one may restrict this infimum to spaces \(U\) satisfying \(\chi(U) = 1\) and \(A : E \rightarrow U\) to be an isometric embedding.

The \textit{Gordon-Lewis constant} of \(E\), denoted by \(\text{gl}(E)\), is defined to be

\[
\text{gl}(E) = \inf \{c > 0; \gamma_1(A) \leq c \Pi_1(A); F \text{ a Banach space and } A : E \rightarrow F\}.
\]

Here

\[
\gamma_1(A) = \inf \{\|\|\| \|\beta\|; \beta \alpha = i_F A\},
\]

where the infimum ranges over all Banach spaces \(F\) and all \(A : E \rightarrow L^1(\mu)\), \(\beta : L^1(\mu) \rightarrow F''\), \(i_F : F \rightarrow F''\) is the canonical inclusion, and

\[
\Pi_1(A) = \inf \left\{ C > 0; \sum_{i=1}^n \|Ax_i\| \leq C \sup \left\| \sum_{i=1}^n \pm x_i \right\| \right\}
\]

for every \(n \geq 1\) and \(x_1, \ldots, x_n \in E\)

is the classical 1-\textit{absolutely summing norm} of \(A\). It was proved in [GL1] that \(\text{gl}(E) \leq \chi_u(E)\), and it is well known that there exist infinite-dimensional separable Banach spaces \(E\) such that \(\text{gl}(E)\) is finite and \(\chi_u(E)\) is infinite (the first such example is presented in [LSS]). In particular, it follows that there exists an increasing sequence of \(2n\)-dimensional Banach spaces \(E_n\) for which \(\chi_u(E_n) \to \infty\) and \((\text{gl}(E_n))_n\) is bounded.

We prove in the sequel that, for some constant \(c > 0\), one has

\[
c_n = \sup \left\{ \frac{\chi_u(E_n)}{\text{gl}(E_n)}; E_n \text{ is an } n\text{-dimensional normed space} \right\} \geq c \log n.
\]
To do this, we refine the proof, given by [JLS], that the Kalton-Peck space $Z_2$ (see [KP]) has bounded gl-constant but that $\chi_u(Z_2)$ is infinite. The spaces $E_n$ mentioned above are the subspaces $Z_2^n$ of $Z_2$ spanned by first 2n coordinates. Note that always $c_n \leq \sqrt{n}$. An interesting problem is, how big can $c_n$ be? For example, is $c_n \geq cn^{\alpha}$ for some $\alpha > 0$ and $c$ an absolute positive constant? Perhaps this is true even with $\alpha = 1/2$.

$Z_2$ (and $Z_p$) has unconditional finite-dimensional decomposition into 2-dimensional subspaces (2-UFDD). Many of the concrete examples of spaces with GL and without l.u.st. possess such structure (see [KT] for references). The paper [CK] presents a general treatment of spaces having uniform UFDD (that is, UFDD for which the dimensions of the blocks are uniformly bounded). It is shown there that such a (infinite-dimensional) space either has an unconditional basis or fails to have l.u.st. Observing the computations made in the works mentioned above, it is plausible that if $c_n \geq cn^{\alpha}$ for some $\alpha > 0$, then the examples showing this would not be with uniform UFDD. In this respect we may quote a conjecture that Nigel Kalton sent us, together with other helpful suggestions, in response to a preprint version of the present paper:

**Conjecture.** If we restrict the definition of $c_n$ to $kn$-dimensional initial blocks of spaces with a $k$-UFDD, then $c_n$ is equivalent to $(\log n)^{k-1}$.

The concept of asymmetry of an n-dimensional Banach space was introduced in [GG] and generalized the notion of the asymmetry of a basis. This was followed up in [GL1] as well. Here we study related notions of asymmetry and apply them to $Z_2^n$.

For general terminology concerning the geometry of Banach spaces we refer the reader to [LT1, LT2]. Terminology concerning normed ideals of operators may be found in [P] and [T].

Finally, we wish to thank Alexander Litvak for very helpful discussions.

2. THE L.U.S.T. CONSTANT OF $Z_2^n$

Let $Z_2^n$ be the 2n-dimensional (real) Banach space which is the subspace of the Kalton-Peck [KP] space $Z_2$, spanned by the first 2n coordinates. More precisely, for $a, b \in \mathbb{R}^n$, $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$, we define

$$||(a, b)||_{Z_2^n} = ||(a_1, \ldots, a_n, b_1, \ldots, b_n)||_{Z_2^n} = \left( \sum_{j=1}^{n} (a_j e_j + b_j f_j) \right)^{1/2}$$

As it is proved in [KP], this quasi-norm on $\mathbb{R}^{2n}$ is (uniformly in $n$) equivalent to a norm.

As in [JLS], we consider $Z_2^n$ as an unconditional sum

$$Z_2^n = \sum_{j=1}^{n} \bigoplus E_j,$$

where $E_j = [e_j, f_j]$. 

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It was proved in [JLS] that $Z_2$ fails to have l.u.s.t. They observed that if a Banach space $E$ is of the form $E = \bigoplus_{j=1}^{\infty} E_j$, a 1-unconditional sum of finite-dimensional subspaces, then

$$\text{gl}(E) \leq \operatorname{sup} \dim(E_j),$$

and thus $\text{gl}(Z_2) \leq 2c$. Hence it follows that $\text{gl}(Z_2^n) \leq 2c$, but that the l.u.s.t. constants of $Z_2^n$ tend to infinity with $n$. Here we establish the order of growth of these constants.

**Theorem 1.** One has $\text{gl}(Z_2^n) \leq 2c$, and the l.u.s.t. constant $\chi_u(Z_2^n)$ of $Z_2^n$ satisfies

(2)

$$\chi_u(Z_2^n) \sim \log n.$$

Throughout the proof the letters $C$, $c$, $c_1, c_2, \ldots$, etc. will denote absolute constants which do not depend on $n$. The same letter $c$ (etc.) may denote different constants in different lines.

It is shown in [KP] that the Banach-Mazur distance $d(Z_2^n, \ell_2^n)$ from $Z_2^n$ to $\ell_2^n$ is of the order of $\log n$. Using Proposition 2 of [JLS], there exists a Banach space $Y_n$ with a 1-unconditional basis $\{y_{j,i}, j=1, \ldots, n, \ i=1, \ldots, k_j\}$, such that

- $Z_2^n$ is a subspace of $Y_n$ and, for each $1 \leq j \leq n$, $E_j \subset \{y_{j,i}, \ i=1, \ldots, k_j\}$.
- There exists a projection $P_n : Y_n \to Z_2^n$ such that $P_n(\{y_{j,i}, \ i=1, \ldots, k_j\}) = E_j$ for all $j, \ 1 \leq j \leq n$. ($[A]$ denotes the span of $A$.)
- $\|P_n\| = K_n \leq c\chi_u(Z_2^n)$. To justify this, one first shows (see [FJT]) that for any Banach space $E$,

$$\chi_u(E) = \inf \|P\|_{L \to E},$$

where the infimum is taken over all Banach lattices $L$ such that $E$ is isometrically embedded in $L$ and all projections $P : L \to E$. Then observing that $E = Z_2^n$ have bounded cotype-$q$ constants for a fixed $q < \infty$, we use [K] to reduce to the case when moreover $L$ is supposed to have the same bound on its cotype-$q$ constant. Then, we follow the lines of Proposition 2 of [JLS], using Rademacher embedding.

We shall show that $K_n \geq c \log n$, thus proving (2) because it is clear that $K_n \leq d(Z_2^n, \ell_2^n) \sim \log n$.

**Notation.** Let $T : Z_2^n \to Z_2^n$ be a linear operator that satisfies $T(E_j) \subset E_j$ for $1 \leq j \leq n$. We say that $T$ splits through $\{E_j\}$ or, for short, that $T$ splits. If $T$ splits, the matrices representing $T|_{E_j}$ in the basis $\{e_j, f_j\}$, $1 \leq j \leq n$, will be denoted by

$$\begin{pmatrix}
\alpha_j & \beta_j \\
\gamma_j & \delta_j
\end{pmatrix}.$$

If $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, the support of $a$ is $\text{supp}(a) := \{j : 1 \leq j \leq n, \ a_j \neq 0\}$. If $a, b \in \mathbb{R}^n$, $a = (a_1, \ldots, a_n)$, $b = (a_1, \ldots, b_n)$, we denote $(a, b) = \sum_{j=1}^{n} (a_je_j + b_jf_j) \in Z_2^n$. We also define a vector $ab \in \mathbb{R}^n$ by $ab = (a_1b_1, \ldots, a_nb_n)$.

**Lemma 2.** For any $T$ that splits we have

$$\max_j \max |\alpha_j|, |\beta_j|, |\gamma_j|, |\delta_j| \leq \|T\|.$$

**Proof.** If $x = e_j, 1 \leq j \leq n$, then $\|Tx\| = |\gamma_j| + |\alpha_j| \leq \|T\||x|| = \|T\|$. If $x = f_j$, one similarly obtains $|\delta_j| + |\beta_j| \leq \|T\|$. \qed
Lemma 3. With the preceding notation, there exists $C > 0$ such that for every $T : Z_2^n \to Z_2^n$ that splits and for all $(a, b) \in \mathbb{R}^{2n}$, we have

$$\left( \sum_{j=1}^{n} \left( \delta_j a_j - \alpha_j a_j + \gamma_j a_j \log \frac{|a_j|}{\|a\|_2} \right)^2 \right)^{\frac{1}{2}} \leq C \left( \|T\| \|(a, b)\|_{Z_2^n} + (\max_{j \in \text{supp}(a)} |\gamma_j|) \|a\|_2 \right).$$

Proof. It is established in [KP] that the function $F : \ell_2^n \to \ell_2^n$ given by

$$F(b) = \left( b_1 \log \left| b_1 \right| / \|b\|_2, \ldots, b_n \log \left| b_n \right| / \|b\|_2 \right)$$

for $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ is quasi-linear in the sense that for every $a, b \in \mathbb{R}^n$,

$$\|F(a + b) - F(a) - F(b)\|_2 \leq C (\|a\|_2 + \|b\|_2).$$

We have

$$\|(\gamma F(a), \gamma a)\|_{Z_2^n} = \|\gamma a\|_2 + \|\gamma F(a) - F(\gamma a)\|_2,$$

so that

$$\|\gamma F(a) - F(\gamma a)\|_2 \leq \|(\gamma F(a), \gamma a)\|_{Z_2^n}.$$ 

Hence

$$\|(\delta - \alpha)a + \gamma F(a)\|_2 \leq \|(\delta - \alpha)a + F(\gamma a)\|_2 + \|(\gamma F(a), \gamma a)\|_{Z_2^n}.$$ 

Also, unconditionality of the sum (1) implies for every $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$,

$$\|(\theta a, \theta b)\|_{Z_2^n} \leq C \max_j |\theta_j| \|(a, b)\|_{Z_2^n}.$$ 

Defining $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^b$, $\beta$, $\gamma$ and $\delta$ related to $T$ as above, one has

$$\|(a, b)\|_{Z_2^n} = \|b\|_2 + \|a - F(b)\|_2,$$

$$\|T(a, b)\|_{Z_2^n} = \|\gamma a + \delta b\|_2 + \|\alpha a + \beta b - F(\gamma a + \delta b)\|_2$$

and

$$\|F(\gamma a + \delta b) - \alpha a\|_2 \leq 2\|T\| \|(a, b)\|_{Z_2^n},$$

because

$$\|F(\gamma a + \delta b) - \alpha a\|_2 \leq \|\beta b\|_2 + \|\alpha a + \beta b - F(\gamma a + \delta b)\|_2 + \|\gamma a + \delta b\|_2 \leq \|T(a, b)\|_{Z_2^n} + \max_j \|b\|_2 \leq 2\|T\| \|(a, b)\|_{Z_2^n}.$$ 

Applying Lemma 2, 3, 6, 6 and the fact that $\|(F(a), a)\|_{Z_2^n} = \|a\|_2$, we get from (4)

$$\|(\delta - \alpha)a + \gamma F(a)\|_2 \leq \|(\delta - \alpha)a + F(\gamma a)\|_2 + \|(\gamma F(a), \gamma a)\|_{Z_2^n}$$

$$\leq \|F(\gamma a) + F(\delta b) - F(\gamma a + \delta b)\|_2 + \|F(\gamma a + \delta b) - F(\alpha a + \beta b)\|_2$$

$$+ (\max_{j \in \text{supp}(a)} |\gamma_j|) \|(F(a), a)\|_{Z_2^n}$$

$$\leq C \left( \|\gamma a\|_2 + \|\delta b\|_2 \right) + 2\|T\| \|(a, b)\|_{Z_2^n} + \max_j |\gamma_j| \|a\|_2$$

$$\leq C \left( \|T\| \|(a, b)\|_{Z_2^n} + \max_{j \in \text{supp}(a)} |\gamma_j| \|a\|_2 \right).$$
**Lemma 4.** In the context of the preceding lemmas, for every subset \( A \subset \{1, \ldots, n\} \), \(|A| = k > 1\), one has
\[
\left( \frac{1}{k} \sum_{j \in A} \gamma_j^2 \right)^{\frac{1}{2}} \leq \frac{4\|T\|}{\log k}.
\]

**Proof.** Let \( x = (a, b) \in Z_2^n \), where \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) are given by
\[
a_j = \frac{\log \frac{1}{\sqrt{k}}}{\sqrt{k}}, \quad b_j = \frac{1}{\sqrt{k}} \text{ if } j \in A, \quad a_j = b_j = 0 \text{ otherwise}.
\]
Then \( \|x\|_{Z_2^n} = 1 \) and therefore
\[
\|T\| \geq \|Tx\|_{Z_2^n} \geq \left( \frac{1}{k} \sum_{j \in A} (-\gamma_j \log \sqrt{k} + \delta_j)^2 \right)^{\frac{1}{2}} \geq \log \sqrt{k} \left( \frac{1}{k} \sum_{j \in A} \gamma_j^2 \right)^{\frac{1}{2}} - \max_j |\delta_j|.
\]
We now use Lemma 3\footnote{Corollary 5} to complete the proof. \( \square \)

**Corollary 5.** In the context of the preceding lemmas, let \( A \subset \{1, \ldots, n\} \), \(|A| = k > 1\). Then
a) There exists a subset \( A' \subset A \), with \(|A'| \geq \frac{k}{2}\), such that
\[
(7) \quad \max_{j \in A'} |\gamma_j| \leq \frac{4\sqrt{2}\|T\|}{\log k}.
\]
b) If \( \text{supp}(a) \subset A' \), where \( A' \subset \{1, \ldots, n\} \), \(|A'| \leq k \) and (7) is satisfied, then
\[
\left( \sum_{j=1}^n \left( \delta_j a_j - \alpha_j a_j + \gamma_j a_j \log \frac{|a_j|}{|a||a|} \right)^2 \right)^{\frac{1}{2}} \leq C\|T\||a, b\|_{Z_2^n}.
\]

**Proof.** Let \( C = \{j \in A; |\gamma_j| > \frac{4\sqrt{2}\|T\|}{\log k}\} \). To prove a), it is enough to show that \(|C| \leq k/2\). If it was not true, one would have
\[
\left( \frac{1}{k} \sum_{j \in C} \gamma_j^2 \right)^{\frac{1}{2}} \geq \left( \frac{1}{k} \sum_{j \in C} \gamma_j^2 \right)^{\frac{1}{2}} > \frac{4\|T\|}{\log k},
\]
which contradicts Lemma 4. Part b) follows from Lemma 3\footnote{Lemma 6} and the fact (see\footnote{Remark 2}) that the norm of the operator \( p_k : Z_2^k \to Z_2^k \), \( p_k(a, b) = a \) is equivalent to \( \log k \). \( \square \)

**Lemma 6.** Let \( A \subset \{1, \ldots, n\} \), \(|A| = k > k_0 > 1\). Then there exists a subset \( A' \subset A \), with \(|A'| \geq \frac{k}{2}\), such that for all \( j \in A' \) we have
\[
|\gamma_j| \leq \frac{C\|T\|}{(\log k)^2} \quad \text{and} \quad |\delta_j - \alpha_j| \leq \frac{C\|T\|}{\log k}.
\]

**Proof.** Let \( A' \subset A \) with \( k' = |A'| \geq \frac{k}{2} \) satisfy (7). Let \( x = (a, b) \in Z_2^n \) be defined as in the proof of Lemma 4 with \( A' \) replacing \( A \). Then, by Corollary 5b), we have (since \( \|x\| = 1 \))
\[
\left( \frac{1}{k'} \sum_{j \in A'} (\delta_j - \alpha_j - \gamma_j \log \sqrt{k'})^2 \right)^{\frac{1}{2}} \leq \frac{C\|T\|}{\log \sqrt{k'}}.
\]
As in the proof of Corollary 5 there exists a subset $B \subset A'$ with $k'' = |B| \geq \frac{k'}{2} \geq \frac{k}{4}$ such that
\begin{equation}
|\delta_j - \alpha_j - \gamma_j \log \sqrt{k'}| \leq \frac{\sqrt{2C ||T||}}{\log \sqrt{k'}} = \frac{2C_1 ||T||}{\log k'} \quad \text{for all } j \in B.
\end{equation}

Now take any subset $B' \subset B$ with $|B'| = k''' = \sqrt{k'} \leq \frac{k'}{2} \leq k''$ and a new vector $x \in Z_n^2$ like the one above, with $B'$ replacing $A'$ (the implicit assumption that $\sqrt{k'}$ is an integer can easily be dealt with). The same argument as above provides a subset $A'' \subset B'$ with
\[ |A''| \geq \frac{k''}{2} = \frac{\sqrt{k'}}{2} \geq \frac{\sqrt{k}}{2\sqrt{2}} \]
such that
\begin{equation}
|\delta_j - \alpha_j - \gamma_j \log \sqrt{k''}| \leq \frac{\sqrt{2C ||T||}}{\log \sqrt{k''}} = \frac{4C_1 ||T||}{\log k''} \quad \text{for all } j \in A''.
\end{equation}

Adding (8) and (9) together we get for $j \in A''$:
\[ \frac{|\gamma_j| \log k}{4} = \left| \frac{1}{2} \gamma_j \log k' - \frac{1}{4} \gamma_j \log k \right| \]
\[ = \left| (-\delta_j + \alpha_j + \gamma_j \log \sqrt{k'}) + (\delta_j - \alpha_j - \gamma_j \log \sqrt{k''}) \right| \leq \frac{6C_1 ||T||}{\log k''}, \]
and thus
\begin{equation}
|\gamma_j| \leq \frac{24C_1 ||T||}{(\log k'')^2} \leq \frac{C_2 ||T||}{(\log k)^2}.
\end{equation}

From (9) and (10) together we get
\[ |\delta_j - \alpha_j| \leq \frac{4C_1 ||T||}{\log k'} + |\gamma_j| \log \sqrt{k''} \leq \frac{4C_1 ||T||}{\log k'} + \frac{24C_1 ||T||}{(\log k')^2} \cdot \frac{1}{4} \log k' \leq \frac{C_3 ||T||}{\log k} \quad \text{for } j \in A''. \]

\[ \square \]

**Proof of Theorem I.** We now recall the construction from [JLS]: Under the hypotheses that $Z_2$ has l.u.s.t., they constructed an operator $T : Z_2 \to Z_2$ that provided a counterexample to that hypotheses. In the context of $Z_n^2$, this construction, instead of providing a counterexample, will provide a lower bound estimate of $\chi_u(Z_n^2)$.

Let the Banach space $Y_n$ with the unconditional basis $\{y_{j,i} : j = 1, \ldots, n, \ i = 1, \ldots, k_j\}$ and the projection $P_n : Y_n \to Z_n^2$ be those presented at the opening. For a fixed $1 \leq j \leq n$ a set $A_j \subset \{1, \ldots, k_j\}$ is selected and an operator $T_{A_j} : [e_j, f_j] \to [e_j, f_j]$ is defined by
\[ T_{A_j}(x) = P_n \left( \sum_{i \in A_j} y_{j,i}(x)y_{j,i}^* \right), \]
where $\{y_{j,i}^*\} \subset Y_n^*$ are the bi-orthogonal functionals of $\{y_{j,i}\}$. Thus $T_{A_j} = \sum_{i \in A_j} T_{j,i}$, where $T_{j,i}$ is the rank-1 operator $T_{j,i}(x) = P_n (y_{j,i}^*(x)y_{j,i})$. We may assume that $T_{j,i} \neq 0$, otherwise $y_{j,i}$ can be dropped from the basis. Being a rank-1 operator, the matrix representing $T_{j,i}$ is of the form
\[ \begin{pmatrix} a_i & b_i \\ \theta_i a_i & \theta_i b_i \end{pmatrix} \quad \text{(case I)} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ a_i & b_i \end{pmatrix} \quad \text{(case II)} \]
Lemma 6. Let (note that we dropped the index $j$ for simplicity). In case II we define $\theta_i$ to be 1.

We have $\sum_{i=1}^{k_j} \theta_i b_i = 1$ (because $\sum_i T_j, f_j = (\sum_i \theta_i b_i) f_j = f_j$) and $\sum_{i=1}^{k_j} |b_i| \leq K_n$. Define

$$B_j := \left\{ i \in \{1, \ldots, k_j\} : |\theta_i| \geq \frac{1}{2K_n} \right\}.$$ 

We have

$$\sum_{i \in B_j} \theta_i b_i > \frac{1}{2}.$$ 

For each $i \in B_j$ at least one of the following four possibilities holds:

1. $\theta_i a_i > \frac{1}{4K_n} |\theta_i b_i|$, 
2. $-\theta_i a_i > \frac{1}{4K_n} |\theta_i b_i|$, 
3. $\alpha_i - \theta_i b_i \geq \frac{1}{2} |\theta_i b_i|$, or $(-\theta_i b_i \geq \frac{1}{2} |\theta_i b_i|)$ in case II), 
4. $\theta_i b_i - \alpha_i \geq \frac{1}{2} |\theta_i b_i|$, or $|\theta_i b_i| \geq \frac{1}{2} |\theta_i b_i|$ in case II).

In fact, if $|\theta_i a_i| > \frac{1}{4K_n} |\theta_i b_i|$, then (1) or (2) holds. Otherwise $|\theta_i a_i| \leq \frac{1}{4K_n} |\theta_i b_i|$. Since $i \in B_j$ this implies in case I that

$$\frac{1}{2K_n} |a_i| \leq \frac{1}{4K_n} |\theta_i b_i|,$$

which is

$$\frac{|a_i|}{\theta_i} \leq \frac{1}{2} |b_i|.$$ 

We thus have

$$|\frac{a_i}{\theta_i} - b_i| \geq |b_i| - |\frac{a_i}{\theta_i}| \geq |b_i| - \frac{1}{2} |b_i| = \frac{1}{2} |b_i|,$$

which is either (3) or (4). The remarks about case II are trivial.

It follows that there exists a subset $A_j \subset B_j$ such that one and the same possibility out of (1)-(4) holds for all $i \in A_j$ and $\sum_{i \in A_j} \theta_i b_i > \frac{1}{8}$. The operator $T : Z_n \to Z_n^\perp$ that splits and is defined by $T|_E = T_{A_j}$ satisfies $\|T\| \leq K_n$. Assume that the matrix representing $T_{A_j}$ in the basis $\{e_j, f_j\}$ is, as before,

$$\begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix}.$$ 

From the above it follows that $\delta_j > \frac{1}{8}$ for all $j$.

If (1) or (2) is satisfied for all $i \in A_j$, then we have

$$|\gamma_j| > \frac{1}{4K_n} |\delta_j| > \frac{1}{32K_n}.$$ 

If (3) or (4) is satisfied for all $i \in A_j$, then we have

$$|\alpha_j - \delta_j| > \frac{1}{2} |\delta_j| > \frac{1}{16}.$$ 

We can now select a subset $D \subset \{1, \ldots, n\}$, with $|D| \geq \frac{n}{2}$, such that either (11) holds for all $j \in D$ or (12) holds for all $j \in D$. We take $D$ to be the set $A$ of Lemma \ref{lem:4}. Let $A'' \subset D$ be as in Lemma \ref{lem:4} and $j \in A''$.

If the members of $D$ satisfy (11), then we have

$$\frac{1}{32K_n} \leq |\gamma_j| \leq \frac{C\|T\|}{(\log \frac{n}{2})^2} \leq \frac{C_1 K_n}{(\log n)^2},$$
and we conclude
\begin{equation}
K_n \geq c \log n
\end{equation}
for some positive constant $c$.

If the members of $D$ satisfy (12), then we have
\[ \frac{1}{16} \leq |\delta_j - \alpha_j| \leq \frac{C\|T\|}{\log \frac{n}{2}} \leq \frac{C_1 K_n}{\log n}; \]
thus
\begin{equation}
K_n \geq c \log n. \quad \Box
\end{equation}

3. The asymmetry of $Z^n_2$

Definition. We say that a subgroup $G$ of $GL_k(\mathbb{R})$ is rich if it is a compact group
and that every operator in $L(\mathbb{R}^k)$ which commutes with all elements of $G$ is a scalar
multiple of the identity on $\mathbb{R}^k$. It is well known that for any compact subgroup $G$
of $GL_k(\mathbb{R})$, there exists a compact subgroup $H$ of $O_k$ (the orthogonal group) such
that $G = \{V^{-1}hV; h \in H\}$, where $V \in L(\mathbb{R}^k)$ is some linear invertible operator.
If $G$ is a rich subgroup of $GL_k(\mathbb{R})$, we define the measure of symmetry $s_G(E)$ of a
$k$-dimensional normed space $E$ with respect to $G$ by
\[ s_G(E) := \int_G \|g\|_{E \rightarrow E} d\mu(g), \]
where $\mu$ is the normalized Haar measure on $G$. We define
\[ s(E) = \inf \{s_G(E); G \text{ a rich subgroup}\}. \]
Note that
\[ 1 \leq s(E) \leq S(E) := \inf \{\sup_{g \in G} \|g\|_{E \rightarrow E}; G \text{ a rich subgroup}\}. \]

The quantity $S(E)$, called the asymmetry constant of $E$, was defined originally
in [GG] and further discussed in [GL1]. If $S(E) = 1$ we say that $E$ has enough
symmetries. This is the case for spaces with 1-symmetric basis (take the rich group
of permutations and changes of signs on the basis). In connection with this quantity
we have the following partial result:

Theorem 7. For every rich subgroup $G$ of $O_{2n}$ one has
\begin{equation}
s_G(Z^n_2) = \int_G \|g\|_{Z^n_2 \rightarrow Z^n_2} d\mu(g) \sim \log n \sim \max_{g \in G} \|g\|_{Z^n_2 \rightarrow Z^n_2},
\end{equation}
where $\mu$ denotes the normalized Haar measure on $G$.

We use the following lemma.

Lemma 8. Let a couple $E$ and $F$ of $k$-dimensional normed spaces, an ideal norm
$\alpha$ and a rich subgroup $G$ of $GL_k(\mathbb{R})$ be given. Then for every invertible operator
$S \in L(\mathbb{R}^k)$, one has
\begin{equation}
k \leq \alpha(S : E \rightarrow F) \alpha^* (S^{-1} : F \rightarrow E) \leq k \int_G \|S^{-1}gS\|_{E \rightarrow E} \|g^{-1}\|_{F \rightarrow F} d\mu(g),
\end{equation}
where \( \alpha^* \) denotes the conjugate ideal norm of \( \alpha \). In particular, if all elements of \( G \) are isometries of \( F \), then

\[
(17) \quad k \leq \alpha(S : E \to F) \alpha^*(S^{-1} : F \to E) \leq k \int_G \|S^{-1}gS\|_{E \to E} \; d\mu(g).
\]

Proof. We first prove for any \( k \)-dimensional normed spaces \( D \) and \( F \) on \( \mathbb{R}^n \) that if \( I : \mathbb{R}^k \to \mathbb{R}^k \) denotes the identity operator, then

\[
(18) \quad k \leq \alpha(I : D \to F) \alpha^*(I^{-1} : F \to D) \leq k \int_G \|g\|_{D \to D} \|g^{-1}\|_{F \to F} \; d\mu(g).
\]

By the definition of \( \alpha^* \), there exists \( U \in L(\mathbb{R}^k) \) such that

\[
\alpha(U : D \to F) \alpha^*(I^{-1} : F \to D) = \text{trace}(UI^{-1}) = \text{trace}(U).
\]

Let \( V = \int_G g^{-1}Ug \; d\mu(g) \) be considered as an operator from \( D \) to \( F \). Since \( G \) is a rich subgroup, one has \( V = cI \) for some \( c \in \mathbb{R} \), so that \( \text{trace}(U) = \text{trace}(V) = kc \). One clearly has

\[
\alpha(I : D \to F) = \alpha(V : D \to F) \leq \int_G \alpha(g^{-1}Ug : D \to F) \; d\mu(g)
\]

\[
\leq \alpha(U : D \to F) \int_G \|g^{-1}\|_{F \to F} \|g\|_{D \to D} \; d\mu(g).
\]

It follows that

\[
\alpha(I : D \to F) \alpha^*(I^{-1} : F \to D)
\]

\[
\leq \frac{\alpha(U : D \to F) \alpha^*(I^{-1} : F \to D)}{c} \int_G \|g\|_{D \to D} \|g^{-1}\|_{F \to F} \; d\mu(g)
\]

\[
= k \int_G \|g\|_{D \to D} \|g^{-1}\|_{F \to F} \; d\mu(g).
\]

This proves (18). We define a normed space \( D \) by setting \( \|x\|_D = \|S^{-1}x\|_E \). Then \( S^{-1} : D \to E \) and \( S : E \to D \) are isometries, and one has

\[
\alpha(I : D \to F) = \alpha(S : E \to F) \cdot \alpha^*(I^{-1} : F \to D) = \alpha^*(S^{-1} : F \to E)
\]

and

\[
\|g\|_{D \to D} = \|S^{-1}gS\|_{E \to E}.
\]

Together with (18) this implies (19). \( \square \)

Proof of Theorem 7. We shall use Lemma 8 with \( S = I : \ell_2^n \to \ell_2^n \), where \( I \) is the identity sending the basis \( e_j, f_j \) \( 1 \leq j \leq n \) on the canonical basis of \( \ell_2^n \). Here we present two proofs: A and B.

A. Let \( \alpha = \Pi_1 \) and \( \alpha^* = \gamma_\infty \), the ideal norm of factorization through \( L_\infty \).

a. We claim that

\[
\Pi_1(I : \ell_2^n) \geq c \sqrt{n} \log n.
\]

In fact, define, for \( 1 \leq j \leq n \), \( x_j \in \ell_2^n \) by

\[
x_j = \frac{1}{\sqrt{n}} \left( \log \frac{1}{\sqrt{n}} \right) e_j + \frac{1}{\sqrt{n}} f_j.
\]
Then \(|I(x_j)|\|_2 \sim \frac{\log n}{\sqrt{n}}\), and for every choice of signs \(\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n\), one has
\[
\left\| \sum_{j=1}^n \varepsilon_j x_j \right\|_{Z_2^n} = 1.
\]
Hence
\[
\Pi_1(I : Z_2^n \to \ell_2^{2n}) \geq \frac{\sum_{j=1}^n \|x_j\|_2}{\sup_{\varepsilon} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|_{Z_2^n}} \geq c\sqrt{n} \log n.
\]

**b.** Define \(P : Z_2^n \to \ell_2^n\) by \(P(a, b) = b\). Then \(|P|_{Z_2^n \to \ell_2^n} = 1\), and if \(W : \ell_2^n \to \ell_2^{2n}\) is the embedding defined by \(W(b) = (0, b)\), then
\[
\gamma_\infty(I^{-1} : \ell_2^{2n} \to Z_2^n) = |P|_{Z_2^n \to \ell_2^n} \gamma_\infty(I^{-1} : \ell_2^{2n} \to Z_2^n)|W|_{\ell_2^n \to \ell_2^{2n}} \geq \gamma_\infty(P^{-1}W : \ell_2^n \to \ell_2^n) \sim \sqrt{n},
\]
because \(P^{-1}W : \ell_2^n \to \ell_2^n\) is actually the identity mapping on \(\ell_2^n\).

**c.** It follows that
\[
\Pi_1(I : Z_2^n \to \ell_2^{2n}) \gamma_\infty(I^{-1} : \ell_2^{2n} \to Z_2^n) \geq cn \log n.
\]

**B.** Let \(\alpha\) be the operator norm; then \(\alpha^* = i_1\), the integral norm. It is easy to see that
\[
|I|_{Z_2^n \to \ell_2^n} \sim \log n
\]
and
\[
i_1(I^{-1} : \ell_2^{2n} \to Z_2^n) = i_1(I^{-1} : \ell_2^{2n} \to Z_2^n)|P|_{Z_2^n \to \ell_2^n} \geq i_1(P^{-1} : \ell_2^n \to \ell_2^n) = \sup_{S \neq 0} \frac{\text{trace}(PI^{-1}S)}{|S|_{\ell_2^n \to \ell_2^{2n}}} \geq n.
\]

It follows that
\[
|I|_{Z_2^n \to \ell_2^n} i_1(I^{-1} : \ell_2^{2n} \to Z_2^n) \geq cn \log n.
\]

Using either **A. a.** or **B.**, together with Lemma [8] with \(F = \ell_2^{2n}\), \(E = Z_2^n\) and \(S = I\), we get that
\[
\int_G \|g\|_{Z_2^n \to Z_2^n} \, d\mu(g) \geq c \log n.
\]

It follows that
\[
\log n \leq \frac{1}{c} \int_G \|g\|_{Z_2^n \to Z_2^n} \, d\mu(g) \leq \frac{1}{c} \max_{g \in G} \|g\|_{Z_2^n \to Z_2^n} \leq d(Z_2^n, \ell_2^{2n}) \sim \log n,
\]
where here \(d\) denotes the Banach-Mazur distance. This proves (15). \(\square\)

**Remark.** One can prove in the same way as **A. a.** in the proof of the last theorem that if \(I : Z_2^n \to \ell_2^{2n}\) is the identity mapping sending the basis \((e_j, \ell_2^n)\), \(1 \leq j \leq n\), on the canonical basis of \(\ell_2^{2n}\), then
\[
\Pi_1(I : Z_2^n \to \ell_2^{2n}) \geq c\sqrt{n} \log n.
\]

We also have that
\[
\gamma_\infty(I^{-1} : \ell_2^{2n} \to Z_2^n) = |I^{-1}|_{\ell_2^{2n} \to Z_2^n} \sim \sqrt{n} \log n.
\]
It follows from Lemma 8 that for every rich subgroup $G$ of $GL_{2n}(\mathbb{R})$, one has
\[ cn(\log n)^2 \leq \Pi_1(I : Z_2^n \rightarrow \ell_2^{2n}) \gamma_\infty(I^{-1} : \ell_\infty^{2n} \rightarrow Z_2^n) \]
\[ \leq n \int_G ||g^{-1}||_{\ell_\infty^{2n} \rightarrow \ell_\infty^{2n}} ||g||_{Z_2^n \rightarrow Z_2^n} d\mu(g). \]
The first inequality is actually an equivalence, in view of the fact that if one takes for $G$ the group of changes of signs and permutations of indices of the basis, one has $||g^{-1}||_{\ell_\infty^{2n} \rightarrow \ell_\infty^{2n}} = 1$ for all $g \in G$ and
\[ \int_G ||g||_{Z_2^n \rightarrow Z_2^n} d\mu(g) \leq ||I : Z_2^n \rightarrow \ell_\infty^{2n}|| ||I^{-1} : \ell_\infty^{2n} \rightarrow Z_2^n|| \leq c(\log n)^2, \]
as $||I : Z_2^n \rightarrow \ell_\infty^{2n}|| \sim \log n$.

**Added in proof.** After the paper was processed, we realized that answers to some of the questions posed at the end of the introduction had already been given in the paper [KT]. In particular, examples of $n$-dimensional Banach spaces with bounded $GL$-constant and l.u.s.t constant greater than $cn^\alpha$ (with $\alpha$ as large as $1/6$) are given there.

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