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# THE GL-L.U.ST. CONSTANT AND ASYMMETRY OF THE KALTON-PECK TWISTED SUM IN FINITE DIMENSIONS

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(Communicated by Nigel J. Kalton)

*Dedicated to the memory of Nigel J. Kalton*

ABSTRACT. We prove that the Kalton-Peck twisted sum $Z_2^n$ of $n$-dimensional Hilbert spaces has a GL-l.u.st. constant of order $\log n$ and bounded GL constant. This is the first concrete example which shows different explicit orders of growth in the GL and GL-l.u.st. constants. We also discuss the asymmetry constants of $Z_2^n$.

## 1. Introduction

Local unconditional structure, or l.u.s.t., is an important notion in the study of the geometry of Banach spaces (see for instance [GL1], [KT], [PW]). The variant of l.u.st. that we investigate here was introduced by Gordon and Lewis [GL1] and is frequently referred to as *GL-l.u.st.*; another, formally more restrictive, notion of l.u.st. was introduced in [FJT].

This notion was not, however, fully studied in the finite-dimensional case, where the asymptotic values of the various constants that it involves are not yet fully understood. We show in this paper that in the finite-dimensional setting of $n$-dimensional normed spaces, the GL constant and the l.u.s.t. constant can be of significantly different orders of magnitude, by considering the $2n$-dimensional Kalton-Peck twisted sum of $n$-dimensional Hilbert spaces.

Let us now recall some definitions ([GL1]). A basis $B = (b_i)_{i\in I}$ for a Banach space $E$ is called *unconditional* if there is a constant $C > 0$ such that for every $x \in E$, $x = \sum_{i\in I} \xi_i b_i$ and every choice of signs $\varepsilon_i = \pm 1$, with $\varepsilon_i = 1$ for all but a finite number of $i \in I$, one has

$$||\sum_{i\in I} \varepsilon_i \xi_i b_i|| \leq C||x||.$$

The smallest $C$ satisfying this is called *the unconditional constant of* $B$ and denoted by $\chi(B)$. The *unconditional constant of* $E$ is

$$\chi(E) := \inf\{\chi(B);\ B \text{ is a basis of } E\}.$$

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More generally, define the local unconditional structure constant of $E$ (see [GL1]), $\chi_u(E)$, by

$$\chi_u(E) := \sup_{F\subset E} \big( \inf_{A,B,U} ||B||\chi(U)||A||\big),$$

where the sup is taken over all finite-dimensional subspaces $F$ of $E$ and the infimum over all Banach spaces $U$ and all continuous operators $A : F \to U$ and $B : U \to E$ such that $BA = I_F$, the identity on $F$. Clearly $\chi_u(E) \le \chi(E)$. By [FJT], $\chi_u(E) = \chi_u(E')$ and $\chi_u(E)$ is finite iff $E''$ is isomorphic to a complemented subspace of a Banach lattice.

Given a subset $\{x_i; i \in I\}$ of $E$, we denote

$$\varepsilon_1(\{x_i; i \in I\}) := \sup\{\sum_{i\in I} |x'(x_i)|; x' \in E', ||x'|| \le 1\}.$$

Let $E$ and $F$ be Banach spaces. The *weakly nuclear norm* of a linear operator $u : E \to F$, which has a representation $u = \sum_{i\in I} x'_i \otimes y_i$, which converges unconditionally in $L(E, F)$, is defined by

$$\eta(u) = \inf \varepsilon_1(\{x'_i \otimes y_i, i \in I\}),$$

where the infimum is taken over all such representations of $u$.

By Proposition 1.2. of [GL2], one has for all $u \in L(E, F)$

$$\eta(u) = \inf ||A||\chi(U)||B||,$$

where the infimum is taken over all factorizations $u = BA$, with $A : E \to U$ compact, $B : U \to F$, and $U$ is a Banach space with an unconditional basis. Hence $\eta(I_E) = \chi_u(E)$. It is easy to show, using e.g. techniques from [R], that when $E = F$ is finite dimensional and $u = I_E$, one may restrict this infimum to spaces $U$ satisfying $\chi(U) = 1$ and $A : E \to U$ to be an isometric embedding.

The *Gordon-Lewis constant* of $E$, denoted by $\mathrm{gl}(E)$, is defined to be

$$\mathrm{gl}(E) = \inf\{c > 0;\ \gamma_1(A) \le c\Pi_1(A);\ F \text{ a Banach space and } A : E \to F\}.$$

Here

$$\gamma_1(A) = \inf\{||\alpha||\ ||\beta||\ ; \beta\alpha = i_F A\},$$

where the infimum ranges over all Banach spaces $F$ and all $\alpha : E \to L^1(\mu)$, $\beta : L^1(\mu) \to F''$, $i_F : F \to F''$ is the canonical inclusion, and

$$\Pi_1(A) = \inf\Big\{C > 0; \sum_{i=1}^{n} ||Ax_i|| \le C \sup_{\pm} ||\sum_{i=1}^{n} \pm\, x_i|| \quad \text{for every } n \ge 1 \text{ and } x_1, \dots, x_n \in E\ \Big\}$$

is the classical 1-*absolutely summing norm* of $A$. It was proved in [GL1] that $\mathrm{gl}(E) \le \chi_u(E)$, and it is well known that there exist infinite-dimensional separable Banach spaces $E$ such that $\mathrm{gl}(E)$ is finite and $\chi_u(E)$ is infinite (the first such example is presented in [JLS]). In particular, it follows that there exists an increasing sequence of $2n$-dimensional Banach spaces $E_n$ for which $\chi_u(E_n) \to \infty$ and $\big(\mathrm{gl}(E_n)\big)_n$ is bounded.

We prove in the sequel that, for some constant $c > 0$, one has

$$c_n = \sup\left\{\frac{\chi_u(E_n)}{\mathrm{gl}(E_n)}; E_n \text{ is an } n\text{-dimensional normed space}\right\} \ge c \log n\,.$$

To do this, we refine the proof, given by [JLS], that the Kalton-Peck space $Z_2$ (see [KP]) has bounded gl-constant but that $\chi_u(Z_2)$ is infinite. The spaces $E_n$ mentioned above are the subspaces $Z_2^n$ of $Z_2$ spanned by first $2n$ coordinates. Note that always $c_n \leq \sqrt{n}$. An interesting problem is, how big can $c_n$ be? For example, is $c_n \geq cn^{\alpha}$ for some $\alpha > 0$ and $c$ an absolute positive constant? Perhaps this is true even with $\alpha = 1/2$.

$Z_2$ (and $Z_p$) has unconditional finite-dimensional decomposition into 2-dimensional subspaces (2-UFDD). Many of the concrete examples of spaces with GL and without l.u.st. possess such structure (see [KT] for references). The paper [CK] presents a general treatment of spaces having uniform UFDD (that is, UFDD for which the dimensions of the blocks are uniformly bounded). It is shown there that such a (infinite-dimensional) space either has an unconditional basis or fails to have l.u.st. Observing the computations made in the works mentioned above, it is plausible that if $c_n \geq cn^{\alpha}$ for some $\alpha > 0$, then the examples showing this would not be with uniform UFDD. In this respect we may quote a conjecture that Nigel Kalton sent us, together with other helpful suggestions, in response to a preprint version of the present paper:

**Conjecture.** *If we restrict the definition of $c_n$ to $kn$-dimensional initial blocks of spaces with a $k$-UFDD, then $c_n$ is equivalent to* $(\log n)^{k-1}$.

The concept of asymmetry of an $n$-dimensional Banach space was introduced in [GG] and generalized the notion of the asymmetry of a basis. This was followed up in [GL1] as well. Here we study related notions of asymmetry and apply them to $Z_2^n$.

For general terminology concerning the geometry of Banach spaces we refer the reader to [LT1, LT2]. Terminology concerning normed ideals of operators may be found in [P] and [T].

Finally, we wish to thank Alexander Litvak for very helpful discussions.

## 2. The l.u.st. constant of $Z_2^n$

Let $Z_2^n$ be the $2n$-dimensional (real) Banach space which is the subspace of the Kalton-Peck [KP] space $Z_2$, spanned by the first $2n$ coordinates. More precisely, for $a, b \in \mathbb{R}^n$, $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n) \in \mathbb{R}^n$, we define

$$||(a,b)||_{Z_2^n} = \|(a_1, \dots, a_n, b_1, \dots, b_n)\|_{Z_2^n} = ||\sum_{j=1}^{n}(a_j e_j + b_j f_j)||_{Z_2^n}$$

$$= \left(\sum_{j=1}^{n} b_j^2\right)^{1/2} + \left(\sum_{j=1}^{n}\left(a_j - b_j \log\left(|b_j|\left(\sum_{i=1}^{n} b_i^2\right)^{-1/2}\right)\right)^2\right)^{1/2}.$$

As it is proved in [KP], this quasi-norm on $\mathbb{R}^{2n}$ is (uniformly in $n$) equivalent to a norm.

As in [JLS] we consider $Z_2^n$ as an unconditional sum

$$Z_2^n = \sum_{j=1}^{n} \bigoplus E_j, \tag{1}$$

where $E_j = [e_j, f_j]$.

It was proved in [JLS] that $Z_2$ fails to have l.u.st. They observed that if a Banach space $E$ is of the form $E = \bigoplus_{j=1}^{\infty} E_j$, a 1-unconditional sum of finite-dimensional subspaces, then

$$\mathrm{gl}(E) \leq \sup_j \dim(E_j),$$

and thus $gl(Z_2) \leq 2c$. Hence it follows that $gl(Z_2^n) \leq 2c$, but that the l.u.st. constants of $Z_2^n$ tend to infinity with $n$. Here we establish the order of growth of these constants.

**Theorem 1.** *One has $gl(Z_2^n) \leq 2c$, and the l.u.st. constant $\chi_u(Z_2^n)$ of $Z_2^n$ satisfies*

$$\chi_u(Z_2^n) \sim \log n. \tag{2}$$

Throughout the proof the letters $C$, $c$, $c_1, c_2, \ldots$, etc. will denote absolute constants which do not depend on $n$. The same letter $c$ (etc.) may denote different constants in different lines.

It is shown in [KP] that the Banach-Mazur distance $d(Z_2^n, \ell_2^{2n})$ from $Z_2^n$ to $\ell_2^{2n}$ is of the order of $\log n$. Using Proposition 2 of [JLS], there exists a Banach space $Y_n$ with a 1-unconditional basis $\{y_{j,i}, j = 1, \ldots, n, \quad i = 1, \ldots, k_j\}$, such that

- $Z_2^n$ is a subspace of $Y_n$ and, for each $1 \leq j \leq n$, $E_j \subset [\{y_{j,i}\},\ i = 1, \ldots, k_j]$.
- There exists a projection $P_n : Y_n \to Z_2^n$ such that $P_n([\{y_{j,i},\ i = 1, \ldots, k_j\}]) = E_j$ for all $j$, $1 \leq j \leq n$. ($[A]$ denotes the span of $A$.)
- $\|P_n\| = K_n \leq c\chi_u(Z_2^n)$. To justify this, one first shows (see [FJT]) that for any Banach space $E$,

$$\chi_u(E) = \inf\ ||P||_{L \to E},$$

where the infimum is taken over all Banach lattices $L$ such that $E$ is isometrically embedded in $L$ and all projections $P : L \to E$. Then observing that $E = Z_2^n$ have bounded cotype-$q$ constants for a fixed $q < \infty$, we use [R] to reduce to the case when moreover $L$ is supposed to have the same bound on its cotype-$q$ constant. Then, we follow the lines of Proposition 2 of [JLS], using Rademacher embedding.

We shall show that $K_n \geq c \log n$, thus proving (2) because it is clear that $K_n \leq d(Z_2^n, \ell_2^{2n}) \sim \log n$.

*Notation.* Let $T : Z_2^n \to Z_2^n$ be a linear operator that satisfies $T(E_j) \subset E_j$ for $1 \leq j \leq n$. We say that $T$ *splits through* $\{E_j\}$ or, for short, that $T$ *splits.* If $T$ splits, the matrices representing $T|_{E_j}$ in the basis $\{e_j, f_j\}$, $1 \leq j \leq n$, will be denoted by

$$\begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix}.$$

If $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, the *support* of $a$ is $\mathrm{supp}(a) := \{j : 1 \leq j \leq n,\ a_j \neq 0\}$. If $a, b \in \mathbb{R}^n$, $a = (a_1, \ldots, a_n)$, $b = (a_1, \ldots, b_n)$, we denote $(a, b) = \sum_1^n (a_j e_j + b_j f_j) \in Z_2^n$. We also define a vector $ab \in \mathbb{R}^n$ by $ab = (a_1 b_1, \ldots, a_n b_n)$.

**Lemma 2.** *For any $T$ that splits we have*

$$\max_j \max\{|\alpha_j|, |\beta_j|, |\gamma_j|, |\delta_j|\} \leq \|T\|.$$

*Proof.* If $x = e_j$, $1 \leq j \leq n$, then $\|Tx\| = |\gamma_j| + |\alpha_j| \leq \|T\|\|x\| = \|T\|$. If $x = f_j$, one similarly obtains $|\delta_j| + |\beta_j| \leq \|T\|$. □

**Lemma 3.** *With the preceding notation, there exists $C > 0$ such that for every $T : Z_2^n \to Z_2^n$ that splits and for all $(a,b) \in \mathbb{R}^{2n}$, we have*

$$\left(\sum_{j=1}^{n}\left(\delta_j a_j - \alpha_j a_j + \gamma_j a_j \log \frac{|a_j|}{\|a\|_2}\right)^2\right)^{\frac{1}{2}} \leq C\left(\|T\|\|(a,b)\|_{Z_2^n} + (\max_{j\in \operatorname{supp}(a)} |\gamma_j|)\|a\|_2\right).$$

*Proof.* It is established in [KP] that the function $F : \ell_2^n \to \ell_2^n$ given by

$$F(b) = \left(b_1 \log \frac{|b_1|}{\|b\|_2}, \ldots, b_n \log \frac{|b_n|}{\|b\|_2}\right) \text{ for } b = (b_1, \ldots, b_n) \in \mathbb{R}^n$$

is *quasi-linear* in the sense that for every $a, b \in \mathbb{R}^n$,

$$\|F(a+b) - F(a) - F(b)\|_2 \leq C\left(\|a\|_2 + \|b\|_2\right). \tag{3}$$

We have

$$||(\gamma F(a), \gamma a)||_{Z_2^n} = ||\gamma a||_2 + ||\gamma F(a) - F(\gamma a)||_2\,,$$

so that

$$||\gamma F(a) - F(\gamma a)||_2 \leq ||(\gamma F(a), \gamma a)||_{Z_2^n}\,.$$

Hence

$$||(\delta - \alpha)a + \gamma F(a)||_2 \leq ||(\delta - \alpha)a + F(\gamma a)||_2 + ||(\gamma F(a), \gamma a)||_{Z_2^n}. \tag{4}$$

Also, unconditionality of the sum (1) implies for every $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$,

$$\|(\theta a, \theta b)\|_{Z_2^n} \leq C \max_j |\theta_j|\ \|(a,b)\|_{Z_2^n}\,. \tag{5}$$

Defining $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^b$, $\beta$, $\gamma$ and $\delta$ related to $T$ as above, one has

$$||(a,b)||_{Z_2^n} = ||b||_2 + ||a - F(b)||_2\,,$$
$$||T(a,b)||_{Z_2^n} = ||\gamma a + \delta b||_2 + ||\alpha a + \beta b - F(\gamma a + \delta b)||_2$$

and

$$\|F(\gamma a + \delta b) - \alpha a\|_2 \leq 2\|T\|\|(a,b)\|_{Z_2^n}, \tag{6}$$

because

$$\begin{aligned}\|F(\gamma a + \delta b) - \alpha a\|_2 &\leq \|\beta b\|_2 + \|\alpha a + \beta b - F(\gamma a + \delta b)\|_2 + \|\gamma a + \delta b\|_2\\ &\leq \|T(a,b)\|_{Z_2^n} + \max_j |\beta_j|\|b\|_2 \leq 2\|T\|\|(a,b)\|_{Z_2^n}\,.\end{aligned}$$

Applying Lemma 2, (3), (5), (6) and the fact that $\|(F(a),a)\|_{Z_2^n} = \|a\|_2$, we get from (4)

$$\begin{aligned}||(\delta - \alpha)a + \gamma F(a)||_2 &\leq ||(\delta - \alpha)a + F(\gamma a)||_2 + ||(\gamma F(a), \gamma a)||_{Z_2^n}\\ &\leq \|F(\gamma a) + F(\delta b) - F(\gamma a + \delta b)\|_2 + \|F(\gamma a + \delta b) - \alpha a\|_2 + \|\delta a - F(\delta b)\|_2\\ &\quad + (\max_{j\in\operatorname{supp}(a)} \gamma_j)\|(F(a),a)\|_{Z_2^b}\\ &\leq C\left(||\gamma a||_2 + ||\delta b||_2\right) + 2\|T\|\|(a,b)\|_{Z_2^n} + 2\|T\|\|(a,b)\|_{Z_2^n} + \max_{j\in\operatorname{supp}(a)} |\gamma_j|\ \|a\|_2\\ &\leq C\left(\|T\|\|(a,b)\|_{Z_2^n} + \max_{j\in\operatorname{supp}(a)} |\gamma_j|\|a\|_2\right). \qquad \square\end{aligned}$$

**Lemma 4.** *In the context of the preceding lemmas, for every subset $A \subset \{1,\dots,n\}$, $|A| = k > 1$, one has*

$$\left(\frac{1}{k}\sum_{j\in A}\gamma_j^2\right)^{\frac{1}{2}} \le \frac{4\|T\|}{\log k}\,.$$

*Proof.* Let $x = (a,b) \in Z_2^n$, where $a = (a_1,\dots,a_n)$ and $b = (b_1,\dots,b_n)$ are given by

$$a_j = \frac{\log\frac{1}{\sqrt{k}}}{\sqrt{k}},\ b_j = \frac{1}{\sqrt{k}} \text{ if } j\in A,\ a_j = b_j = 0 \text{ otherwise}\,.$$

Then $\|x\|_{Z_2^n} = 1$ and therefore

$$\|T\| \ge \|Tx\|_{Z_2^n} \ge \left(\frac{1}{k}\sum_{j\in A}(-\gamma_j\log\sqrt{k}+\delta_j)^2\right)^{\frac{1}{2}} \ge \log\sqrt{k}\left(\frac{1}{k}\sum_{j\in A}\gamma_j^2\right)^{\frac{1}{2}} - \max_j|\delta_j|\,.$$

We now use Lemma 2 to complete the proof. □

**Corollary 5.** *In the context of the preceding lemmas, let $A \subset \{1,\dots,n\}$, $|A| = k > 1$. Then*

a) *There exists a subset $A' \subset A$, with $|A'| \ge \frac{k}{2}$, such that*

$$\max_{j\in A'}|\gamma_j| \le \frac{4\sqrt{2}\|T\|}{\log k}\,. \tag{7}$$

b) *If $\text{supp}(a) \subset A'$, where $A' \subset \{1,\dots,n\}$, $|A'| \le k$ and (7) is satisfied, then*

$$\left(\sum_{j=1}^n\left(\delta_j a_j - \alpha_j a_j + \gamma_j a_j \log\frac{|a_j|}{\|a\|_2}\right)^2\right)^{\frac{1}{2}} \le C\|T\|\|(a,b)\|_{Z_2^n}\,.$$

*Proof.* Let $C = \{j\in A; |\gamma_j| > \frac{4\sqrt{2}\|T\|}{\log k}\}$. To prove a), it is enough to show that $|C| \le k/2$. If it was not true, one would have

$$\left(\frac{1}{k}\sum_{j\in A}\gamma_j^2\right)^{\frac{1}{2}} \ge \left(\frac{1}{k}\sum_{j\in C}\gamma_j^2\right)^{\frac{1}{2}} > \frac{4||T||}{\log k}\,,$$

which contradicts Lemma 4. Part b) follows from Lemma 3 and the fact (see [KP]) that the norm of the operator $p_k : Z_2^k \to \ell_2^k$, $p_k(a,b) = a$ is equivalent to $\log k$. □

**Lemma 6.** *Let $A \subset \{1,\dots,n\}$, $|A| = k > k_0 > 1$. Then there exists a subset $A'' \subset A$, with $|A''| \ge \frac{\sqrt{k}}{3}$, such that for all $j \in A''$ we have*

$$|\gamma_j| \le \frac{C\|T\|}{(\log k)^2} \quad \textit{and} \quad |\delta_j - \alpha_j| \le \frac{C\|T\|}{\log k}\,.$$

*Proof.* Let $A' \subset A$ with $k' = |A'| \ge \frac{k}{2}$ satisfy (7). Let $x = (a,b) \in Z_2^n$ be defined as in the proof of Lemma 4, with $A'$ replacing $A$. Then, by Corollary 5b), we have (since $\|x\| = 1$)

$$\left(\frac{1}{k'}\sum_{j\in A'}(\delta_j - \alpha_j - \gamma_j\log\sqrt{k'})^2\right)^{\frac{1}{2}} \le \frac{C\|T\|}{\log\sqrt{k'}}\,.$$

As in the proof of Corollary 5, there exists a subset $B \subset A'$ with $k'' = |B| \geq \frac{k'}{2} \geq \frac{k}{4}$ such that

$$|\delta_j - \alpha_j - \gamma_j \log \sqrt{k'}| \leq \frac{\sqrt{2}C\|T\|}{\log \sqrt{k'}} = \frac{2C_1\|T\|}{\log k'} \text{ for all } j \in B. \tag{8}$$

Now take any subset $B' \subset B$ with $|B'| = k''' = \sqrt{k'} \leq \frac{k'}{2} \leq k''$ and a new vector $x \in Z_2^n$ like the one above, with $B'$ replacing $A'$ (the implicit assumption that $\sqrt{k'}$ is an integer can easily be dealt with). The same argument as above provides a subset $A'' \subset B'$ with

$$|A''| \geq \frac{k'''}{2} = \frac{\sqrt{k'}}{2} \geq \frac{\sqrt{k}}{2\sqrt{2}}$$

such that

$$|\delta_j - \alpha_j - \gamma_j \log \sqrt{k'''}| \leq \frac{\sqrt{2}C\|T\|}{\log \sqrt{k'''}} = \frac{4C_1\|T\|}{\log k'} \text{ for all } j \in A''. \tag{9}$$

Adding (8) and (9) together we get for $j \in A''$:

$$\frac{|\gamma_j| \log k'}{4} = \left| \frac{1}{2}\gamma_j \log k' - \frac{1}{4}\gamma_j \log k' \right|$$

$$= |(-\delta_j + \alpha_j + \gamma_j \log \sqrt{k'}) + (\delta_j - \alpha_j - \gamma_j \log \sqrt{k'''})| \leq \frac{6C_1\|T\|}{\log k'},$$

and thus

$$|\gamma_j| \leq \frac{24C_1\|T\|}{(\log k')^2} \leq \frac{C_2\|T\|}{(\log k)^2}. \tag{10}$$

From (9) and (10) together we get

$$|\delta_j - \alpha_j| \leq \frac{4C_1\|T\|}{\log k'} + |\gamma_j| \log \sqrt{k'''} \leq \frac{4C_1\|T\|}{\log k'} + \frac{24C_1\|T\|}{(\log k')^2} \cdot \frac{1}{4} \log k' \leq \frac{C_3\|T\|}{\log k} \text{ for } j \in A''.$$

□

*Proof of Theorem* 1. We now recall the construction from [JLS]: Under the hypotheses that $Z_2$ has l.u.st., they constructed an operator $T : Z_2 \to Z_2$ that provided a counterexample to that hypotheses. In the context of $Z_2^n$, this construction, instead of providing a counterexample, will provide a lower bound estimate of $\chi_u(Z_2^n)$.

Let the Banach space $Y_n$ with the unconditional basis $\{y_{j,i};\ j = 1, \ldots, n,\ i = 1, \ldots, k_j\}$ and the projection $P_n : Y_n \to Z_2^n$ be those presented at the opening. For a fixed $1 \leq j \leq n$ a set $A_j \subset \{1, \ldots, k_j\}$ is selected and an operator $T_{A_j} : [e_j, f_j] \to [e_j, f_j]$ is defined by

$$T_{A_j}(x) = P_n \left( \sum_{i \in A_j} y_{j,i}^*(x) y_{j,i} \right),$$

where $\{y_{j,i}^*\} \subset Y_n^*$ are the bi-orthogonal functionals of $\{y_{j,i}\}$. Thus $T_{A_j} = \sum_{i \in A_j} T_{j,i}$, where $T_{j,i}$ is the rank-1 operator $T_{j,i}(x) = P_n(y_{j,i}^*(x) y_{j,i})$. We may assume that $T_{j,i} \neq 0$, as otherwise $y_{j,i}$ can be dropped from the basis. Being a rank-1 operator, the matrix representing $T_{j,i}$ is of the form

$$\begin{pmatrix} a_i & b_i \\ \theta_i a_i & \theta_i b_i \end{pmatrix} \text{ (case I) or } \begin{pmatrix} 0 & 0 \\ a_i & b_i \end{pmatrix} \text{ (case II)}$$

(note that we dropped the index $j$ for simplicity). In case II we define $\theta_i$ to be 1. We have $\sum_{i=1}^{k_j} \theta_i b_i = 1$ (because $\sum_i T_{j,i} f_j = (\sum_i \theta_i b_i) f_j = f_j$) and $\sum_{i=1}^{k_j} |b_i| \le K_n$.

Define

$$B_j := \left\{ i \in \{1, \ldots, k_j\};\ |\theta_i| \ge \frac{1}{2K_n} \right\} .$$

We have

$$\sum_{i \in B_j} \theta_i b_i > \frac{1}{2} .$$

For each $i \in B_j$ at least one of the following four possibilities holds:

($\iota$) $\theta_i a_i > \frac{1}{4K_n} |\theta_i b_i|$,
($\iota\iota$) $-\theta_i a_i > \frac{1}{4K_n} |\theta_i b_i|$,
($\iota\iota\iota$) $a_i - \theta_i b_i \ge \frac{1}{2} |\theta_i b_i|$ ($-\theta_i b_i \ge \frac{1}{2} |\theta_i b_i|$ in case II),
($\iota\nu$) $\theta_i b_i - a_i \ge \frac{1}{2} |\theta_i b_i|$ ($\theta_i b_i \ge \frac{1}{2} |\theta_i b_i|$ in case II).

In fact, if $|\theta_i a_i| > \frac{1}{4K_n} |\theta_i b_i|$, then ($\iota$) or ($\iota\iota$) holds. Otherwise $|\theta_i a_i| \le \frac{1}{4K_n} |\theta_i b_i|$. Since $i \in B_j$ this implies in case I that

$$\frac{1}{2K_n} |a_i| \le \frac{1}{4K_n} |\theta_i b_i|,$$

which is

$$\left| \frac{a_i}{\theta_i} \right| \le \frac{1}{2} |b_i| .$$

We thus have

$$\left| \frac{a_i}{\theta_i} - b_i \right| \ge |b_i| - \left| \frac{a_i}{\theta_i} \right| \ge |b_i| - \frac{1}{2} |b_i| = \frac{1}{2} |b_i|,$$

which is either ($\iota\iota\iota$) or ($\iota\nu$). The remarks about case II are trivial.

It follows that there exists a subset $A_j \subset B_j$ such that one and the same possibility out of ($\iota$)–($\iota\nu$) holds for all $i \in A_j$ and $\sum_{i \in A_j} \theta_i b_i > \frac{1}{8}$. The operator $T : Z_2^n \to Z_2^n$ that splits and is defined by $T|_{E_j} = T_{A_j}$ satisfies $\|T\| \le K_n$. Assume that the matrix representing $T_{A_j}$ in the basis $\{e_j, f_j\}$ is, as before,

$$\begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix} .$$

From the above it follows that $\delta_j > \frac{1}{8}$ for all $j$.

If ($\iota$) or ($\iota\iota$) is satisfied for all $i \in A_j$, then we have

$$|\gamma_j| > \frac{1}{4K_n} \delta_j > \frac{1}{32K_n} . \tag{11}$$

If ($\iota\iota\iota$) or ($\iota\nu$) is satisfied for all $i \in A_j$, then we have

$$|\alpha_j - \delta_j| > \frac{1}{2} \delta_j > \frac{1}{16} . \tag{12}$$

We can now select a subset $D \subset \{1, \ldots, n\}$, with $|D| \ge \frac{n}{2}$, such that either (11) holds for all $j \in D$ or (12) holds for all $j \in D$. We take $D$ to be the set $A$ of Lemma 6. Let $A'' \subset D$ be as in Lemma 6 and $j \in A''$.

If the members of $D$ satisfy (11), then we have

$$\frac{1}{32K_n} \le |\gamma_j| \le \frac{C\|T\|}{\left(\log \frac{n}{2}\right)^2} \le \frac{C_1 K_n}{(\log n)^2},$$

and we conclude

$$K_n \geq c \log n \tag{13}$$

for some positive constant $c$.

If the members of $D$ satisfy (12), then we have

$$\frac{1}{16} \leq |\delta_j - \alpha_j| \leq \frac{C\|T\|}{\log \frac{n}{2}} \leq \frac{C_1 K_n}{\log n};$$

thus

$$K_n \geq c \log n\,. \tag{14}$$

□

## 3. The asymmetry of $Z_2^n$

**Definition.** We say that a subgroup $G$ of $GL_k(\mathbb{R})$ is *rich* if it is a compact group and that every operator in $L(\mathbb{R}^k)$ which commutes with all elements of $G$ is a scalar multiple of the identity on $\mathbb{R}^k$. It is well known that for any compact subgroup $G$ of $GL_k(\mathbb{R})$, there exists a compact subgroup $H$ of $\mathcal{O}_k$ (the orthogonal group) such that $G = \{V^{-1}hV;\, h \in H\}$, where $V \in L(\mathbb{R}^k)$ is some linear invertible operator. If $G$ is a rich subgroup of $GL_k(\mathbb{R})$, we define the *measure of symmetry* $s_G(E)$*of a* $k$*-dimensional normed space* $E$ *with respect to* $G$ by

$$s_G(E) := \int_G ||g||_{E\to E}\, d\mu(g)\,,$$

where $\mu$ is the normalized Haar measure on $G$. We define

$$s(E) = \inf\{s_G(E); G \text{ a rich subgroup}\}.$$

Note that

$$1 \leq s(E) \leq S(E) := \inf\{\sup_{g\in G} ||g||_{E\to E}; G \text{ a rich subgroup}\}\,.$$

The quantity $S(E)$, called the *asymmetry constant of* $E$, was defined originally in [GG] and further discussed in [GL1]. If $S(E) = 1$ we say that $E$ has *enough symmetries*. This is the case for spaces with 1-symmetric basis (take the rich group of permutations and changes of signs on the basis). In connection with this quantity we have the following partial result:

**Theorem 7.** *For every rich subgroup* $G$ *of* $\mathcal{O}_{2n}$ *one has*

$$s_G(Z_2^n) = \int_G ||g||_{Z_2^n\to Z_2^n}\; d\mu(g) \sim \log n \sim \max_{g\in G} ||g||_{Z_2^n\to Z_2^n}\,, \tag{15}$$

*where* $\mu$ *denotes the normalized Haar measure on* $G$.

We use the following lemma.

**Lemma 8.** *Let a couple* $E$ *and* $F$ *of* $k$*-dimensional normed spaces, an ideal norm* $\alpha$ *and a rich subgroup* $G$ *of* $GL_k(\mathbb{R})$ *be given. Then for every invertible operator* $S \in L(\mathbb{R}^k)$*, one has*

(16)

$$k \leq \alpha(S : E \to F)\ \alpha^*(S^{-1} : F \to E) \leq k \int_G ||S^{-1}gS||_{E\to E}\ \|g^{-1}||_{F\to F}\ d\mu(g),$$

*where $\alpha^*$ denotes the conjugate ideal norm of $\alpha$. In particular, if all elements of $G$ are isometries of $F$, then*

$$k \le \alpha(S : E \to F)\ \alpha^*(S^{-1} : F \to E) \le k \int_G ||S^{-1}gS||_{E\to E}\ \ d\mu(g). \tag{17}$$

*Proof.* We first prove for any $k$-dimensional normed spaces $D$ and $F$ on $\mathbb{R}^n$ that if $I : \mathbb{R}^k \to \mathbb{R}^k$ denotes the identity operator, then

$$k \le \alpha(I : D \to F)\ \alpha^*(I^{-1} : F \to D) \le k \int_G ||g||_{D\to D}\ ||g^{-1}||_{F\to F}\ d\mu(g). \tag{18}$$

By the definition of $\alpha^*$, there exists $U \in L(\mathbb{R}^k)$ such that

$$\alpha(U : D \to F)\alpha^*(I^{-1} : F \to D) = \operatorname{trace}(UI^{-1}) = \operatorname{trace}(U)\,.$$

Let $V = \int_G g^{-1}Ug\ d\mu(g)$ be considered as an operator from $D$ to $F$. Since $G$ is a rich subgroup, one has $V = cI$ for some $c \in \mathbb{R}$, so that $\operatorname{trace}(U) = \operatorname{trace}(V) = kc$. One clearly has

$$\begin{aligned} c\alpha(I : D \to F) = \alpha(V : D \to F) &\le \int_G \alpha(g^{-1}Ug : D \to F)\ d\mu(g) \\ &\le\ \alpha(U : D \to F)\ \int_G \|g^{-1}\|_{F\to F}\|g\|_{D\to D}\ d\mu(g)\,. \end{aligned}$$

It follows that

$$\begin{aligned} &\alpha(I : D \to F)\ \alpha^*(I^{-1} : F \to D) \\ &\le \frac{\alpha(U : D \to F)\alpha^*(I^{-1} : F \to D)}{c} \int_G\ \|g\|_{D\to D}\|g^{-1}\|_{F\to F}\ \ d\mu(g) \\ &= k \int_G\ \|g\|_{D\to D}\ \|g^{-1}\|_{F\to F}\ d\mu(g)\,. \end{aligned}$$

This proves (18). We define a normed space $D$ by setting $||x||_D = ||S^{-1}x||_E$. Then $S^{-1} : D \to E$ and $S : E \to D$ are isometries, and one has

$$\alpha(I : D \to F) = \alpha(S : E \to F)\ ,\ \alpha^*(I^{-1} : F \to D) = \alpha^*(S^{-1} : F \to E)$$
$$\text{and } \|g\|_{D\to D} = ||S^{-1}gS||_{E\to E}.$$

Together with (18) this implies (16) . □

*Proof of Theorem* 7. We shall use Lemma 8 with $S = I : Z_2^n \to \ell_2^{2n}$, where $I$ is the identity sending the basis $e_j, f_j$ $1 \le j \le n$ on the canonical basis of $\ell_2^{2n}$. Here we present two proofs: **A** and **B**.

**A.** Let $\alpha = \Pi_1$ and $\alpha^* = \gamma_\infty$, the ideal norm of factorization through $L_\infty$.

**a.** We claim that

$$\Pi_1(I : Z_2^n \to \ell_2^{2n}) \ge c\sqrt{n}\log n.$$

In fact, define, for $1 \le j \le n$, $x_j \in Z_2^n$ by

$$x_j = \frac{1}{\sqrt{n}}\left(\log\frac{1}{\sqrt{n}}\right)e_j + \frac{1}{\sqrt{n}}\,f_j.$$

Then $||I(x_j)||_2 \sim \frac{\log n}{\sqrt{n}}$, and for every choice of signs $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$, one has

$$||\sum_{j=1}^{n} \varepsilon_j x_j||_{Z_2^n} = 1.$$

Hence

$$\Pi_1(I : Z_2^n \to \ell_2^{2n}) \geq \frac{\sum_{j=1}^n ||x_j||_2}{\sup_\varepsilon ||\sum_{j=1}^n \varepsilon_j x_j||_{Z_2^n}} \geq c\sqrt{n}\log n.$$

**b.** Define $P : Z_2^n \to \ell_2^n$ by $P(a, b) = b$. Then $||P||_{Z_2^n \to \ell_2^n} = 1$, and if $W : \ell_2^n \to \ell_2^{2n}$ is the embedding defined by $W(b) = (0, b)$, then

$$\begin{aligned}\gamma_\infty(I^{-1} : \ell_2^{2n} \to Z_2^n) &= ||P||_{Z_2^n \to \ell_2^n} \;\; \gamma_\infty(I^{-1} : \ell_2^{2n} \to Z_2^n)||W||_{\ell_2^n \to \ell_2^{2n}} \\ &\geq \gamma_\infty(PI^{-1}W : \ell_2^n \to \ell_2^n) \sim \sqrt{n}\,,\end{aligned}$$

because $PI^{-1}W : \ell_2^n \to \ell_2^n$ is actually the identity mapping on $\ell_2^n$.

**c.** It follows that

$$\Pi_1(I : Z_2^n \to \ell_2^{2n})\gamma_\infty(I^{-1} : \ell_2^{2n} \to Z_2^n) \geq cn\log n\,.$$

**B.** Let $\alpha$ be the operator norm; then $\alpha^* = i_1$, the integral norm. It is easy to see that

$$||I||_{Z_2^n \to \ell_2^{2n}} \sim \log n$$

and

$$\begin{aligned}i_1(I^{-1} : \ell_2^{2n} \to Z_2^n) &= i_1(I^{-1} : \ell_2^{2n} \to Z_2^n)||P||_{Z_2^n \to \ell_2^n} \\ &\geq i_1(PI^{-1} : \ell_2^{2n} \to \ell_2^n) = \sup_{S \neq 0} \frac{\text{trace}(PI^{-1}S)}{||S||_{\ell_2^n \to \ell_2^{2n}}} \geq n\,.\end{aligned}$$

It follows that

$$||I||_{Z_2^n \to \ell_2^{2n}} \;\; i_1(I^{-1} : \ell_2^{2n} \to Z_2^n) \geq cn\log n\,.$$

Using either **A** or **B**, together with Lemma 8 with $F = \ell_2^{2n}$, $E = Z_2^n$ and $S = I$, we get that

$$\int_G ||g||_{Z_2^n \to Z_2^n} \; d\mu(g) \geq c\log n\,.$$

It follows that

$$\log n \leq \frac{1}{c}\int_G ||g||_{Z_2^n \to Z_2^n} \; d\mu(g) \leq \frac{1}{c}\max_{g \in G} ||g||_{Z_2^n \to Z_2^n} \leq d(Z_2^n, \ell_2^{2n}) \sim \log n,$$

where here $d$ denotes the Banach-Mazur distance. This proves (15). □

*Remark.* One can prove in the same way as **A.a.** in the proof of the last theorem that if $I : Z_2^n \to \ell_\infty^{2n}$ is the identity mapping sending the basis $(e_j, f_j), 1 \leq j \leq n$, on the canonical basis of $\ell_\infty^{2n}$, then

$$\Pi_1(I : Z_2^n \to \ell_\infty^{2n}) \geq c\sqrt{n}\log n\,.$$

We also have that

$$\gamma_\infty(I^{-1} : \ell_\infty^{2n} \to Z_2^n) = ||I^{-1}||_{\ell_\infty^{2n} \to Z_2^n} \sim \sqrt{n}\log n\,.$$

It follows from Lemma 8 that for every rich subgroup $G$ of $GL_{2n}(\mathbb{R})$, one has

$$cn(\log n)^2 \le \Pi_1(I : Z_2^n \to \ell_\infty^{2n})\gamma_\infty(I^{-1} : \ell_\infty^{2n} \to Z_2^n)$$
$$\le n \int_G ||g^{-1}||_{\ell_\infty^{2n}\to\ell_\infty^{2n}} ||g||_{Z_2^n \to Z_2^n} \, d\mu(g) \, .$$

The first inequality is actually an equivalence, in view of the fact that if one takes for $G$ the group of changes of signs and permutaions of indices of the basis, one has $||g^{-1}||_{\ell_\infty^{2n}\to\ell_\infty^{2n}} = 1$ for all $g \in G$ and

$$\int_G ||g||_{Z_2^n \to Z_2^n} \, d\mu(g) \le \|I : Z_2^n \to \ell_\infty^{2n}\| \|I^{-1} : \ell_\infty^{2n} \to Z_2^n\| \le c \, (\log n)^2 \, ,$$

as $\|I : Z_2^n \to \ell_\infty^{2n}\| \sim \log n$.

**Added in proof.** After the paper was processed, we realized that answers to some of the questions posed at the end of the introduction had already been given in the paper [KT]. In particular, examples of $n$-dimensional Banach spaces with bounded GL-constant and l.u.st constant greater than $cn^\alpha$ (with $\alpha$ as large as 1/6) are given there.

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