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A REMARK ON THE MAXIMAL OPERATOR FOR RADIAL MEASURES

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ABSTRACT. The purpose of this paper is to prove that there exist measures $d\mu(x) = \gamma(x)dx$, with $\gamma(x) = \gamma_0(|x|)$ and γ_0 being a decreasing and positive function, such that the Hardy-Littlewood maximal operator, $\mathcal{M}_\mu$, associated to the measure μ does not map $L^p_\mu(\mathbb{R}^n)$ into weak $L^p_\mu(\mathbb{R}^n)$, for every $p < \infty$. This result answers an open question of P. Sjögren and F. Soria.

1. STATEMENT OF RESULTS

Let μ be a non-negative measure in $\mathbb{R}^n$, finite on compact sets. If $f \in L^1_{loc}(\mu)$, we define the maximal operator

$$\mathcal{M}_\mu f(x) = \sup \frac{1}{\mu(B)} \int_B |f| d\mu,$$

where the supremum is taken over all balls containing the point x. If we consider only balls centered at x, we obtain that the operator-associated maps from $L^1(d\mu)$ into $L^{1,\infty}(d\mu)$ are said to be of weak type (1,1). This can be proved using the Besicovitch covering lemma. If μ is a doubling measure, then $\mathcal{M}_\mu$ is of weak type (1,1). This can be proved using the Vitali covering lemma. For $n \geq 2$, if $d\mu(x) = e^{-|x|^2}dx$, then $\mathcal{M}_\mu$ does not map $L^1(d\mu)$ into weak $L^1(d\mu)$; see [2]. A. Vargas in [4] showed that $\mathcal{M}_\mu$ being of weak type (1,1) is equivalent to μ being a doubling measure away from the origin. In [1] it is shown that the operator associated with the measure $d\mu(x) = e^{-|x|^2}dx$ is bounded on L^p_μ, $n \geq 2$, for $1 < p \leq \infty$. P. Sjögren and F. Soria [3] proved, for a family of absolutely continuous measure with respect to the Lebesgue measure $d\mu(x) = \gamma(x)dx$, that the maximal operator associated to $\mathcal{M}_\mu$ is bounded on the Lebesgue space L^p_μ, $p < \infty$. These measures are such that $\gamma(x)$ is a radial function, where $\gamma(x) = \gamma_0(|x|)$ with γ_0 strictly decreasing, continuous and $\lim_{t\to 0} \gamma_0(t) < \infty$. For this measure the function $\phi : (0, \infty) \to (0, \infty)$ is defined by

$$\gamma_0(t + \phi(t)) = \frac{1}{2}\gamma_0(t).$$

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The result of P. Sjögren and F. Soria establishes that if $\frac{\phi(t)}{t}$ decreases on the interval $(0, \infty)$ and

$$\lim_{t\to\infty} \frac{\phi(t)}{t} = 0,$$

then $\mathcal{M}_\mu$ is bounded on L^p_μ for $1 < p \leq \infty$.

The aim of this paper is to answer the question left open by P. Sjögren and F. Soria whether the monotonicity condition on $\frac{\phi(t)}{t}$ can be eliminated. This paper gives a negative answer to this question, by giving an example where the operator is not of weak type (p, p), for any $p < \infty$.

There are measures μ for which $\mathcal{M}_\mu$ is only bounded on L^∞. An example mentioned in the work of Vargas [4] is $d\mu = dx + d\sigma$, namely the Lebesgue measure plus the area measure over the unitary sphere S^{n-1}. We shall now verify the properties of this example in dimension two.

We observe that for a ball B^1 of radius r which does not intersect S^1,

$$\mu(B^1) \sim r^2.$$

If the ball B^2 of radius $r \ll 1$ is such that its center is in S^1, then

$$\mu(B^2) \sim r^2 + r \sim r \qquad \text{if } r \ll 1.$$

Let B_r be a fixed ball of radius $r \ll 1$ and center x_0, with $|x_0| = 1 + 2r$. We define the function

$$f(x) = \chi_{B_r}(x).$$

Given $\epsilon > 0$, with $\epsilon \ll r$ chosen later, we consider the family

$$\mathcal{A}_\epsilon = \left\{ \widetilde{B} \text{ ball} : \ B_r \subset \widetilde{B},\ |x_{\widetilde{B}}| = 1 + 2r,\ r_{\widetilde{B}} = 2r + \epsilon \right\},$$

where $r_{\widetilde{B}}$ is the radius of $\widetilde{B}$ and $x_{\widetilde{B}}$ its center. If $\widetilde{B} \in \mathcal{A}_\epsilon$, then $\mu(\widetilde{B}) \sim r^2 + \sqrt{\epsilon r}$ and we have

$$\frac{1}{\mu(\widetilde{B})} \int_{\widetilde{B}} f(y) d\mu(y) = \frac{\mu(B_r)}{\mu(\widetilde{B})} \sim \frac{r^2}{r^2 + \sqrt{\epsilon r}}.$$

Taking $\epsilon < r^3$ and defining $D = \bigcup_{\widetilde{B} \in \mathcal{A}_\epsilon} \widetilde{B}$ we have $D \subset \{x : \mathcal{M}_\mu f(x) \geq C_0\}$ for some C_0 independent of ϵ and r. Besides, $\mu(D) \sim r$.

If $\mathcal{M}_\mu$ would be of weak type (p, p), for some $p \in [1, \infty)$, we would obtain some constant $c = c_p$ such that

$$r \sim \mu(D) \leq \frac{c}{C_0^p} \int |f|^p d\mu = \frac{c\mu(B)}{C_0^p} \sim r^2.$$

This is a contradiction for $r \to 0$. This says that $\mathcal{M}_\mu$ is not of weak type (p, p) for any $p < \infty$.

In the case of a radial and decreasing measure, $d\mu(x) = \gamma_0(|x|)dx$, we have

(1) The Sjögren and Soria theorem states that if $\frac{\phi(t)}{t}$ decreases to zero, then $\mathcal{M}_\mu : L^p_\mu(\mathbb{R}^n) \to L^p_\mu(\mathbb{R}^n)$, for every $p > 1$.

(2) Otherwise, if there is a constant $C > 0$ such that $\frac{\phi(t)}{t} > C$, for every $t > t_0$, with t_0 large, then $\mathcal{M}_\mu : L^1_\mu(\mathbb{R}^n) \to L^{1,\infty}_\mu(\mathbb{R}^n)$, because the measure μ would be doubling away from the origin and the result of Vargas [4] could be used.

(3) It remains to be answered what happens when $\frac{\phi(t)}{t}$ tends to zero but not in a monotone way.

In the example, $d\mu = dx + d\sigma$ is not a decreasing measure, and thus arises the question, stated by Sjögren and Soria in [3], if the monotonicity of the measure μ is enough to conclude that the operator $\mathcal{M}_\mu$ is bounded in some space $L^p_\mu(\mathbb{R}^n)$, with $p \neq \infty$.

However, it can be proved that there is a γ_0 that decreases to zero, such that $\frac{\phi(t)}{t} \to 0$ when $t \to \infty$, but $\mathcal{M}_\mu$ does not map $L^p_\mu(\mathbb{R}^n)$ into weak $L^p_\mu(\mathbb{R}^n)$, for any $p < \infty$. This result, which answers (3), is stated in the following theorem.

Theorem 1.1. *There is a radial and decreasing measure $d\mu(x) = \gamma(x)dx$, such that $\lim_{t\to\infty} \frac{\phi(t)}{t} = 0$, and the maximal associated operator $\mathcal{M}_\mu$ does not map $L^p_\mu(\mathbb{R}^n)$ into weak $L^p_\mu(\mathbb{R}^n)$, for any $p \neq \infty$.*

This result has also been obtained independently by Liljendahl.

2. Proof of Theorem 1.1. (Counterexample)

We define the density function γ by

$$\gamma(x) = \gamma_0(|x|) = \sum_{k=0}^{\infty} h_{k+1}\chi_{I_k}(|x|),$$

where I_k is the interval $[a_k, a_{k+1})$, with $a_{k+1} \geq 4 + a_k$ such that the annulus

$$\{y \in \mathbb{R}^n \colon a_k \leq |y| < a_{k+1}\}$$

contains a ball of radius 2 and $\{h_k\}_{k=1}^{\infty}$ is a sequence of decreasing numbers with $h_{k+1} \geq 2h_{k+2}$.

Observe that γ_0 defined in that way is not strictly decreasing; nevertheless, via linear interpolation between the points $(a_k, h_{k+1} + \epsilon_k)$ and (a_{k+1}, h_{k+1}), it will become strictly decreasing. This change, taking ϵ_k sufficiently small, does not modify the calculations that will be made, including the value of $\phi(t)$.

The numbers a_k can be chosen such that $\lim_{t\to\infty} \frac{\phi(t)}{t} = 0$, for example $a_k = k^2$. In fact, in this case $\phi(t) \sim a_{k+1} - t$ for $t \in I_k$ and it can be verified that

$$\frac{\phi(t)}{t} \sim \frac{a_{k+1} - t}{t} \leq C\frac{a_{k+1} - a_k}{a_k} \sim \frac{(k+1)^2 - k^2}{k^2} \sim \frac{2k+1}{k^2} \to 0.$$

Suppose that the sequence $\{h_k\}_{k=1}^{\infty}$ satisfies $\sup_k \frac{h_k}{h_{k+1}} = \infty$. (For example, we can take $h_k = e^{-k^2}$.)

Given k, let B_k be the ball of radius 1 and center on the point $(a_k + 2, 0, \ldots, 0)$. Consider the function

$$f(x) = \chi_{B_k}(x).$$

Observe that $\mu(B_k) \sim h_{k+1}$.

Given $0 < \epsilon < 1$, the set $\mathcal{B}$ is defined by

$$\mathcal{B} = \{B \text{ ball: } B_k \subset B,\ r_B = 2 + \epsilon,\ |x_B| = a_k + 2\},$$

where r_B and x_B denote the radius and the center of B respectively.

If $E = \bigcup_{B\in\mathcal{B}} B \cap \{y : |y| \leq a_k\}$, then it can be deduced that $\mu(E) \sim h_k\epsilon$. For every $B \in \mathcal{B}$ we can verify that

$$\mu(B) \sim h_{k+1} + h_k\epsilon^{\frac{n+1}{2}}.$$

For $x \in B \in \mathcal{B}$ we have

$$\begin{aligned} \mathcal{M}_\mu f(x) &\geq \frac{1}{\mu(B)} \int_B |f| d\mu = \frac{\mu(B_k)}{\mu(B)} = C(\epsilon) \\ &\sim \frac{h_{k+1}}{h_{k+1} + h_k \epsilon^{\frac{n+1}{2}}} \\ &\sim \frac{1}{1 + \left(\frac{h_k}{h_{k+1}}\right) \epsilon^{\frac{n+1}{2}}}. \end{aligned}$$

Defining $D = \bigcup_{B \in \mathcal{B}} B$, we have

$$D \subset \{x \colon \mathcal{M}_\mu f(x) \geq C(\epsilon)\}.$$

As $\mu(D) \sim h_k\epsilon + h_{k+1}$, if we suppose that $\mathcal{M}_\mu$ is of weak type (p,p) for some p, we obtain

$$h_k \epsilon + h_{k+1} \sim \mu(D) \leq \frac{C_p}{[C(\epsilon)]^p} \int |f|^p d\mu = \frac{C_p}{[C(\epsilon)]^p} \mu(B_k).$$

Taking $\epsilon = \left(\frac{h_{k+1}}{h_k}\right)^{\left(\frac{2}{n+1}\right)}$, we have $C(\epsilon) \sim 1$ and so

$$h_k \epsilon + h_{k+1} \leq \frac{C_p}{C^p(\epsilon)} \mu(B_k) \sim C'_p h_{k+1}.$$

This implies that $\frac{h_k}{h_{k+1}} \epsilon \leq C'_p$. So

$$\left(\frac{h_k}{h_{k+1}}\right)^{1 - \frac{2}{n+1}} = \left(\frac{h_k}{h_{k+1}}\right)^{\frac{n-1}{n+1}} \leq C'_p.$$

This contradicts the condition $\sup_k \frac{h_k}{h_{k+1}} = \infty$, except when $n = 1$, and finishes the proof of Theorem 1.1.

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