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WEIGHTED SOBOLEV INEQUALITIES UNDER LOWER RICCI CURVATURE BOUNDS

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Abstract. We obtain sharp weighted Poincaré and Sobolev inequalities over complete, noncompact Riemannian manifolds with polynomial volume growth and a quadratically decaying lower bound on Ricci. This improves and extends earlier work of Tian-Yau and Minerbe. We deduce a sharp existence result for bounded solutions of the Poisson equation on such manifolds, highlighting the well-known distinction between spaces of volume growth ≤ 2 and > 2 in terms of their Green's functions. We also show that if the manifold is nonparabolic and carries a smooth function which behaves like the radius function of a cone, then these solutions almost decay at the rates expected from a cone.

1. Introduction

Let (M, g) be a complete, noncompact Riemannian manifold of dimension $n > 2$. We assume throughout that M satisfies a fairly mild and natural polynomial growth condition, SOB(β) (see Definition 1.1), which is supposed to capture behavior often found in Ricci-flat spaces, including the property that $|B(x_0, r)| \sim r^\beta$ for some base point $x_0 \in M$, all $r \gg 1$, and some fixed $\beta > 0$, which need not be an integer.

Our main point is to show that, under this condition, there exist global weighted Poincaré and Sobolev inequalities for functions on M, with weights that are sharp already on simple model spaces such as flat $\mathbb{R}^\beta \times T^{n-\beta}$. Moreover, we demonstrate that these inequalities nicely capture some basic aspects of potential theory on M, in particular, the distinction between parabolic and not. More specifically, we will construct uniformly bounded solutions u to $\Delta u = f$ whenever $|f| \leq Cr^{-\mu}$ for some $\mu > 2$ (and in addition $\int f = 0$ in the parabolic case, $\beta \leq 2$), and we will show that $|u| \leq C_\varepsilon r^{2-\mu+\varepsilon}$ for every $\varepsilon > 0$ if $2 < \mu < \beta$ and if in addition r is comparable with a smooth positive function which behaves like the radius function on a cone. This meshes well with the Green's function estimates proved by Li and Tam [13].

Being based on Moser iteration, these results also apply to other equations with a reasonable divergence structure, notably to the complex Monge-Ampère equation if M is Kähler. This problem was discussed by Tian and Yau [19, 20], using various ad-hoc weighted Sobolev inequalities. For example, if M is asymptotic to a minimal cone in some $\mathbb{R}^N$, cf. [20], they apply the Michael-Simon inequality [15] to conclude that M has the same Sobolev inequality as $\mathbb{R}^n$. It is then natural to wonder whether the minimal surface property is responsible for this or really only the asymptotic

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conicality. Our initial motivation was to find a simple framework for dealing with such questions, and thus to strengthen the Tian-Yau existence results.

Definition 1.1. (M,g) is called SOB(β) if there exist $x_0 \in M$ and $C \geq 1$ such that $B(x_0,s)\setminus \bar{B}(x_0,t)$ is connected for all $s > t \geq C$, $|B(x_0,s)| \leq Cs^\beta$ for all $s \geq C$, and $|B(x,(1-\frac{1}{C})r(x))| \geq \frac{1}{C}r(x)^\beta$ and $\mathrm{Ric}(x) \geq -Cr(x)^{-2}$ if $r(x) := \mathrm{dist}(x_0,x) \geq C$.

Theorem 1.2. *Suppose that (M,g) satisfies* SOB(β) *for some $\beta \in \mathbb{R}^+$.*

(i) *For all $\varepsilon > 0$ there exists a positive step function $\psi_\varepsilon \sim (1+r)^{-\max\{\beta,2\}-\varepsilon}$ on M such that for all $\alpha \in [1, \frac{n}{n-2}]$ and all $u \in C_0^\infty(M)$,*

$$\left(\int_M |u-u_\varepsilon|^{2\alpha}\,(1+r)^{\alpha(\min\{\beta-2,0\}-\varepsilon)-\beta}\,d\mathrm{vol}\right)^{\frac{1}{\alpha}} \leq C_\varepsilon \int_M |\nabla u|^2\,d\mathrm{vol}, \tag{1.1}$$

where u_ε denotes the average of u with respect to the finite measure $\psi_\varepsilon\,d\mathrm{vol}$.

(ii) *If $\beta > 2$, then, for all $\alpha \in [1, \frac{n}{n-2}]$ and all $u \in C_0^\infty(M)$,*

$$\left(\int_M |u|^{2\alpha}\,(1+r)^{\alpha(\beta-2)-\beta} d\mathrm{vol}\right)^{\frac{1}{\alpha}} \leq C \int_M |\nabla u|^2\,d\mathrm{vol}. \tag{1.2}$$

Corollary 1.3. *Let (M,g) be such that there exist a compact set $K \subset M$, a compact manifold (N,h), a diffeomorphism $\Phi : (1,\infty)\times N \to M\setminus K$, and $C \geq 1$ such that $\frac{1}{C}g_{\mathrm{cone}} \leq \Phi^* g \leq C g_{\mathrm{cone}}$ for the metric $g_{\mathrm{cone}} = dt^2 \oplus t^2 h$ on $\mathbb{R}^+ \times N$. Then*

$$\left(\int_M |u|^{2\alpha}(1+r)^{\alpha(n-2)-n}\,d\mathrm{vol}\right)^{\frac{1}{\alpha}} \leq C \int_M |\nabla u|^2\,d\mathrm{vol} \tag{1.3}$$

for all $u \in C_0^\infty(M)$ and $\alpha \in [1, \frac{n}{n-2}]$.

Tian-Yau [19] obtain an estimate such as (1.1) in a rather more flexible setting, but with weaker exponents that may depend on lower bounds for $|B(x,1)|$, $x \to \infty$, and (1.2) overlaps with results of Minerbe [16] for $\mathrm{Ric} \geq 0$, one of whose methods, originating from [7], we borrow. The novelty in our approach here is that we revisit foundational work on isoperimetry due to Gromov [8] to first prove a sharp Sobolev inequality with Dirichlet boundary conditions on annuli (Corollary 2.6(ii)). We can then apply a patching scheme based on [7] and [16]. Recent, independent work due to van Coevering [21] contains a proof of (1.3) via *scaling* and patching.

Remark 1.4. SOB(β) and (1.2) are closely related to nonparabolicity. Specifically, SOB(β) implies Li-Tam's condition (VC) from [13]. Thus, from their results, M is nonparabolic if and only if $\beta > 2$, in which case all Green's functions are $\sim r^{2-\beta}$ at infinity. Also, from Carron [3], a complete Riemannian manifold M is nonparabolic if and only if there exists $\varphi > 0$ with $\int u^2\varphi \leq C\int |\nabla u|^2$ for all $u \in C_0^\infty(M)$.

Suppose in addition that $|\mathrm{Rm}| \leq C$. Thus, for all $x \in M$ there exists a smooth covering map Φ_x from the unit ball $B \subset \mathbb{R}^n$ onto a neighborhood of x, $\Phi_x(0) = x$, with $\frac{1}{C}g_{\mathbb{R}^n} \leq \Phi_x^* g \leq C g_{\mathbb{R}^n}$ and $\|\Phi_x^* g\|_{C^{1,\alpha}(B)} \leq C$; see Petersen [17, Theorem 4.1]. This enables us to work with the Laplacian in the global Hölder space $C^{2,\alpha}(M)$.

Theorem 1.5. *Let $f \in C^{0,\alpha}(M)$ satisfy $|f| \leq Cr^{-\mu}$ on $\{r > 1\}$ for some $\mu > 2$. If $\beta \leq 2$, then assume in addition that $\int f\,d\mathrm{vol} = 0$. Then there exists a $u \in C^{2,\alpha}(M)$ such that $\Delta u = f$. If $\beta \leq 2$, then moreover $\int |\nabla u|^2\,d\mathrm{vol} < \infty$.*

Theorem 1.6. *Suppose there exists a smooth $\rho \sim 1+r$ with $|\nabla\rho| + \rho|\Delta\rho| \leq C$. If $2 < \mu < \beta$, then the above solution u satisfies $|u| \leq C_\varepsilon \rho^{2-\mu+\varepsilon}$ for all $\varepsilon > 0$.*

We understand Theorems 1.5 and 1.6 as saying that the Poisson equation on M largely behaves like the Poisson equation on $\mathbb{R}^\beta$, at least for $\beta \in \mathbb{N}$. Indeed, if $\beta > 2$, then, by [13], all Green's functions $G(x_0, x) \sim_{x_0} \mathrm{dist}(x_0, x)^{2-\beta}$ if $\mathrm{dist}(x_0, x) \gg_{x_0} 1$, whereas in the parabolic case, $\beta \leq 2$, there exist unbounded, sign-changing Green's functions of $\log r$ and $r^{2-\beta}$ growth if $\beta = 2$ and $\beta < 2$, respectively. In this case, the condition $\int f = 0$ is necessary to obtain a bounded solution, even if $f \in C_0^\infty(M)$.

If M is Kähler of complex dimension m with Kähler form ω, then these theorems have close analogs for the complex Monge-Ampère equation $(\omega + i\partial\bar{\partial}u)^m = e^f\omega^m$. The only differences are that the lowest regularity we can deal with now is $f \in C^{2,\alpha}$ $\Rightarrow u \in C^{4,\bar{\alpha}}$ for some $\bar{\alpha} \in (0, \alpha]$ if $|\mathrm{Rm}| + |\nabla \mathrm{Scal}| + |\nabla^2 \mathrm{Scal}| \leq C$, that the correct integrability condition to impose when $\beta \leq 2$ is $\int (e^f - 1)\omega^m = 0$, and that we need to assume $\rho|\partial\bar{\partial}\rho| \leq C$ in the counterpart of Theorem 1.6. Altogether, this answers, in some generality, a question raised by Tian-Yau [19, p. 581].

Example 1.7. (i) The conditions of Theorems 1.2 and 1.5 are satisfied if, outside a compact subset, M is isometric to the product of a Riemannian cone of dimension $\beta \in \{2, ..., n\}$, or a half-line, and a closed manifold with $\mathrm{Ric} \geq 0$. If $\beta > 2$, Theorem 1.6 applies as well, with ρ given by the radius function on the cone factor. We refer to the Ricci-flat manifolds constructed in [11, 19] for some more involved examples of a roughly similar flavor, with $\beta \leq 2$, and $\beta \in \mathbb{Q} \setminus \mathbb{Z}$ in most cases. Their tangent cones at infinity are again Riemannian cones.

(ii) Very recently, Hattori [9] examined the geometry of complete 4-dimensional hyperkähler manifolds obtained from an infinite Gibbons-Hawking ansatz. In one of these examples, we have a 4-manifold M whose $H_2(M, \mathbb{Z})$ is not finitely generated. Fix $\sigma \in \mathbb{R}^+$. M carries an S^1-action with quotient map $p : M \to \mathbb{R}^3$, and an explicit complete hyperkähler metric g of bounded curvature, such that p is a Riemannian submersion from (M, g) onto $(\mathbb{R}^3, Fg_{\mathbb{R}^3})$, where $F : \mathbb{R}^3 \to (0, \infty]$ is given by

$$F(x) = \sum_{n \in \mathbb{Z}} \frac{1}{|x - n|n|^\sigma v|}$$

for some fixed unit vector v. The fiber $p^{-1}(x)$ has length proportional to $F(x)^{-1/2}$. M satisfies $\mathrm{SOB}(4 - \frac{2}{2+\sigma})$ by [9, Theorem 1.1], and a candidate for ρ would be

$$\rho(x)^2 = 1 + \sum_{n \in \mathbb{Z}} \frac{|x|^2}{|x| + |n|^{1+\sigma}}.$$

It is easy to check that this satisfies the required bounds if $|\langle x, v\rangle| \leq \gamma|x|$ for a fixed $\gamma \in (0, 1)$, but I do not know if the constants must blow up as $\gamma \to 1$. The proof of Theorem 1.6 can be localized to yield $|u| \leq C_{\varepsilon,\gamma}\rho^{2-\mu+\varepsilon}$ in this region. It would be very interesting to see whether or not this can be made independent of γ.

Notation. We write $|X|$ for the Hausdorff measure of $X \subset M$ in the right dimension, u_X for the average and $\|u\|_{X,p}$ for the L^p norm of u over X if X is open, $\|u\|_{M,p} = \|u\|_p$, $B = B(x, r)$ for balls, $A = A(x, r, s)$ for metric annuli, $\lambda B = B(x, \lambda r)$ if $\lambda > 0$ and $\lambda A = A(x, \lambda^{-1}r, \lambda s)$ if $\lambda > 1$. For $\delta \geq 0$, $v_\delta(r) = |B(r)|$ and $a_\delta(r) = |\partial B(r)|$ in the model space of constant curvature $-\delta^2$. Finally, $\alpha_{np} = \frac{n}{n-p}$ if $1 \leq p < n$.

2. Weighted Sobolev inequalities

The key to the proof is a volume comparison method introduced by Gromov [8]. Buser [2], Maheux and Saloff-Coste [14], and Cheeger-Colding [4] have applied this

method to derive various L^p Poincaré and Sobolev inequalities of Neumann type, for the most part on geodesic balls, and there is a Dirichlet-type L^1 Sobolev inequality for geodesic balls, due to Anderson [1], based on the same principle. In Sections 2.1 and 2.2, we show that the same idea also yields Dirichlet-type inequalities on more general domains, especially on annuli (Corollary 2.6). This quickly yields a global Sobolev inequality under lower Ricci bounds (Corollary 2.8), which implies Gallot's inequality [6] on compact manifolds. In Section 2.3, we then combine Corollary 2.6 with patching methods from Grigor'yan and Saloff-Coste [7] and Minerbe [16], and with the Cheeger-Colding segment inequality [4], to prove Theorem 1.2.

2.1. **Isoperimetric estimates à la Gromov and Anderson.** Let M denote an n-dimensional Riemannian manifold without boundary, which may be incomplete. The goal of this section is to prove a general estimate (Lemma 2.2) for the volume of a domain in M in terms of the area of its boundary, based on an idea of Gromov [8, §6.C, Appendix C] and Anderson [1, Section 4].

Definition 2.1. Let X be a metric-measure space, $Y \subset X$. Let $\mathcal{B}$ be a covering of Y by metric balls. We say that $\mathcal{B}$ is (ε, r_0)-*good*, $0 < \varepsilon \leq \frac{1}{2}$, $r_0 > 0$, if $\mathrm{center}(B) \in Y$, $\min\{|B \setminus Y|, |B \cap Y|\} \geq \varepsilon |B|$, and $\mathrm{radius}(B) \leq r_0$ for all $B \in \mathcal{B}$.

Lemma 2.2. *Let $\Omega \subset M$ be open with a smooth boundary. Let $\mathcal{B}$ be an (ε, r_0)-good covering of Ω. Let N be the open $5r_0$-neighborhood of Ω, and assume $\bar{N}$ is compact. If* $\mathrm{Ric} \geq -(n-1)\delta^2$ *on N with $0 \leq r_0\delta \leq \Lambda$, and if $\alpha \geq 1$, then*

$$\frac{|\Omega|^{\frac{1}{\alpha}}}{|\partial\Omega|} \leq C(n, \varepsilon, \Lambda) \sup_{B \in \mathcal{B}} |B|^{\frac{1}{\alpha}-1} \mathrm{radius}(B). \tag{2.1}$$

Proof. Choose finitely many balls $B_i = B(x_i, r_i) \in \mathcal{B}$, $i = 1, ..., k$, such that the $2B_i$ are pairwise disjoint but still $\Omega \subset \bigcup 5B_i$. This can be achieved through a standard Vitali type procedure: Since $\bar{\Omega}$ is compact and $B \setminus \Omega \neq \emptyset$ for all $B \in \mathcal{B}$, there exists a finite subcollection $\mathcal{B}' \subset \mathcal{B}$ that still covers Ω. We choose a $B_1 \in \mathcal{B}'$ of maximal radius, and we take $B_{i+1} \in \mathcal{B}'$ to be of maximal radius among all those $B \in \mathcal{B}'$ for which $2B$ is disjoint from $2B_1, ..., 2B_i$. Then, indeed, $\bigcup 5B_i \supset \bigcup \mathcal{B}' \supset \Omega$.

Key estimate: We have $|B_i| \leq C(n, \varepsilon, \Lambda) r_i |\partial\Omega \cap 2B_i|$ for all $i = 1, ..., k$.

If this is true, then (2.1) follows immediately by noting that

$$|\Omega|^{\frac{1}{\alpha}} \leq C(n, \Lambda) \sum |B_i|^{\frac{1}{\alpha}} \leq C(n, \varepsilon, \Lambda) \sum |B_i|^{\frac{1}{\alpha}-1} r_i |\partial\Omega \cap 2B_i|.$$

Proof of the key estimate. The basic idea is as follows. Fix $z \in B_i \cap \Omega$ and project $B_i \setminus \Omega$ onto $\partial\Omega$ along minimal geodesics emanating from z. Define $\Sigma^{\mathrm{first}} \subset \partial\Omega \cap 2B_i$ to consist of the first points of entry into $B_i \setminus \Omega$ of the "light rays" involved in this projection. By integrating the infinitesimal version of the Bishop-Gromov volume comparison inequality along the maximal subsegment with endpoints in $\bar{B}_i \setminus \Omega$ of each such light ray, and then integrating across Σ^{first}, one eventually finds that

$$\varepsilon |B_i| \leq |B_i \setminus \Omega| \leq \int_{\Sigma^{\mathrm{first}}} \frac{v_\delta(2r_i) - v_\delta(\mathrm{dist}(z, y))}{a_\delta(\mathrm{dist}(z, y))} \, d\mathrm{area}(y).$$

This proves the key estimate if $r_i \leq C(n, \varepsilon, \Lambda) \, \mathrm{dist}(z, \Sigma^{\mathrm{first}})$. But if this inequality *fails* for *all* $z \in B_i \cap \Omega$, then intuitively one should be able to find points $z' \in B_i \setminus \Omega$ such that the argument does go through with $z, B_i \setminus \Omega$ replaced by $z', B_i \cap \Omega$.

We now work out the details. Specifically, we show that $\min\{|B_i \setminus \Omega|, |B_i \cap \Omega|\} \leq 2(v_\delta(4r_i)/a_\delta(2r_i))|\partial\Omega \cap 2B_i|$, and this then suffices by (ε, r_0)-goodness.

Define $X_1, X_2 \subset (B_i \cap \Omega) \times (B_i \setminus \Omega)$ as follows: $X_1 := \{(z, z') :$ *there is a unique minimal geodesic* γ *from* z *to* z', *and this has the following properties: it intersects* $\partial\Omega$ *only transversely, and if* y *denotes the first point along* γ, *counted from* z, *where* γ *intersects* $\partial\Omega$, *then* $\operatorname{dist}(z, y) \geq \operatorname{dist}(y, z')\}$, and almost verbatim for X_2, with the only difference that the inequality is now reversed. Then $X_1 \cup X_2$ has full measure in $(B_i \cap \Omega) \times (B_i \setminus \Omega)$, and so one of X_1, X_2 must have at least half measure.

• X_1 *has at least half measure.* By Fubini, there must be a $z \in B_i \cap \Omega$ such that $Z' := \{z' \in B_i \setminus \Omega : (z, z') \in X_1\}$ has at least half measure in $B_i \setminus \Omega$. We now bound $|Z'|$ above by projecting onto $\partial\Omega$ along minimal geodesics from z, and integrating the infinitesimal Bishop-Gromov inequality along these geodesics.

Let Σ^{first} be the set of all $y \in \partial\Omega \cap 2B_i$ which occur as the first intersection with $\partial\Omega$ of the geodesic γ from z to some $z' \in Z'$ as in the definition of X_1. Thus, for all $y \in \Sigma^{\text{first}}$ there exists a unique minimal geodesic γ_y from z to y, and we can write $\gamma_y(t) = \exp_z(v_y t)$, where $v_y \in T_z M$ is a uniquely determined unit vector.

Define $d_1, d_2 : \Sigma^{\text{first}} \to \mathbb{R}^+$ by $d_1(y) := \operatorname{dist}(z, y)$ and $d_2(y) := \min\{\sup\{t > 0 : \gamma_y(d_1(y) + t) \in Z'\}, \sup\{t > 0 : \gamma_y$ *is minimal on* $[0, d_1(y) + t]\}\}$.

For every $z' \in Z'$ then, there exists a unique $y \in \Sigma^{\text{first}}$ such that $z' = \gamma_y(t)$ for some $d_1(y) < t \leq d_1(y) + d_2(y)$. Thus, if we define an imbedding $\Phi : U \hookrightarrow M$ with $U := \{(y, t) \in \Sigma^{\text{first}} \times \mathbb{R}^+ : d_1(y) < t < d_1(y) + d_2(y)\}$ and $\Phi(y, t) := \gamma_y(t)$, then $\operatorname{clos}(\Phi(U))$ contains Z'. On the other hand, a fairly standard calculation yields

$$\Phi^*(d\mathrm{vol}_M)|_{(y,t)} = \frac{J(tv_y)}{J(d_1(y)v_y)} \cos\alpha_y \, d\mathrm{area}_{\partial\Omega} \wedge dt,$$

where $J(w) := |w|^{n-1} \det d\exp_z|_w$ for all $w \in T_z M$, and where α_y denotes the angle between $\dot\gamma_y(d_1(y))$ and the exterior unit normal to $\partial\Omega$ at y.

We integrate over U and apply relative volume comparison:

$$\begin{aligned}
|\Phi(U)| &\leq \int_{\Sigma^{\text{first}}} \int_{d_1(y)}^{d_1(y)+d_2(y)} \frac{J(tv_y)}{J(d_1(y)v_y)} \, dt \, d\mathrm{area}(y) \\
&\leq \int_{\Sigma^{\text{first}}} \int_{d_1(y)}^{d_1(y)+d_2(y)} \frac{a_\delta(t)}{a_\delta(d_1(y))} \, dt \, d\mathrm{area}(y) \\
&= \int_{\Sigma^{\text{first}}} \frac{v_\delta(d_1(y) + d_2(y)) - v_\delta(d_1(y))}{a_\delta(d_1(y))} \, d\mathrm{area}(y).
\end{aligned}$$

We now estimate the integrand as follows:

$$\frac{v_\delta(d_1 + d_2) - v_\delta(d_1)}{a_\delta(d_1)} \leq \frac{v_\delta(d_1 + d_2)}{a_\delta(d_1)} \leq \frac{v_\delta(2d_1)}{a_\delta(d_1)} \leq \frac{v_\delta(4r_i)}{a_\delta(2r_i)},$$

because $d_2 \leq d_1 \leq 2r_i$ and because $s \mapsto v_\delta(2s)/a_\delta(s)$ is nondecreasing. Altogether then, $|B_i \setminus \Omega| \leq 2|Z'| \leq 2|\Phi(U)| \leq 2(v_\delta(4r_i)/a_\delta(2r_i))|\Sigma^{\text{first}}|$, as needed.

• X_2 *has at least half measure.* By Fubini again, there now exists a $z' \in B_i \setminus \Omega$ such that $Z := \{z \in B_i \cap \Omega : (z, z') \in X_2\}$ has at least half measure in $B_i \cap \Omega$. Let Σ^{last} denote the set of all $y \in \partial\Omega \cap 2B_i$ which occur as the *last* intersection of γ^{-1} with $\partial\Omega$, where γ is the geodesic from some $z \in Z$ to z' as in the definition of X_2. For each $y \in \Sigma^{\text{last}}$, there then exists a unique minimal geodesic $\gamma_y(t) = \exp_{z'}(tv_y)$ from z' to y, and the rest of the argument will be the same as above up to replacing $z, Z', \Sigma^{\text{first}}$ by $z', Z, \Sigma^{\text{last}}$ and switching the interior and exterior normals of Ω. □

Remark 2.3. A similar sort of reasoning yields the following result: If $B = B(x, r) \subset M$ is such that $3r < \operatorname{diam}(M)$, $4\bar{B}$ is compact, and $\operatorname{Ric} \geq -\Lambda r^{-2}$ on $4B$ with $\Lambda \geq 0$,

then $\frac{1}{C}r|\partial B| \leq |B| \leq Cr|\partial B|$ with a uniform $C = C(n,\Lambda)$ which in particular does not depend on the collapsedness of B. Here, the upper bound follows as before, by projecting B onto ∂B along minimal geodesics issuing from a point on $\partial(3B)$. For the lower bound, we sweep out a subset $B^* \subset B$ by joining x to all smooth points of ∂B, express $|B^*|$ in polar coordinates, and then use Bishop-Gromov in the form $J(tv)/J(rv) \geq a_\delta(t)/a_\delta(r)$, $v \in T_xM$, $|v| = 1$, $t \leq r$, to estimate from below.

2.2. **Dirichlet-type Sobolev inequalities on balls and annuli.** Under lower Ricci bounds, subsets of geodesic balls or annuli admit (ε, r_0)-good ball coverings for controlled values of ε, r_0 (Lemma 2.4). By Lemma 2.2, this implies a Dirichlet-isoperimetric, hence a Dirichlet-Sobolev inequality (Corollary 2.6). As a corollary, we obtain a global Gallot- or Varopoulos-type inequality (Corollary 2.8).

Note that M is still not required to be complete. In the following lemma, we fix a point $x_0 \in M$, and we put $B(r) := B(x_0, r)$, $A(r_1, r_2) := A(x_0, r_1, r_2)$.

Lemma 2.4. (i) *For all $\Lambda \geq 0$ there exists $\varepsilon = \varepsilon(n,\Lambda) > 0$ such that if $B := B(s)$, $4s < \operatorname{diam}(M)$, $9\bar{B}$ is compact, and $\operatorname{Ric} \geq -\Lambda s^{-2}$ on $9B$, then for all open $\Omega \subset B$ and $x \in \Omega$ there exists $0 < r_{x,\Omega} \leq 4s$ such that $|B(x, r_{x,\Omega}) \setminus \Omega| = \varepsilon |B(x, r_{x,\Omega})|$.*

(ii) *For all $\Lambda, N \geq 0$ there exists $\varepsilon = \varepsilon(n,\Lambda,N) > 0$ such that if $A := A(r, r+s)$, $r > 6t$, $s \leq Nt$, $\bar{B}(r+2s+2t)$ is compact, and $\operatorname{Ric} \geq -\Lambda t^{-2}$ on $A(r-6t, r+2s+2t)$, then for all open $\Omega \subset A$ and $x \in \Omega$ there exists $0 < r_{x,\Omega} \leq r_x^* := \operatorname{dist}(x_0, x) - r + 2t$ such that $|B(x, r_{x,\Omega}) \setminus \Omega| = \varepsilon |B(x, r_{x,\Omega})|$.*

Proof. (i) Fix a minimal geodesic γ from x to some point $x'' \in \partial B(x, 4s)$ and put $x' := \gamma(3s)$. Then, by volume comparison, $|B(x,4s) \setminus B| \geq |B(x', s)| \geq \varepsilon |B(x, 4s)|$ for some definite $\varepsilon = \varepsilon(n,\Lambda) > 0$. The claim then follows by continuity.

(ii) Fix a minimal geodesic γ from x_0 to x. Let $x_i := \gamma(r + (2i-3)t)$, $B_i := B(x_i, t)$ for $i = 1, ..., k$, where k is maximal with $r + (2k-3)t \leq \operatorname{dist}(x_0, x)$; notice that $k \leq \frac{1}{2}(N+3)$. Then $B(x, r_x^*) \setminus A \supset B_1$ and $3B_i \supset B_{i+1}$ for $i = 1, ..., k-1$. A volume comparison shows that $|B_i| \geq \varepsilon |3B_i|$; thus $|B(x, r_x^*) \setminus A| \geq \varepsilon |B_k|$ by induction, where we understand ε as a small generic constant. By volume comparison again, $|B_k| \geq \varepsilon |B(x_k, 2t + r_x^*)|$, and $B(x_k, 2t + r_x^*) \supset B(x, r_x^*)$ by maximality of k. Altogether, $|B(x, r_x^*) \setminus A| \geq \varepsilon |B(x, r_x^*)|$, so we conclude as in (i). □

Thus, for all $\Omega \subset B, A$ in (i), (ii), the covering $\{B(x, r_{x,\Omega}) : x \in \Omega\}$ is (ε, r_0)-good with $\varepsilon = \varepsilon(n,\Lambda)$, $r_0 = 4s$, and $\varepsilon = \varepsilon(n,\Lambda,N)$, $r_0 = s + 2t$, respectively, so then Lemma 2.2 provides a uniform isoperimetric estimate for all such Ω. It is a classical fact that this implies Dirichlet-Sobolev inequalities on B, A; cf. Li [12, Theorem 9.1] and Saloff-Coste [18, Section 3.1.2] for expositions. We recall the relevant result for convenience, and then state the ensuing Sobolev bounds in our setting.

Lemma 2.5. *For all $\Omega_0 \subset M$ open and precompact, $\alpha \geq 1$,*

$$\sup\left\{\frac{\|u\|_\alpha}{\|\nabla u\|_1} : u \in C_0^\infty(\Omega_0),\ u \neq 0\right\} = \sup\left\{\frac{|\Omega|^{\frac{1}{\alpha}}}{|\partial\Omega|} : \Omega \subset \Omega_0 \text{ open, } \partial\Omega \text{ smooth}\right\}.$$

Also, recall that by Hölder's inequality, for all $p \geq 1$ and $\alpha \geq 1$,

$$\operatorname{DS}(\Omega_0, p, \alpha) := \sup \frac{\|u\|_{\alpha p}}{\|\nabla u\|_p} \leq \frac{\alpha p}{\alpha'} \sup \frac{\|u\|_{\alpha'}}{\|\nabla u\|_1} \quad \text{if } 1 - \frac{1}{\alpha'} = \frac{1}{p}\left(1 - \frac{1}{\alpha}\right), \tag{2.2}$$

where the suprema are over all $u \in C_0^\infty(\Omega_0)$ with $u \neq 0$.

Corollary 2.6. (i) *If* $20s < \operatorname{diam}(M)$ *and* $\bar{B}(20s)$ *is compact, and if* $\operatorname{Ric} \geq -\Lambda s^{-2}$ *on* $B(20s)$ *with* $\Lambda \geq 0$*, then, for all* $p \in [1, n)$ *and* $\alpha \in [1, \alpha_{np}]$*,*

$$\operatorname{DS}(B(s), p, \alpha) \leq C(n, p, \Lambda) s |B(s)|^{\frac{1}{p}(\frac{1}{\alpha}-1)}. \tag{2.3}$$

(ii) *If* $20s < \min\{r, \operatorname{diam}(M)\}$ *and* $\bar{B}(r + 20s)$ *is compact, and if* $\operatorname{Ric} \geq -\Lambda s^{-2}$ *on* $A(r - 20s, r + 20s)$ *with* $\Lambda \geq 0$*, then, for all* $p \in [1, n)$ *and* $\alpha \in [1, \alpha_{np}]$*,*

$$\operatorname{DS}(A(r, r + s), p, \alpha) \leq C(n, p, \Lambda) s \sup_{x \in A(r, r+s)} |B(x, s)|^{\frac{1}{p}(\frac{1}{\alpha}-1)}. \tag{2.4}$$

Proof. (i) Combining Lemmas 2.2, 2.4, with Lemma 2.5, (2.2),

$$\operatorname{DS}(B(s), p, \alpha) \leq C(n, p, \alpha, \Lambda) \sup \left\{ r |B(x, r)|^{\frac{1}{p}(\frac{1}{\alpha}-1)} : x \in B(s), r \in (0, 4s] \right\}$$

for all $p \geq 1$, $\alpha \geq 1$. The right-hand side is finite iff either $p \in [1, n)$ and $\alpha \in [1, \alpha_{np}]$, or $p \geq n$. The second case is covered by the first, and in the first case, (2.3) follows by volume comparison. The proof of (ii) is similar. □

Remark 2.7. Corollary 2.6(i) reproduces Anderson [1, Theorem 4.1], which has its roots in Gromov [8]. We have streamlined Anderson's proof and tweaked it so that it applies to more general domains. Note that (2.3) simply means that if $B = B(s)$, then, for all $\alpha \in [1, \alpha_{np}]$ and all $u \in C_0^\infty(B)$,

$$\left(\fint_B |u|^{\alpha p} \right)^{\frac{1}{\alpha p}} \leq C(n, p, \Lambda) s \left(\fint_B |\nabla u|^p \right)^{\frac{1}{p}}. \tag{2.5}$$

Croke's sharp isoperimetric inequality from [5] would imply (2.5) with an additional collapsedness factor of $s^n |B|^{-1}$ on the right; the improvement afforded by (2.5) was crucial for the main ε-regularity theorem proved in [1]. By Maheux and Saloff-Coste [14, Théorème 1.1], (2.5) holds as well for all $u \in C^\infty(B)$ with mean value zero.

We conclude with a global Sobolev inequality similar to the ones given in Hebey [10, Theorem 3.14, Theorem 3.22] which follows immediately from Corollary 2.6. In the compact case, this implies a familiar result of Gallot [6, Théorème 6.16].

Corollary 2.8. *Let* M^n *be a complete Riemannian manifold. If* $r, \varepsilon > 0$ *and* $\Lambda \geq 0$ *are such that* $100r < \operatorname{diam}(M)$*,* $\operatorname{Ric} \geq -\Lambda r^{-2}$*, and* $|B(x, r)| \geq \varepsilon$ *for all* $x \in M$*, and if* $p \in [1, n)$ *and* $\alpha \in [1, \alpha_{np}]$*, then*

$$\|u\|_{\alpha p} \leq C(n, p, \Lambda) \varepsilon^{\frac{1}{p}(\frac{1}{\alpha}-1)} (r \|\nabla u\|_p + \|u\|_p), \tag{2.6}$$

for all $u \in C_0^\infty(M)$ *if* M *is open, and for all* $u \in C^\infty(M)$ *if* M *is closed.*

Proof. Fix $x_0 \in M$ and define $r_m := (1 + \frac{m}{100}) r$, $A_m := A(x_0, r_m, r_{m+1})$ for $m \in \mathbb{N}_0$. Take $\chi_0 \in C_0^\infty(B(x_0, r_2))$ with $0 \leq \chi_0 \leq 1$, $\chi_0 \equiv 1$ on $B(x_0, r_1)$, $|\nabla \chi_0| \leq 200 r^{-1}$. For $m \geq 1$ take $\chi_m \in C_0^\infty(A_{m-1} \cup \bar{A}_m \cup A_{m+1})$ with $0 \leq \chi_m \leq 1$, $\chi_m \equiv 1$ on A_m, $|\nabla \chi_m| \leq 200 r^{-1}$. By Corollary 2.6, for all $m \in \mathbb{N}_0$,

$$\|u \chi_m\|_{\alpha p} \leq C(n, p, \Lambda) r \varepsilon^{\frac{1}{p}(\frac{1}{\alpha}-1)} \|\nabla (u \chi_m)\|_p.$$

Take p-th powers, sum over m, and take p-th roots. □

2.3. **Proof of the weighted Sobolev inequalities.** This section concludes the proof of Theorem 1.2 by combining Corollary 2.6 and some basic analysis on graphs, similar to what was developed in [7, 16] in much greater generality.

The key step in passing from the Dirichlet-Sobolev estimates in Corollary 2.6 to the global estimates in Theorem 1.2 is a certain Neumann-type Poincaré inequality. We need both a continuous and a discrete version. The continuous one follows from a special case of the Cheeger-Colding segment inequality [4, Theorem 2.11]:

Lemma 2.9 (Cheeger-Colding)**.** *Let $B = B(x,r) \subset (M^n, g)$. If $2B$ is precompact and* $\mathrm{Ric} \geq -\Lambda r^{-2}$ *on $2B$ with $\Lambda \geq 0$, then, for all $u \in C^\infty(B)$,*

$$\int_B |u - u_B|^2 \leq C(n,\Lambda) r^2 \int_{2B} |\nabla u|^2. \tag{2.7}$$

Passing from B to $2B$ is inevitable in their proof because a segment between two points in B will usually only be contained in $2B$. Buser [2, Lemma 5.1] gives an L^p Neumann-type Poincaré inequality for every $p \geq 1$ which does not require doubling the radius, but (2.7) yields all we need here and is surprisingly simple to show.

Also, (2.7) has a useful discrete counterpart with a closely related proof. If V is a set and E is a set of 2-element subsets of V, both countable, we call $G = (V,E)$ a *graph*. We say G is *connected* if any two vertices $x, y \in V$, $x \neq y$, can be joined by a *path*, i.e. a set $\gamma \subset V$ which can be listed as $\gamma = \{\gamma_0, ..., \gamma_m\}$ such that $\{\gamma_0, \gamma_m\} = \{x, y\}$ and $\{\gamma_{i-1}, \gamma_i\} \in E$ for $i = 1, ..., m$. If $u : V \to \mathbb{C}$, then we write $u_x := u(x)$, and we define $|\nabla u|^2 : V \to \mathbb{R}$ by setting $|\nabla u|_x^2 := \sum_{y \in V : \{x,y\} \in E} |u_x - u_y|^2$.

Lemma 2.10. *Let G be connected and let $w : V \to \mathbb{R}^+$ satisfy $\sum w_x = 1$. For all $x, y \in V$, $x \neq y$, fix a path $\gamma = \gamma^{\{x,y\}}$ as above, and write $m = m^{\{x,y\}}$. Introduce $\bar{w} : V \to \mathbb{R}^+$ as $\bar{w}_z := \sum_{\{x,y\} : z \in \gamma^{\{x,y\}}} m^{\{x,y\}} w_x w_y$. Then, for all $u : V \to \mathbb{C}$,*

$$\sum_{x \in V} w_x u_x = 0 \Longrightarrow \sum_{x \in V} w_x |u_x|^2 \leq \sum_{x \in V} \bar{w}_x |\nabla u|_x^2. \tag{2.8}$$

Indeed, multiply the left-hand side by $\sum w_x = 1$ and use $\sum w_x u_x = 0$ to obtain

$$\sum_x w_x |u_x|^2 = \sum_{\{x,y\}} w_x w_y |u_x - u_y|^2 \leq \sum_{\{x,y\}} m^{\{x,y\}} w_x w_y \sum_{i=1}^{m^{\{x,y\}}} |u_{\gamma_i^{\{x,y\}}} - u_{\gamma_{i-1}^{\{x,y\}}}|^2.$$

Proof of Theorem 1.2. During this proof, both C_0 and C will denote large constants that are only allowed to depend on the geometry of M, but not on the function u. However, C_0 is fixed once and for all, whereas C may change from line to line.

Fix $\alpha \in [1, \alpha_{n2}]$ and let $\eta := 1 + \frac{1}{C_0}$. Then Corollary 2.6 yields

$$B := B(x_0, C_0) \Longrightarrow \mathrm{DS}(\eta B, 2, \alpha) \leq C, \tag{2.9}$$

$$A := A(x_0, r, \eta r),\ r \geq C_0 \Longrightarrow \mathrm{DS}(\eta A, 2, \alpha) \leq C r^{1 + \frac{\beta}{2}(\frac{1}{\alpha} - 1)}, \tag{2.10}$$

where we recall that $\mu A(x_0, r, s) := A(x_0, \mu^{-1} r, \mu s)$ if $\mu > 1$, $r < s$. The remainder of the proof is in three steps. In Step 0, we use Lemmas 2.9, 2.10 to establish weak Neumann-Poincaré inequalities, (2.11), (2.12), on slightly larger domains ηB_κ, ηA_κ, which together with (2.9), (2.10) imply weak Neumann-Sobolev inequalities (2.15), (2.16) for ηB, ηA. In Steps 1 and 2, we apply these four Neumann-type inequalities from Step 0, and Lemma 2.10 again, to prove (1.1) and (1.2), respectively.

Step 0: Weak Neumann-type Poincaré and Sobolev on certain balls and annuli. For $\kappa \geq 1$ define $B_\kappa := B(x_0, C_0\kappa)$ and $A_\kappa := A(x_0, r, \eta\kappa r)$. We first show that

$$\|u - u_{\eta B_\kappa}\|_{\eta B_\kappa, 2} \leq C(\kappa)\|\nabla u\|_{\eta^2 B_\kappa, 2}, \tag{2.11}$$

$$\|u - u_{\eta A_\kappa}\|_{\eta A_\kappa, 2} \leq C(\kappa) r\|\nabla u\|_{\eta^2 A_\kappa, 2}. \tag{2.12}$$

We write out a detailed argument for the annulus case (2.12) only. Pick a maximal $r/2000C_0$-separated set $x_1, ..., x_m$ in ηA_κ, so that the $B_i := B(x_i, r/1000C_0)$ cover ηA_κ, but the $\frac{1}{2}B_i$ are disjoint. Notice that $|B_i| \sim r^\beta$ from SOB(β), so $m \leq C(\kappa)$. Then, from SOB(β) and the segment inequality (2.7), for all i and $1 \leq \lambda \leq 10$,

$$\int_{\lambda B_i} |u - u_{\lambda B_i}|^2 \leq Cr^2 \int_{2\lambda B_i} |\nabla u|^2. \tag{2.13}$$

Then for any $\mu \in \mathbb{R}$,

$$\int_{\eta A_\kappa} |u - u_{\eta A_\kappa}|^2 \leq \int_{\eta A_\kappa} |u - \mu|^2 \leq 2\sum \int_{B_i} |u - u_{B_i}|^2 + 2\sum |B_i||u_{B_i} - \mu|^2. \tag{2.14}$$

The first sum can be bounded by using (2.13) with $\lambda = 1$. For the second, we apply Lemma 2.10, as follows. Construct a graph $G = (V, E)$ by setting $V := \{1, ..., m\}$, and for $i \neq j$, $\{i, j\} \in E$ iff $B_i \cap B_j \neq \emptyset$. Then G is connected because ηA_κ is. Let $w \equiv \frac{1}{m}$, and for $i \neq j$, let $\gamma^{\{i,j\}}$ be any path without loops joining i and j. Thus, by Lemma 2.10, if $\mu := \frac{1}{m}\sum u_{B_i}$, then the second sum in (2.14) is bounded by

$$\sum |B_i||u_{B_i} - \mu|^2 \leq Cr^\beta \sum_i \sum_{j: B_i \cap B_j \neq \emptyset} |u_{B_i} - u_{B_j}|^2.$$

Next, for any constant $\nu \in \mathbb{R}$, by Cauchy-Schwarz,

$$\begin{aligned} |u_{B_i} - u_{B_j}|^2 &\leq \frac{1}{|B_i||B_j|} \int_{B_i \times B_j} |u(x) - u(y)|^2 \, dx \, dy \\ &\leq 4\frac{|B_i \cup B_j|}{|B_i||B_j|} \int_{B_i \cup B_j} |u(x) - \nu|^2 \, dx. \end{aligned}$$

Now $B_i \cup B_j \subset 3B_i$ since $B_i \cap B_j \neq \emptyset$, so put $\nu = u_{3B_i}$ and apply (2.13), $\lambda = 3$:

$$|u_{B_i} - u_{B_j}|^2 \leq Cr^{2-\beta} \int_{6B_i} |\nabla u|^2.$$

This bounds the second sum in (2.14), concluding the proof of (2.12). The proof of (2.11) is entirely similar, dropping the r-dependence everywhere. Observe that we could apply Lemma 2.9 to ηB_κ directly in that case, but it will be convenient later on to be integrating over $\eta^2 B_\kappa$ rather than $2B_\kappa$ on the right-hand side of (2.11).

To conclude, we deduce weak Neumann-type Sobolev inequalities for ηB and ηA. Construct cutoff functions $\chi_B \in C_0^\infty(\eta B)$, $0 \leq \chi_B \leq 1$, $\chi_B \equiv 1$ on B, $|\nabla \chi_B| \leq C$, and $\chi_A \in C_0^\infty(\eta A)$, $0 \leq \chi_A \leq 1$, $\chi_A \equiv 1$ on A, $|\nabla \chi_A| \leq Cr^{-1}$. Then, setting $\kappa = 1$, (2.9), (2.11) and (2.10), (2.12) easily imply

$$\|u - u_{\eta B}\|_{B, 2\alpha} \leq \|\chi_B(u - u_{\eta B})\|_{\eta B, 2\alpha} \leq C\|\nabla u\|_{\eta^2 B, 2}, \tag{2.15}$$

$$\|u - u_{\eta A}\|_{A, 2\alpha} \leq \|\chi_A(u - u_{\eta A})\|_{\eta A, 2\alpha} \leq Cr^{1+\frac{\beta}{2}(\frac{1}{\alpha}-1)}\|\nabla u\|_{\eta^2 A, 2}. \tag{2.16}$$

Together with (2.11), (2.12), these are what we need for Steps 1 and 2 below. □

Put $r_i := \eta^i C_0$ ($i \in \mathbb{N}_0$), $A_0 := B = B(x_0, r_0)$, $A_i := A(x_0, r_{i-1}, r_i)$ ($i \in \mathbb{N}$). Fix $\varphi \in C^\infty(M)$, $\varphi > 0$, to be determined, and use $\|..\|_{X,\varphi,p}$ to denote the L^p norm on X with respect to $\varphi \, d\text{vol}$. Let $u_i := u_{\eta A_i}$ and $\varphi_i := \sup_{A_i} \varphi$ ($i \in \mathbb{N}_0$).

Step 1: Proof of (1.1). For $\mu \in \mathbb{R}$ to be determined, consider (all sums over $\mathbb{N}_0$)

$$\|u-\mu\|^2_{M,\varphi,2\alpha} \le 2\sum \|u-u_i\|^2_{A_i,\varphi,2\alpha} + 2\sum (\varphi_i|A_i|)^{\frac{1}{\alpha}}|u_i-\mu|^2. \tag{2.17}$$

The first sum can be bounded by (2.15) for $i=0$ and (2.16), $A=A_i$, for $i\ge 1$:

$$\sum \|u-u_i\|^2_{A_i,\varphi,2\alpha} \le C\sum \varphi_i^{\frac{1}{\alpha}} r_i^{2+\beta(\frac{1}{\alpha}-1)} \|\nabla u\|^2_{\eta^2 A_i,2}.$$

For the second sum in (2.17), we apply Lemma 2.10 to the graph $A_0 - A_1 - A_2 - \cdots$ with weights $w_i := \tilde{w}_i/\tilde{w}$, where $\tilde{w}_i := (\varphi_i|A_i|)^{1/\alpha}$ and $\tilde{w} := \sum \tilde{w}_i$, assuming that this series converges. Thus, if we choose

$$\mu := \sum w_i u_i = \int_M u\psi \, d\mathrm{vol},\ \psi := \sum \frac{w_i}{|\eta A_i|}\chi_{\eta A_i}, \int_M \psi \, d\mathrm{vol} = 1, \tag{2.18}$$

then we can continue to estimate the second sum in (2.17) as follows:

$$\sum (\varphi_i|A_i|)^{\frac{1}{\alpha}}|u_i-\mu|^2 \le \frac{C}{\tilde{w}}\sum(\bar{w}_k+\bar{w}_{k+1})|u_k-u_{k+1}|^2,\ \bar{w}_k := \sum_{i\le k\le j}(j-i)\tilde{w}_i\tilde{w}_j.$$

As before, for all $k\ge 0$ and any $\nu\in\mathbb{R}$, by Cauchy-Schwarz,

$$\begin{aligned} |u_k-u_{k+1}|^2 &\le \frac{1}{|\eta A_k||\eta A_{k+1}|}\int_{\eta A_k\times \eta A_{k+1}} |u(x)-u(y)|^2\,dx\,dy \\ &\le 4\frac{|\eta A_k\cup \eta A_{k+1}|}{|\eta A_k||\eta A_{k+1}|}\int_{\eta A_k\cup \eta A_{k+1}} |u(x)-\nu|^2\,dx. \end{aligned} \tag{2.19}$$

For $k=0$, apply (2.11), with $\kappa=\eta$, and for $k\ge 1$, (2.12), with $r=r_{k-1}$, $\kappa=\eta^2$, choosing ν to be the average of u over the appropriate domain in each case. Thus, $\|u-\mu\|_{M,\varphi,2\alpha}\le C\|\nabla u\|_2$ with μ as in (2.18), provided $\varphi: M\to\mathbb{R}^+$ satisfies

$$\sup_{i\in\mathbb{N}_0}\varphi_i^{\frac{1}{\alpha}} r_i^{2+\beta(\frac{1}{\alpha}-1)} \le C,\ \frac{1}{C}\le \tilde{w}\le C,\ \sup_{k\in\mathbb{N}_0}(\bar{w}_k+\bar{w}_{k+1})r_k^{2-\beta}\le C.$$

The first condition checks if $\varphi\le C(1+r)^{\alpha(\beta-2)-\beta}$, the second if $\varphi|_{A_0}\ge C^{-1}$ and $\varphi\le C(1+r)^{-\beta-\varepsilon}$ $(\varepsilon>0)$, and the third if $\varphi\le C(1+r)^{\alpha(\beta-2)-\beta-\varepsilon}$ $(\varepsilon>0)$. □

Step 2: Proof of (1.2). We begin as in Step 1, splitting (and summing over $\mathbb{N}_0$)

$$\|u\|^2_{M,\varphi,2\alpha} \le 2\sum \|u-u_i\|^2_{A_i,\varphi,2\alpha} + 2\sum(\varphi_i|A_i|)^{\frac{1}{\alpha}}|u_i|^2.$$

The first sum can be bounded by (2.15) for $i=0$ and (2.16), $A=A_i$, for $i\ge 1$:

$$\sum \|u-u_i\|^2_{A_i,\varphi,2\alpha} \le C\sum \varphi_i^{\frac{1}{\alpha}} r_i^{2+\beta(\frac{1}{\alpha}-1)} \|\nabla u\|^2_{\eta^2 A_i,2}. \tag{2.20}$$

To estimate the second sum, fix $K\in\mathbb{N}$ and consider

$$\sum(\varphi_i|A_i|)^{\frac{1}{\alpha}}|u_i|^2 \le 2\sum(\varphi_i|A_i|)^{\frac{1}{\alpha}}|u_i-u_{i+K}|^2 + 2\sum(\varphi_i|A_i|)^{\frac{1}{\alpha}}|u_{i+K}|^2. \tag{2.21}$$

The first term here can again be bounded in terms of the Dirichlet energy:

$$\sum(\varphi_i|A_i|)^{\frac{1}{\alpha}}|u_i-u_{i+K}|^2 \le CK\sum(\varphi_i|A_i|)^{\frac{1}{\alpha}} r_i^{2-\beta}\int_{\eta^2 A_i\cup\ldots\cup\eta^2 A_{i+K}}|\nabla u|^2. \tag{2.22}$$

To see this, write u_i-u_{i+K} as a telescoping sum, use Cauchy-Schwarz, and estimate each term as in (2.19). As a result, if $\varphi\sim(1+r)^{\alpha(\beta-2)-\beta}$, then both (2.20), (2.22)

are bounded by $CK^2 \int |\nabla u|^2$. The remaining term in (2.21) can be absorbed:

$$\sum (\varphi_i |A_i|)^{\frac{1}{\alpha}} |u_{i+K}|^2 = \sum \frac{(\varphi_i |A_i|)^{\frac{1}{\alpha}}}{(\varphi_{i+K}|A_{i+K}|)^{\frac{1}{\alpha}}} (\varphi_{i+K}|A_{i+K}|)^{\frac{1}{\alpha}} |u_{i+K}|^2$$
$$\leq C\eta^{(2-\beta)K} \sum (\varphi_i |A_i|)^{\frac{1}{\alpha}} |u_i|^2,$$

so it suffices to make K sufficiently large, depending only on η, β, and C. □

3. Applications to the Poisson equation

3.1. **Existence.** The basic idea in proving Theorem 1.5 is to solve boundary value problems on larger and larger domains. Then, by local elliptic theory, we only need a uniform L^∞ bound to pass to the limit. To obtain such an estimate, we will apply Moser iteration based on Theorem 1.2. If $\beta > 2$, the estimate becomes fairly simple thanks to (1.2). If $\beta \leq 2$, only (1.1) is available, and we need to assume $\int f = 0$ in order to balance the subtraction on the left-hand side in (1.1).

This overall approach would not work for the complex Monge-Ampère equation because then we may not be able to solve any boundary value problems at all. The main idea in Tian-Yau [19] is to first solve $(\omega + i\partial\bar{\partial} u_\varepsilon)^m = e^{f+\varepsilon u_\varepsilon}\omega^m$ for all $\varepsilon > 0$, which is substantially easier, and then exploit the exponential decay of u_ε.

• If $\beta > 2$, suppose that $\Delta u = f|_\Omega$ over a smooth and bounded domain $\Omega \subset M$, with $u = 0$ on $\partial\Omega$, where $f \in C^{0,\alpha}(M)$ is as in the statement of Theorem 1.5. For $p > 1$, multiply $\Delta u = f$ by $u|u|^{p-2}$ and integrate by parts to obtain

$$\int_\Omega |\nabla |u|^{\frac{p}{2}}|^2 = -\frac{p^2}{4(p-1)} \int_\Omega u|u|^{p-2} f. \tag{3.1}$$

Define $\rho := 1 + r$, $C_0 := \sup \rho^\mu |f| < \infty$, and $\|u\|_{p,\alpha} := (\int_M |u|^p \rho^{\alpha(\beta-2)-\beta})^{1/p}$. By (3.1), (1.2), Hölder, and SOB(β), we then have the following two facts. First of all, for every $\alpha \in [1, \alpha_{n2}]$ and $0 < \varepsilon \leq \frac{1}{C}$ such that $\mu - 2 > \varepsilon(\alpha - 1)$,

$$p > 1,\ p \geq \frac{\beta - 2 + \varepsilon}{\mu - 2 - \varepsilon(\alpha - 1)} \Longrightarrow \|u\|_{\alpha p,\alpha} \leq \frac{C_0 C p^2}{\varepsilon\alpha(p-1)}, \tag{3.2}$$

so this weighted $L^{\alpha p}$ norm is bounded independent of Ω and we can start to iterate at αp. Second, for every $\alpha \in [1, \alpha_{n2}]$ with $\alpha(\beta - 2) > \beta - \mu$, and $0 < \varepsilon \leq \frac{1}{C}$,

$$p > 1,\ p \geq \frac{\alpha(\beta - 2 + \varepsilon)}{\mu + \alpha(\beta - 2) - \beta} \Longrightarrow \|u\|_{\alpha p,\alpha} \leq \left(\frac{C_0 C p^2}{\varepsilon\alpha(p-1)}\right)^{\frac{1}{p}} \|u\|_{p,\alpha}^{1-\frac{1}{p}},$$

so we can keep iterating from any such p. This proves Theorem 1.5 if $\beta > 2$.

• If $\beta \leq 2$, suppose instead that $\Delta u = f|_\Omega - (f|_\Omega)_\Omega$ on Ω with $u = 0$ on $\partial\Omega$. For any $\lambda \in \mathbb{R}$, the function $u + \lambda$ has the same Laplacian as u and is equal to λ on the boundary, which implies that (3.1) holds for $u + \lambda$ as well since $\int_\Omega \Delta u = 0$. Now define ρ and C_0 as before. For any $\delta > 0$, put

$$\|u\|_{p,\alpha,\delta} := \left(\int_M |u|^p \rho^{\alpha(\beta-2-\delta)-\beta}\right)^{\frac{1}{p}},$$

and recall the weight $\psi_\delta \sim \rho^{-2-\delta}$ needed for Theorem 1.2(i). Also, before iterating, observe that f_Ω can be controlled: If $B(x_0, r) \subset \Omega \subset B(x_0, 2r)$ with $r \geq C$, then

$$|f_\Omega| \leq \frac{1}{|\Omega|} \int_{M\setminus\Omega} |f| \leq C_0 C r^{-\mu}. \tag{3.3}$$

Then, by (3.1) with $p = 2$ applied to $u - u_\psi$ as explained above, (3.3), and (1.1), if $\alpha \in [1, \alpha_{n2}]$ and $0 < \varepsilon \le \frac{1}{C}$ satisfy $\mu - 2 > \varepsilon(\alpha - 1) + \delta$ and $\varepsilon \le \delta$, then

$$\|u - u_{\psi_\delta}\|_{2\alpha,\alpha,\delta} \le \frac{C_0 C_\delta}{\varepsilon\alpha}. \tag{3.4}$$

Next, by (3.1) applied to $u - u_\psi$, but for $p \ge 2$, and by (1.1) again,

$$\left\| |u - u_{\psi_\delta}|^{\frac{p}{2}} - (|u - u_{\psi_\delta}|^{\frac{p}{2}})_{\psi_\delta} \right\|^2_{2\alpha,\alpha,\delta} \le \frac{C_\delta p^2}{p-1} \int_\Omega |u - u_{\psi_\delta}|^{p-1} |f - f_\Omega|. \tag{3.5}$$

To proceed, we note the following simple, general inequality:

$$\|(|v|^{\frac{p}{2}})_{\psi_\delta}\|_{2\alpha,\alpha,\delta} \le \left(\int_M \psi_\delta\right)^{-1} \left(\int_M \frac{\psi_\delta^2}{\varphi_{\alpha,\delta}}\right)^{\frac{1}{2}} \left(\int_M \varphi_{\alpha,\delta}\right)^{\frac{1}{2\alpha}} \|v\|^{\frac{p}{2}}_{p,\alpha,\delta}, \tag{3.6}$$

where $\varphi_{\alpha,\delta} := \rho^{\alpha(\beta-2-\delta)-\beta}$ and the product of the three coefficients on the right is bounded by $C/(\varepsilon\alpha(2 - \beta + \delta))$ if $1 \le \alpha \le 2 - \varepsilon/(2 - \beta + \delta)$ and $0 < \varepsilon \le \frac{1}{C}$. Then, combining (3.5) and (3.6), and assuming that $\alpha(2 - \beta + \delta) \le \mu - \beta$ and $\varepsilon \le \delta$,

$$\|u - u_{\psi_\delta}\|_{\alpha p,\alpha,\delta} \le \left(\frac{C_0 C_{\alpha,\delta,\varepsilon} p^2}{p-1}\right)^{\frac{1}{p}} \|u - u_{\psi_\delta}\|^{1-\frac{1}{p}}_{p,\alpha,\delta} + C^{\frac{1}{p}}_{\alpha,\delta,\varepsilon} \|u - u_{\psi_\delta}\|_{p,\alpha,\delta}.$$

This concludes the proof as before. Observe that we were forced to start with $p = 2$, but then on the plus side we obtain $\int |\nabla u|^2 < \infty$ from the proof of (3.4).

3.2. **Decay.** We now prove Theorem 1.6. Assume $\Delta u = f|_\Omega$ on a smooth bounded domain Ω with $u = 0$ on $\partial\Omega$. Multiply the equation by $\zeta u|\zeta u|^{p-2}\zeta$, where $\zeta := \rho^l$ for some fixed $l \in \mathbb{R}$, and $p > 1$. Commute the spare factor of ζ past the Laplacian and integrate by parts. Using $|\nabla\rho| + \rho|\Delta\rho| \le C$ and (1.2), for any $\alpha \in [1, \alpha_{n2}]$,

$$\left(\int \rho^{\alpha(\beta-2)-\beta} |\zeta u|^{\alpha p}\right)^{\frac{1}{\alpha}} \le \frac{Cp^2}{p-1} \left(\int |\zeta u|^{p-1}\zeta|f| + \frac{|l|(|l|+1)p}{p-1} \int \rho^{-2}|\zeta u|^p\right).$$

Let $C_0 := \sup \rho^\mu |f| < \infty$. Fix $p_0 > 1$ and let $p_k := \alpha^k p_0$ and $\zeta_k := \rho^{\alpha^k(\beta-2)-\beta}$ for $k \in \mathbb{N}$. Then, if $p_0(\mu - 2) \ge \beta - 2 + \delta$ for some $0 < \delta \le \frac{1}{C}$, then, for all $k \in \mathbb{N}$,

$$\left(\int \zeta_{k+1}|u|^{p_{k+1}}\right)^{\frac{1}{p_{k+1}}} \le \left(\frac{C_0 C_\delta p_k^3}{(p_k - 1)^2}\right)^{\frac{1}{p_k}} \max\left\{1, \left(\int \zeta_k |u|^{p_k}\right)^{\frac{1}{p_k}}\right\}.$$

The necessary bound for $\int \zeta_1 |u|^{p_1}$ follows from (3.2) if α is close enough to 1.

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