LARGE CHARACTER DEGREES OF SOLVABLE $3'$-GROUPS

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Abstract. We prove that if $G$ is a finite solvable group and $3 \nmid |G : F(G)|$, then the index of the Fitting subgroup of $G$ is at most the square of the largest irreducible character degree of $G$.

1. Introduction

Let $G$ be a finite group and denote by $b(G) = \max \{ \psi(1) \mid \psi \in \text{Irr}(G) \}$ the largest degree of an irreducible character of $G$. In [5] Gluck proves that in all finite groups the index of the Fitting subgroup $F(G)$ in $G$ is bounded by a polynomial function of $b(G)$. For a solvable group, Gluck further shows that $|G : F(G)| \leq b(G)^{13/2}$ and conjectures that $|G : F(G)| \leq b(G)^2$. This has been verified by Espuelas [1] for $G$ of odd order. Espuelas’ result has been extended in [4] to $G$ a solvable group with abelian Sylow 2-subgroups by Dolfi and Jabara. The best general bound $|G : F(G)| \leq b(G)^3$ is given by Moretó and Wolf in [6]. In this note we prove Gluck’s conjecture for all solvable groups with order not divisible by 3.

2. Gluck’s conjecture for solvable $3'$-groups

Theorem 2.1. Suppose that a finite solvable group $G$ acts faithfully, irreducibly and quasi-primitively on a finite vector space $V$. By [9, Theorem 2.2], $G$ will have a uniquely determined normal subgroup $E$ which is a direct product of extraspecial $p$-groups for various $p$ and $e = \sqrt{|E/Z(E)|}$. Assume $e = 5, 7$ or $e \geq 10$ and $e \neq 16$; then $G$ will have at least 5 regular orbits on $V$.

Proof. This follows from [9, Theorem 3.1] and [10, Theorem 3.1].

Theorem 2.2. Suppose that $G$ is a finite solvable group and $V$ is a faithful, irreducible and quasi-primitive $FG$-module and $\text{char}(F) = r$. Assume $3 \nmid |G|$; then $G$ has at least 3 regular orbits on $V \oplus V$.

Proof. By [9, Theorem 2.2], $G$ will have a uniquely determined normal subgroup $E$ which is a direct product of extraspecial $p$-groups for various $p$ and $e = \sqrt{|E/Z(E)|}$.

Since $3 \nmid |G|$, $3 \nmid e$ and $G$ will have at least 5 regular orbits on $V$ unless $e = 1, 2, 4, 8, 16$ by Theorem 2.1. Since $3 \nmid |G|$, $G$ will have at least $r$ regular orbits on $V \oplus V$ by [3, Theorem 3.4]. Assume $e = 2, 4, 8, 16$; then $r \geq 3$ and $G$ will have at least 3 regular orbits on $V \oplus V$.
Thus we may assume \( e = 1 \) and \( r = 2 \). As in [7], if \( V \) is a finite vector space of dimension \( n \) over \( \text{GF}(q) \), where \( q \) is a prime power, we denote by \( \Gamma(q^n) = \Gamma(V) \) the semilinear group of \( V \), i.e.,

\[
\Gamma(V) = \{ x \mapsto ax^\sigma | x \in \text{GF}(q^n), a \in \text{GF}(q^n)^\times, \sigma \in \text{Gal}(\text{GF}(q^n)/\text{GF}(q)) \}.
\]

Since \( e = 1 \) we have \( G \leq \Gamma(V) \leq \Gamma(2^d) \cong G_1 \) by [7 Corollary 2.3(b)]. For any \( 0 \neq v \in V \), \( |C_{G_1}(v)| = d \). We can hence assume that \( C_{G_1}(v) \) is the Galois group of \( V = \text{GF}(2^d) \). So the elements of \( V \) that do not belong to a regular orbit of \( C_{G_1}(v) \) are in the union of the subfields \( \text{GF}(2^{d/m}) \), \( m \) varying among the prime divisors of \( d \). Since the number of distinct prime divisors of \( d \) is at most \( \log_2(d) \), it is enough to prove that \( f(d) = (2^d - 1) - \log_2(d) \cdot (2^{d/2} - 1) - 2d \) is positive. It is not hard to check that \( f(d) > 0 \) for all \( d \geq 4 \). Thus we are left with the cases when \( d = 1, 2, 3 \):

1. Let \( d = 1 \); then \( G \leq \Gamma(2^1) \) and \( G \) is trivial. The result is clear.
2. Let \( d = 2 \); then \( G \leq \Gamma(2^2) \cong S_3 \). Since \( 3 \nmid |G| \), \( G \cong \mathbb{Z}_2 \) and the result is clear.
3. Let \( d = 3 \); then \( G \leq \Gamma(2^3) \). Since \( 3 \nmid |G| \), \( G \cong \mathbb{Z}_7 \) and the result is clear.

\( \square \)

**Theorem 2.3.** Suppose that \( G \) is a finite solvable group and \( V \) is a faithful and completely reducible \( G \)-module (possibly of mixed characteristic). Assume \( 3 \nmid |G| \); then \( G \) has at least 3 regular orbits on \( V \oplus V \).

**Proof.** We work by induction on \( |GV| \).

Assume first that \( V = U \oplus W \) with \( U \) and \( W \) proper \( G \)-submodules. Then by inductive hypothesis \( G/C_G(U) \) has 3 regular orbits on \( U \oplus U \) and \( G/C_G(W) \) has 3 regular orbits on \( W \oplus W \). Since \( C_G(U) \cap C_G(W) = 1 \), it follows that \( G \) has 3 regular orbits on \( U \oplus U \oplus W \oplus W \cong V \oplus V \).

Therefore, we can assume that \( V \) is irreducible.

Assume \( V \) is quasi-primitive; then the result follows from Theorem 2.2.

Now we assume that \( V \) is not quasi-primitive, then there exists \( N \) normal in \( G \) such that \( V_N = V_1 \oplus \cdots \oplus V_m \) for \( m > 1 \) homogeneous components \( V_i \) of \( V_N \). If \( N \) is maximal with this property, then \( S = G/N \) primitively permutes the \( V_i \). Also \( V = V_1^G \), induced from \( N_G(V_1) \). If \( H = N_G(V_1)/C_G(V_1) \), then \( H \) acts faithfully and irreducibly on \( V_1 \) and \( G \) is isomorphic to a subgroup of \( H \rtimes S \).

By induction \( H \) will have at least 3 regular orbits on \( V_1 \oplus V_1 \). \( S \) is a solvable primitive permutation group on \( \Omega = \{ V_1, \ldots, V_m \} \). By [8] Proposition 3.2(2), \( G \) will have at least 5 regular orbits on \( V \oplus V \) unless \( m \leq 4 \). Since \( 3 \nmid |S| \), the only case left is when \( |\Omega| = 2 \) and \( S \cong S_2 \). In this case \( G \) will have at least 3 regular orbits on \( V \oplus V \).

\( \square \)

**Corollary 2.4.** Suppose that \( G \) is a finite solvable group and \( V \) is a faithful and completely reducible \( G \)-module (possibly of mixed characteristic). Assume \( 3 \nmid |G| \); then there exists \( v \in V \) such that \( |C_G(v)| \leq \sqrt{|G|} \).

**Proof.** By Theorem 2.3 there is an element \( (v, u) \in V \oplus V \) such that \( C_G((v, u)) = C_G(v) \cap C_G(u) = 1 \). Thus, we have

\[
|C_G(v)| : |C_G(u)| = \frac{|C_G(v) : C_G(u)|}{|C_G(v) \cap C_G(u)|} = |C_G(v)| C_G(u) \leq |G|.
\]

It follows that either \( |C_G(v)| \leq \sqrt{|G|} \) or \( |C_G(u)| \leq \sqrt{|G|} \).

\( \square \)
Theorem 2.5. Let $G$ be a finite solvable group and $3 
mid |G : F(G)|$. Then $|G : F(G)| \leq b(G)^2$.

Proof. Let $U = F(G)/\Phi(G)$ and $\overline{G} = G/F(G)$. $U$ is a faithful and completely reducible $G$-module by Gaschütz's theorem [7, Theorem 1.12]. Let $V = \text{Irr}(F(G)/\Phi(G))$. $V$ is a faithful and completely reducible $\overline{G}$-module by [7, Proposition 12.1]. By Corollary 2.4, there exists $\lambda \in V$ such that $\overline{I} = I_{\overline{G}}(\lambda) = \{g \in \overline{G} | \lambda^g = \lambda\}$ satisfies $|\overline{I}| \leq |\overline{G}|^{1/2}$. Consider $\lambda$ as a character of $F(G)$ with a kernel containing $\Phi(G)$. Let $I$ be the preimage of $\overline{I}$ in $G$. Now $I = I_G(\lambda) = \{g \in G | \lambda^g = \lambda\}$. Take $\mu \in \text{Irr}(I|\lambda)$. Now $\psi = \mu^G \in \text{Irr}(G)$. Thus we have $|G : F(G)| = |G : I| |I : F(G)| \leq |G : I|^2 \leq \psi(1)^2 \leq b(G)^2$. □

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