HASSE-WEIL ZETA FUNCTION OF ABSOLUTELY IRREDUCIBLE SL_2-REPRESENTATIONS OF THE FIGURE 8 KNOT GROUP

SHINYA HARADA

(Communicated by Matthew A. Papanikolas)

Abstract. Weil-type zeta functions defined by the numbers of absolutely irreducible SL_2-representations of the figure 8 knot group over finite fields are computed explicitly. They are expressed in terms of the congruence zeta functions of reductions of a certain elliptic curve defined over the rational number field. Then the Hasse-Weil type zeta function of the figure 8 knot group is also studied. Its central value is written in terms of the Mahler measures of the Alexander polynomial of the figure 8 knot and a certain family of elliptic curves.

0. Introduction

The character variety of SL_2-representations of a 3-manifold group is an important object in the study of 3-dimensional topology ([5]). It is known that a character variety for a finitely generated group is an algebraic variety defined by a finite number of polynomials with coefficients in the ring of integers \( \mathbb{Z} \). Hence we are naturally led to study the Hasse-Weil type zeta function of the character variety for a 3-manifold group as an arithmetic invariant of the manifold and expect that it encodes interesting information which reflects the topology of the 3-manifold.

In this paper, as a first step in this direction, we compute the Weil-type zeta functions

\[
Z_d(G_K, q, T) := \exp \left( \sum_{n=1}^{\infty} \frac{A_{d,n}}{n} T^n \right)
\]

and a Hasse-Weil type zeta function for the knot group \( G_K := \pi_1(S^3 \setminus K) \) of the figure 8 knot \( K \) (the fundamental group of the complement of \( K \) in the 3-sphere \( S^3 \)) defined by the numbers \( (A_{d,n})_{n \geq 1} \) for the degree \( d = 2 \) case. Here, for a finitely generated group \( G \) and a power \( q \) of a prime number \( p \), \( A_{d,n} \) is the number of \( GL_d(\mathbb{F}_q^n) \)-conjugacy classes of absolutely irreducible representations \( \rho : G \to \text{SL}_d(\mathbb{F}_q^n) \), where \( \mathbb{F}_q^n \) is the finite field of \( q^n \) elements.

We note that the figure 8 knot is a unique arithmetic knot among hyperbolic knots ([14]). In particular, the holonomy representation associated with the complete hyperbolic structure on \( S^3 \setminus K \) gives a representation \( \rho : G_K \to \text{SL}_2(\mathbb{O}_{-3}) \),

Received by the editors June 22, 2010 and, in revised form, August 13, 2010.
2010 Mathematics Subject Classification. Primary 11S40; Secondary 14G10, 57M27.
Key words and phrases. Hasse-Weil zeta function, modular representation, topological invariant, figure 8 knot.

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where $O_{-3}$ is the ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{-3})$. Hence the figure 8 knot is the most appropriate one as a first example for our arithmetic study of the character variety and we may expect that the Weil-type zeta function $Z_d(G_K, q, T)$ for the figure 8 knot $K$ has more interesting information among all the hyperbolic knots.

In Section 1 we prove that Weil-type zeta functions $Z_d(G, q, T)$ are rational functions in $T$ for any finitely generated group $G$ and degree $d$ (Theorem 1.1). In Section 2 we prove that $Z_2(G_K, q, T)$ are expressed in terms of congruence zeta functions of reductions of a certain elliptic curve over $\mathbb{Q}$ (Theorem 2.8). In Section 3 we calculate the Hasse-Weil zeta function of absolutely irreducible SL$_2$-representations of the figure 8 knot group over $\mathbb{Q}$,

$$\zeta_A(s) := \prod_p Z_2(G_K, p, p^{-s}),$$

where $p$ runs through all the prime numbers. Our main result in this paper is the following:

**Theorem (Theorem 3.1).** The function $\zeta_A(s)$ and its completion

$$\xi_A(s) := \frac{4\pi^{3s/2}+1}{10\sqrt{2}r}\Gamma(s/2)^3 \zeta_A(s)$$

are continued meromorphically to the complex plane $\mathbb{C}$ and $\xi_A(s)$ has a functional equation

$$\xi_A(2-s)\xi_Q(2-s)\xi_Q(\sqrt{5})(2-s) = -\xi_A(s)\xi_Q(s)\xi_Q(\sqrt{5})(s),$$

where $\xi_Q(s)$ and $\xi_Q(\sqrt{5})(s)$ are the completed Riemann zeta function and the completed Dedekind zeta function of $\mathbb{Q}(\sqrt{5})$ respectively. The function $\xi_A(s)$ has a simple zero at $s = 1$, and the central value $\xi_A(1)$ is given by

$$\lim_{s \to 1} \frac{\xi_A(s)}{s-1} = \frac{\text{AGM}(\varphi, \varphi-1)}{\sqrt{10\log(\varphi)}},$$

where $\varphi = (\sqrt{5} + 1)/2$ and $\text{AGM}(\varphi, \varphi-1)$ is the arithmetic-geometric mean of $\varphi$ and $\varphi-1$.

Note that $\varphi$ and $\varphi-1$ are square roots of the Alexander polynomial of the figure 8 knot. Hence the central value at $s = 1$ is expressed in terms of topological invariants of the figure 8 knot. There is also another description of the central value of the Hasse-Weil zeta function in terms of the Mahler measures of the Alexander polynomial of the figure 8 knot and certain polynomials which define a family of elliptic curves (see Remarks 3.3 and 4.1).

1. **Rationality of $Z_d(G, q, T)$**

Let $G$ be a finitely generated group and $\mathbb{F}_q$ the finite field with $q$ elements. Let $\text{Rep}_d(G)(k)$ be the set of absolutely irreducible representations of $G$ into $\text{SL}_d(k)$ for any field $k$. Here we say that a representation $\rho : G \to \text{SL}_d(k)$ is absolutely irreducible if the composition of homomorphisms $G \xrightarrow{\rho} \text{SL}_d(k) \xrightarrow{\rho} \text{GL}_d(k)$ is irreducible, where $k$ is an algebraic closure of the field $k$. Then the projective linear group $\text{PGL}_d(k)$ acts on $\text{Rep}_d(G)(k)$ by conjugation. Let $A_{d,n} := #(\text{Rep}_d(G)(\mathbb{F}_q^n) / \text{PGL}_d(\mathbb{F}_q^n))$ be the number of $\text{PGL}_d(\mathbb{F}_q^n)$-conjugacy classes of absolutely irreducible representations $\rho : G \to \text{SL}_d(\mathbb{F}_q^n)$. In this section we prove the following:
Theorem 1.1. Z_d(G, q, T) is a rational function in T.

In [8] we have proved an analogous result for absolutely irreducible GL_d-representations. It is proved by employing the moduli theory of Procesi ([14]) on absolutely irreducible representations of a non-commutative ring into Azumaya algebras. We can prove Theorem [14] by the same argument, but here we give a proof of Theorem [11] by the theory of the character variety of a group over Z by Nakamoto and Saito studied in [12].

Proof. By [12], Theorem 6.18, there exists a separated scheme Ch_d(G) over Z such that for any algebraically closed field \( \Omega \) there is a bijection between \( \text{Rep}_d(G)(\Omega) / \text{PGL}_d(\Omega) \) and \( \text{Ch}_d(G)(\Omega) \), where \( \text{Ch}_d(G)(\Omega) \) is the set of \( \Omega \)-rational points of the scheme \( \text{Ch}_d(G) \). In fact, this induces a bijection between \( \text{Rep}_d(G)(k)/\text{PGL}_d(k) \) and \( \text{Ch}_d(G)(k) \) for any finite field \( k \) by the same argument as in the proof of the GL_d case ([7], Lemma 2.3.1) as follows. Since we consider only absolutely irreducible representations, the natural map

\[
\text{Rep}_d(G)(k)/\text{PGL}_d(k) \to \text{Rep}_d(G)(\overline{k})/\text{PGL}_d(\overline{k})
\]

is an injection by the Skolem-Noether theorem, where \( \overline{k} \) is an algebraic closure of a finite field \( k \). Hence it is sufficient to prove the surjectivity of the following map:

\[
\text{Rep}_d(G)(k)/\text{PGL}_d(k) \to (\text{Rep}_d(G)(\overline{k})/\text{PGL}_d(\overline{k}))^\text{Gal}(\overline{k}/k) \to \text{Ch}_d(G)(k).
\]

Here, for any element \( \sigma \) of \( \text{Gal}(\overline{k}/k) \) the group actions on both sides are defined by \( (\sigma \rho)(g) := \sigma(\rho(g)) \) and \( \sigma x := x \circ \sigma \) respectively for each \( \rho \in \text{Rep}_d(G)(\overline{k}) \) and \( x \in \text{Ch}_d(G)(\overline{k}) \), where \( \sigma : \text{Spec} \overline{k} \to \text{Spec} k \) is the morphism over \( k \) associated with the \( k \)-automorphism \( \sigma \). Since \( \text{Rep}_d(G)(\overline{k})/\text{PGL}_d(\overline{k}) \to \text{Ch}_d(G)(\overline{k}) \), for any point \( x \) of \( \text{Ch}_d(G)(k) \) there is a representation \( \rho' \in \text{Rep}_d(G)(\overline{k}) \) such that \( \rho' \) corresponds to \( x \). For any \( \sigma \in \text{Gal}(\overline{k}/k) \), both of \( \rho' \) and \( \sigma \rho' \) correspond to \( x \). Hence there is a unique element \( s(\sigma) \) of \( \text{PGL}_d(\overline{k}) \) such that \( \sigma \rho' = s(\sigma) \cdot \rho' \), where \( s(\sigma) \cdot \rho' \) is the conjugate of \( \rho' \) by \( s(\sigma) \). This correspondence \( \sigma \mapsto s(\sigma) \) defines a 1-cocycle. Since \( H^1(k, \text{PGL}_d(\overline{k})) = 1 \), there is an element \( A \) of \( \text{PGL}_d(\overline{k}) \) such that \( \sigma(A)A^{-1} = s(\sigma) \) for any \( \sigma \in \text{Gal}(\overline{k}/k) \). Put \( \rho := A^{-1} \cdot \rho' \). Then \( \rho \) is \( \text{Gal}(\overline{k}/k) \)-invariant; i.e., \( \rho \) is contained in \( \text{Rep}_d(G)(k) \) and the image of \( \rho \) in \( \text{Ch}_d(G)(k) \) is \( x \). Thus the zeta function \( Z_d(G, q, T) \) is equal to the congruence zeta function \( Z(\text{Ch}_d(G), T) \) of \( \text{Ch}_d(G) \). Since \( Z(\text{Ch}_d(G), T) \) is a rational function in \( T \) by the work [9] of Dwork, \( Z_d(G, q, T) \) is a rational function in \( T \).

\[ \square \]

2. Weil-type zeta functions of the figure 8 knot group

Let \( K \) be the figure 8 knot and \( G_K \) the knot group \( \pi_1(S^3 \setminus K) \). Here we calculate the Weil-type zeta function \( Z(G_K, q, T) := Z_2(G_K, q, T) \) of the figure 8 knot group for the degree 2 case. It is well known that \( G_K \) has a group presentation such as

\[
G_K = \langle a, b \mid R(a,b) := b^{-1}a^{-1}bab^{-1}aba^{-1}b^{-1}a = 1 \rangle.
\]

Note that the elements \( a \) and \( b \) are conjugate. In fact, we have \( b = dad^{-1} \) for \( d := a^{-1}bab^{-1} \). First we prepare some lemmas.

Lemma 2.1 (Trace Identity, cf. [21], §2). Let \( A \) be a commutative ring and let \( A, B, C \) be elements of \( \text{SL}_2(A) \). Then the following equalities hold:

1. \( \text{Tr}(A) = \text{Tr}(A^{-1}) \).
2. \( \text{Tr}(AB) = \text{Tr}(A)\text{Tr}(B) - \text{Tr}(AB^{-1}) \).
Lemma 2.2 (cf. [21], Lemma 1). Let $A, B \in \text{SL}_2(\mathbb{A})$ and $R := R(A, B)$. Suppose that $x := \text{Tr}(A) = \text{Tr}(B)$ and $z := \text{Tr}(AB)$. Then the following hold:

1. $\text{Tr}(RB) = x$.
2. $\text{Tr}(R) - 2 = (x^2 - z - 2)(z^2 - (1 + x^2)z + 2x^2 - 1)^2$.
3. $\text{Tr}(RA^{-1}) - x = x(2 - z)(x^2 - z - 2)(z^2 - (1 + x^2)z + 2x^2 - 1)$.

Lemma 2.3 (cf. [21], Lemma 2). Let $\mathbf{k}$ be an algebraically closed field of arbitrary characteristic. Let $\rho: G_K \to \text{SL}_2(\mathbf{k})$ be a representation. Then $\rho$ is reducible if and only if $\text{Tr}(\rho(a), \rho(b)) = 2$.

Proposition 2.5. Let $\mathbf{k}$ be an algebraically closed field of arbitrary characteristic. Then there is a bijection between $\text{Rep}_2(G_K)(\mathbf{k})/\text{PGL}_2(\mathbf{k})$ and the set

$$E'(\mathbf{k}) := \{(x, z) \in \mathbf{k}^2 \mid z^2 - (1 + x^2)z + 2x^2 - 1 = 0\} \setminus \{ \pm \sqrt{5}, 3 \}.$$ 

Proof: Let $\rho: G_K \to \text{SL}_2(\mathbf{k})$ be a representation. Put $x := \text{Tr}(\rho(a))$ and $z := \text{Tr}(\rho(ab))$. Since $\text{Tr}(\rho(aba^{-1}b^{-1})) = 2x^2 + z^2 - x^2 z - 2$, the representation $\rho$ is reducible if and only if $z = x^2 - 2$ or $z = 2$ by Lemma 2.2. Hence the set of irreducible representations $\rho: G_K \to \text{SL}_2(\mathbf{k})$ is equal to the set of representations $\{\rho: G_K \to \text{SL}_2(\mathbf{k}) : z \neq x^2 - 2, 2\}$. By Lemma 2.2 (2) we have a map

$$\{\rho: G_K \to \text{SL}_2(\mathbf{k}) : z \neq x^2 - 2, 2\} / \sim \to E'(\mathbf{k})$$

which maps a representation $\rho$ to the pair $(\text{Tr}(a), \text{Tr}(ab))$. Now we prove that the above map is bijective. Since we consider only irreducible representations, we see that it is injective (cf. [12], Theorem 6.12). For any point $(x, z)$ of $E'(\mathbf{k})$, put

$$A = \begin{pmatrix} \alpha & -z + x^2 + 2 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} \alpha & 0 \\ 1 & \alpha^{-1} \end{pmatrix},$$

where $\alpha$ is an element of $\mathbf{k}$ such that $\alpha + \alpha^{-1} = x$. Then we see that $\text{Tr}(A) = \text{Tr}(B) = x$ and $\text{Tr}(AB) = z$, and that $\text{Tr}(R(A, B)) = 2$ and $\text{Tr}(RA) = x$ by Lemma 2.2 (2) and Lemma 2.2 (2), (3). By Lemma 2.3 we have a representation $\rho: G_K \to \text{SL}_2(\mathbf{k})$ which maps $a, b$ to $A, B$ respectively. Note that it is irreducible by Lemma 2.3. Hence we have a bijection between $\text{Rep}_2(G_K)(\mathbf{k})/\text{PGL}_2(\mathbf{k})$ and $E'(\mathbf{k})$. \hfill $\square$
If we take $\mathbb{F}_p$ as $k$, then the bijection in Proposition 2.5 is an isomorphism between $\text{Gal}(\mathbb{F}_p/\mathbb{F}_p)$-sets. Since $H^1(\mathbb{F}_p, \text{PGL}_2(\mathbb{F}_p)) = 1$, we have

$$(\text{Rep}_2(\mathbb{G}_k)(\mathbb{F}_p)/\text{PGL}_2(\mathbb{F}_p))^\text{Gal}(\mathbb{F}_p/\mathbb{F}_p) \cong \text{Rep}_2(\mathbb{G}_k)(\mathbb{F}_{p^n})/\text{PGL}_2(\mathbb{F}_{p^n}).$$

Hence, by taking the Gal($\mathbb{F}_p/\mathbb{F}_{p^n}$)-fixed parts on both sides in Proposition 2.5 we have the following:

**Corollary 2.6.** Let $k$ be a finite field. Then there is a bijection between $\text{Rep}_2(\mathbb{G}_k)(k)/\text{PGL}_2(k)$ and the set

$$E'(k) = \{(x, z) \in k^2 \mid z^2 - (1 + x^2)z + 2x^2 - 1 = 0\} \setminus \{\pm \sqrt{5}, 3\}.$$ 

**Lemma 2.7.** The algebraic curve $E$ defined by the equation

$$(1) \quad X^2 - (1 + Y^2)X + 2Y^2 - 1 = 0$$

is an elliptic curve over $\mathbb{Q}$. The minimal Weierstrass equation is given by

$$Y^2 = X^3 - 2X + 1$$

and its conductor $N_{E/\mathbb{Q}}$ is 40 = $2^3 5$. Moreover, mod $p$ reductions of these curves are isomorphic for any prime $p$.

**Proof.** It is clear that the curve (1) is an elliptic curve over $\mathbb{Q}$ (for instance, $(1, \pm 1)$ are $\mathbb{Q}$-rational points). Next we calculate a Weierstrass form of the equation (1). In the equation

$$X^2 - (1 + Y^2)X + 2Y^2 - 1 = X^2 - X - 1 - Y^2(X - 2),$$

change the variable $X - 2$ by $X'$. Then we have

$$X'^2 + 3X' + 1 - Y^2X' = 0.$$ 

Multiplying $X'$ on each side and replacing $X'Y$ by $Y'$, we have

$$X'^3 + 3X'^2 + X' - Y'^2 = 0.$$ 

Finally by changing the variable $X' + 1$ by $X$ (and $Y' = Y$), we have the minimal Weierstrass form

$$Y^2 = X^3 - 2X + 1.$$ 

Hence we also know that mod $p$ reductions of the curves $X^2 - (1 + Y^2)X + 2Y^2 - 1 = 0$ and $Y^2 - (X^3 - 2X + 1) = 0$ are isomorphic over $\mathbb{F}_p$.

By the Tate Algorithm (cf. [16], Chapter IV, §9) (we used the software PARI-GP for the calculation), we know that its conductor is 40. □

By Corollary 2.6 we have

$$\#(\text{Rep}_2(\mathbb{G}_k)(\mathbb{F}_q)/\text{PGL}_2(\mathbb{F}_q)) = \#E_p(\mathbb{F}_q) - 1 - (*)_q,$$

where $q = p^r$ and $E_p$ is the mod $p$ reduction of the elliptic curve $E$ over $\mathbb{F}_q$ as a projective curve (in the above equation, $-1$ means to remove the point at infinity). The value $(*)_q$ is

$$(*)_q := \begin{cases} 
1 & \text{if } p = 2, 5, \\
2 & \text{if } p \neq 2, 5, \left( \frac{5}{p} \right) = 1 \text{ and } r \geq 1, \\
1 + (-1)^r & \text{if } p \neq 2, 5, \left( \frac{5}{p} \right) = -1 \text{ and } r \geq 1.
\end{cases}$$
Hence we have
\[
\exp \left( \sum_{n=1}^{\infty} \frac{(-1)^n \sigma(n)}{n} T^n \right) = (1 - T)(1 - \left( \frac{20}{q} \right) T)
\]
\[
\begin{cases} 
(1-T) & \text{if } p = 2, 5, \\
(1-T)^2 & \text{if } p \neq 2, 5, \left( \frac{5}{p} \right) = 1 \text{ and } r \geq 1, \\
(1-T)^2 & \text{if } p \neq 2, 5, \left( \frac{5}{p} \right) = -1 \text{ and } 2 \mid r. \\
\end{cases}
\]

Here \( \left( \frac{20}{q} \right) \) is the Jacobi symbol. Thus we have the following:

**Theorem 2.8.** Let \( G_K = \pi_1(S^3 \setminus K) \) be the knot group of the figure 8 knot and let \( q \) be a power of a prime number \( p \). We have

\[
Z(G_K, q, T) := Z_2(G_K, q, T) = Z_{E_p}(q, T)(1 - \left( \frac{20}{q} \right) T).
\]

Here, \( E \) is an elliptic curve over \( \mathbb{Q} \) defined by

\[
E : Y^2 = X^3 - 2X + 1,
\]

\( Z_{E_p}(q, T) \) is the congruence zeta function

\[
Z_{E_p}(q, T) := \exp \left( \sum_{n=1}^{\infty} \frac{\#E_p(F_q^n)}{q^n} T^n \right)
\]

of the mod \( p \) reduction \( E_p \) of the elliptic curve \( E \) over \( \mathbb{F}_q \), and \( \left( \frac{20}{q} \right) \) is the Jacobi symbol.

Now we calculate the congruence zeta functions \( Z_{E_p}(q, T) \) of curves \( E_p \) over \( \mathbb{F}_q \) at the bad primes 2, 5 of \( E \).

**Lemma 2.9.** Let \( E \) be the elliptic curve \( Y^2 = X^3 - 2X + 1 \). Then we have

\[
Z_{E_5}(2^r, T) = \frac{1}{(1-T)(1-2^r T)},
\]

\[
Z_{E_5}(5^r, T) = Z_{E_5}(5^r, T)(1-T) = \frac{1}{1-5^r T}.
\]

Here, \( E_5 \) is the normalization of \( \mathbb{E}_5 \) which is isomorphic to \( \mathbb{P}^1 \) over \( \mathbb{F}_5 \).

**Proof.** Since \( \mathbb{E}_2 : (Y-1)^2 = X^3 \) and the 2-power homomorphism on \( \mathbb{F}_{2^r,n} \) is an isomorphism, we have \( \#E_2(F_{2^r,n}) = 2^{2r} + 1 \). Thus \( Z_{E_2}(2^r, T) = 1/(1-T)(1-2^r T) \).

For the curve \( E_5 \), changing the coordinate \( X \) by \( X + 2 \), we have

\[
\mathbb{E}_5 : Y^2 = X^2(X + 1).
\]

It is well known that the curve \( \mathbb{E}_5 \) has a unique singular point \((0,0)\), which is an ordinary double point of \( \mathbb{E}_5 \) (cf. \cite{10}, 7.5, Example 5.14). The normalization of \( \text{Spec} \ A \), where \( A := \mathbb{F}_5[X,Y]/(Y^2 - X^2(X + 1)) \) is \( \text{Spec} \ A = \text{Spec} \mathbb{F}_{5^r}[u] \), \( u = Y/X \), \( u^2 = 1 + X \) (cf. \cite{10}, 8.2, Example 2.42). If \( \pi : \text{Spec} \ A \rightarrow \text{Spec} \ A \) is the normalization morphism, the fiber \( \pi^{-1}(P) \), where \( P = (x, y) \in \text{Spec} \ A \) is the point corresponding to \((0,0)\), consists of two points \{\((u-1), (u+1)\}\). Hence we have \( Z_{E_5}(5^r, T) = Z_{E_5}(5^r, T)(1-T) \). Since \( \mathbb{E}_5(\mathbb{F}_5) \neq \{P, O\} \), we have \( \mathbb{E}_5(\mathbb{F}_5) \neq \{O\} \). Hence \( \mathbb{E}_5 \) is
isomorphic to \( \mathbb{P}^1 \) over \( \mathbb{F}_5 \) since the arithmetic genus of \( \hat{E}_5 \) is \( p_a(\hat{E}_5) = p_a(\overline{E}_5) - 1 = 0 \) (cf. \cite{10}, 7.5, Proposition 5.4).

**Remark 2.10.** There is a related work of Sink (\cite{17}) on Weil-type zeta functions. As an attempt to define a mod \( p \) analogue of the Casson invariant, Sink has studied the Weil-type zeta function defined by the numbers of \( \text{SL}_2(\mathbb{F}_p) \)-conjugacy classes of non-diagonal \( \text{SL}_2(\mathbb{F}_p) \)-representations of a knot group. In particular he has calculated the Weil-type zeta functions for \((a, 2)\)-torus knot groups.

3. HASSE-WEIL ZETA FUNCTION OF THE FIGURE 8 KNOT GROUP

By the description of the Weil type zeta function obtained in Theorem 2.8 here we study the Hasse-Weil zeta function of \( \text{SL}_2 \)-absolutely irreducible representations of \( G_K \) over \( \mathbb{Q} \),

\[
\zeta_A(s) := \prod_p Z(G_K, p, p^{-s}),
\]

where \( p \) runs through all the prime numbers. We shall prove that the Hasse-Weil zeta function \( \zeta_A(s) \) over \( \mathbb{Q} \) is continued meromorphically to the whole complex plane \( \mathbb{C} \). Moreover, we have a functional equation and calculate its central value at \( s = 1 \).

3.1. \( L \)-series of \( E/\mathbb{Q} \). For the calculation of \( \zeta_A(s) \) it is essential to compute the \( L \)-series \( L_{E/\mathbb{Q}}(s) \) of the elliptic curve \( E \) over \( \mathbb{Q} \). Here we review some properties of the \( L \)-series of the elliptic curve \( E \) over \( \mathbb{Q} \). Since \( E \) has additive reduction at 2 and split multiplicative reduction at 5, the Hasse-Weil \( L \)-series \( L_{E/\mathbb{Q}}(s) \) of \( E/\mathbb{Q} \) is defined (cf. \cite{15}) by

\[
L_{E/\mathbb{Q}}(s) = \prod_p L_p(p^{-s})^{-1},
\]

where if \( p \neq 5, 2 \),

\[
L_p(T) := Z_{\overline{E}_5}(p, T)(1 - T)(1 - pT);
\]

and if \( p = 2, 5 \), put

\[
L_2(T) := 1, \quad L_5(T) := 1 - T.
\]

By the Modularity Theorem (\cite{22}, \cite{18} and \cite{2}), for any elliptic curve over \( \mathbb{Q} \) there is a Hecke eigenform of weight 2 whose \( L \)-function is equal to the Hasse-Weil \( L \)-series of the elliptic curve. In particular, the \( L \)-series of an elliptic curve over \( \mathbb{Q} \) is continued meromorphically to the whole complex plane. By Lemma 2.9 we have

\[
L_{E/\mathbb{Q}}(s)\xi_E(s) = \zeta_E(s)\zeta(Q)(s - 1),
\]

where \( \zeta_E(s) := \prod_p Z_{\overline{E}_5}(p, p^{-s}) \). Let \( \mathcal{L}_{E/\mathbb{Q}}(s) := N_{E/\mathbb{Q}}^{s/2}(2\pi)^{-s}\Gamma(s)L_{E/\mathbb{Q}}(s) \) be the completed Hasse-Weil \( L \)-series, where \( \Gamma(s) \) is the Gamma function. The function \( \mathcal{L}_{E/\mathbb{Q}}(s) \) has a functional equation \( \mathcal{L}_{E/\mathbb{Q}}(s) = w\mathcal{L}_{E/\mathbb{Q}}(2 - s) \), where \( w = \pm 1 \). Since the Mordell-Weil rank and the analytic rank of \( E \) are both zero (see Cremona’s database on elliptic curves) we have \( w = 1 \). By the duplication formula of the Gamma function \( \Gamma(s) \),

\[
\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s + 1}{2}\right) = 2^{1-s}\sqrt{\pi}\Gamma(s),
\]

we have

\[
\mathcal{L}_{E/\mathbb{Q}}(s)\xi_E(s) = \xi_Q(s)\zeta_Q(s - 1) \times \frac{(s - 1)(\sqrt{40})^s}{4\pi},
\]

where \( \xi_Q(s) := (\sqrt{\pi})^{-s}\Gamma(s/2)\zeta(s) \) is the completed Riemann zeta function, which has a functional equation \( \xi_Q(s) = \xi_Q(1 - s) \).
3.2. Calculation of the Hasse-Weil zeta function. Now we calculate the Hasse-Weil zeta function $\zeta_A(s)$ of absolutely irreducible $SL_2$-representations of the figure 8 knot group. Let $\zeta_{Q(\sqrt{5})}(s)$ be the Dedekind zeta function of $Q(\sqrt{5})$ and $\xi_{Q(\sqrt{5})}(s) := \sqrt{5} \pi^{-s} \Gamma(s/2)^2 \zeta_{Q(\sqrt{5})}(s)$ the completed Dedekind zeta function.

It is well known that $\zeta_{Q(\sqrt{5})}(s)$ is written as $\zeta_{Q(\sqrt{5})}(s) = \zeta(s) L\left(\frac{5}{2}, s\right)$, where $L\left(\frac{5}{2}, s\right)$ is the Dirichlet $L$-function defined by the character $\left(\frac{5}{\cdot}\right)$. Hence we have

$$\zeta_A(s) = \prod_p Z(G_K, p, p^{-s}) = \frac{\zeta_E(s)}{\zeta_Q(s) \xi_{Q(\sqrt{5})}(s)}$$

by Theorem 2.8. Thus $\zeta_A(s)$ is also continued meromorphically to the whole complex plane, and the completion

$$\xi_A(s) := \frac{4\pi^{(s/2)+1} \Gamma(s/2)^2}{(10\sqrt{2}) \Gamma(s/2)^2} \zeta_A(s) = \frac{(s-1)\xi_Q(s-1)}{L_{E/Q}(s) \xi_{Q(\sqrt{5})}(s)}$$

has a functional equation $\xi_A(2-s)\xi_Q(2-s)\xi_{Q(\sqrt{5})}(2-s) = -\xi_A(s)\xi_Q(s)\xi_{Q(\sqrt{5})}(s)$. Now we calculate the central value of $\xi_A(s)$. In the expression of $\xi_A(s)$, we know that $L_{E/Q}(1) \neq 0$ since $E$ is an elliptic curve of rank zero. Since $\lim_{s \to 1} (s-1)\xi_Q(s-1) = -1$ and

$$\lim_{s \to 1} (s-1)\xi_{Q(\sqrt{5})}(s) = \lim_{s \to 1} (s-1)\zeta(s) L\left(\frac{5}{2}, s\right) = \frac{\log(\varphi^2)}{\sqrt{5}},$$

where $\varphi := (1 + \sqrt{5})/2$, we have

$$\lim_{s \to 1} (s-1)\xi_{Q(\sqrt{5})}(s) = \log(\varphi^2).$$

It is known that the Birch and Swinnerton-Dyer conjecture holds for the elliptic curve $E$ since $E$ has rank zero. Thus we also have $L_{E/Q}(1) = \frac{1}{2} \Omega(E)$, where $\Omega(E)$ is the real period of the elliptic curve $E$. By [3], Algorithm 7.4.7, we have $\Omega(E) = \frac{2\pi}{\text{AGM}(\varphi, \varphi - 1)}$, where $\text{AGM}(\varphi, \varphi - 1)$ is the arithmetic-geometric mean of $\varphi = (\sqrt{5} + 1)/2$ and $\varphi - 1 = (\sqrt{5} - 1)/2$. Hence we have

$$L_{E/Q}(1) = \frac{\sqrt{10}}{4\pi} \Omega(E) = \frac{\sqrt{10}}{2 \text{AGM}(\varphi, \varphi - 1)}.$$ 

Thus we have

$$\lim_{s \to 1} \frac{\xi_A(s)}{s-1} = \lim_{s \to 1} \frac{\xi_Q(s-1)}{L_{E/Q}(s) \xi_{Q(\sqrt{5})}(s)}$$

$$= -\frac{\text{AGM}(\varphi, \varphi - 1)}{\sqrt{10} \log(\varphi)}.$$ 

**Theorem 3.1.** The function $\zeta_A(s)$ and its completion $\xi_A(s)$ are continued meromorphically to the complex plane $\mathbb{C}$ and $\xi_A(s)$ has a functional equation $\xi_A(2-s)\xi_Q(2-s)\xi_{Q(\sqrt{5})}(2-s) = -\xi_A(s)\xi_Q(s)\xi_{Q(\sqrt{5})}(s)$.
The function $\xi_A(s)$ has a simple zero at $s = 1$ and the central value $\xi_A(1)$ is given by

$$\lim_{s \to 1} \frac{\xi_A(s)}{s - 1} = -\frac{\text{AGM}(\varphi, \varphi - 1)}{\sqrt{10}\log(\varphi)},$$

where $\varphi = (\sqrt{5} + 1)/2$.

**Remark 3.2.** The Alexander polynomial of the figure 8 knot is $\Delta_K(T) = T^2 - 3T + 1$. Thus $\varphi$, $\varphi - 1$ are square roots of the roots of $\Delta_K(T)$. We also note that (at least for representations over $\mathbb{C}$) by [4], Proposition 6 (see also [9], Corollary 4.3) $\varphi$, $\varphi - 1$ are eigenvalues at the meridian $a \in G_K$ of the (non-abelian) reducible $\text{SL}_2$-representations of $G_K$ corresponding to the points $(\pm \sqrt{5}, 3)$ of the $\text{SL}_2$-character variety of $G_K$.

**Remark 3.3.** It is known that $\log(\varphi^2)$ is written as

$$\log(\varphi^2) = m(\Delta_K(T)) = \frac{d}{d\phi} m(P_k(\sqrt{5}))^{-1},$$

where $m(\Delta_K(T))$ is the logarithmic Mahler measure of $\Delta_K(T)$ and $m_n$ is the $n$-fold cyclic covering of $S^3$ branched over the figure 8 knot $K$. Here, for a given Laurent polynomial $P = P(t_1, \ldots, t_n) \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, the logarithmic Mahler measure of $P$ is defined by

$$m(P) := \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi t_1\sqrt{-1}}, \ldots, e^{2\pi t_n\sqrt{-1}})| dt_1 \cdots dt_n.$$

**Remark 3.4.** The arithmetic-geometric mean of $\varphi$, $\varphi - 1$ is also written as

$$\text{AGM}(\varphi, \varphi - 1) = \frac{1}{2} \left(\frac{d}{d\phi} m(P_k(\sqrt{5}))^{-1}\right).$$

Here $P_k$ is a polynomial,

$$x + \frac{1}{x} + y + \frac{1}{y} - 4k,$

which defines an elliptic curve for any complex number $4k \neq 0, 1$. This follows from the following equality ([20], Lemma):

$$m(P_k) = -\frac{1}{2} \Phi(1/k^2) + \log(4)$$

and the relations

$$x \frac{d}{dx} \Phi(x) = 2F_1 \left(\frac{1}{2}, 1; 1; x\right), \quad \text{AGM}(a, b) = \frac{(a + b)\pi}{4K((a - b)/(a + b))},$$

$$K(x) = \frac{\pi}{2} \cdot 2F_1 \left(\frac{1}{2}, 1; 1; x^2\right),$$

where $\Phi(x)$ is defined by

$$\Phi(x) = \log x + \sum_{n=1}^{\infty} \left(\frac{1/2}{n}\right)^2 \frac{x^n}{n},$$

and $2F_1 \left(\frac{1}{2}, \frac{1}{2}; 1; x\right)$ and $K(x)$ are the classical hypergeometric series and the complete elliptic integral of the first kind respectively. The Weierstrass equations of $P_k$ are given by

$$E_m : y^2 = x^3 + 2m(2m - 1)x^2 + m^2 x,$$
where \( m \) is defined by \( m = k^2 \). There is a relation between the logarithmic Mahler measure of the \( A \)-polynomial of the figure 8 knot and the hyperbolic volume of the figure 8 knot complement \([1]\). It would be interesting to find the relation between the figure 8 knot and this family of elliptic curves, especially the meaning of the value \( \sqrt{5} \).

**Remark 3.5.** Can we also have a description of the value

\[
\lim_{s \to 2} (s - 2) \xi_A(s) = \frac{75}{2\sqrt{5\pi^2}L_{E/Q}(2)}
\]

in terms of the Mahler measure of a family of polynomials \( P_k \)? It is numerically shown by R. Villegas ([19], Table 4) that \( L_{E/Q}(2) = L'_{E/Q}(0) = \frac{m(P_{\sqrt{-4}})}{4} \). We note that the elliptic curve defined by \( P_{\sqrt{-4}} \) is isomorphic to \( E \).

**ACKNOWLEDGMENTS**

The author is indebted to Masanori Morishita for his helpful comments and encouragement throughout this work, in particular, for informing him about Remark 3.3. The author also expresses his gratitude to Yuji Terashima for telling him about Remark 3.4. The author also thanks Don Zagier for his comments, which refined the description of the Weil-type zeta function and Hasse-Weil zeta function of the figure 8 knot in this paper. Finally, the author is grateful to the referee for suggestions on the previous manuscript, which improved the exposition of this paper.

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School of Mathematics, Korea Institute for Advanced Study (KIAS), 207-43 Cheongnyangni 2-dong, Dongdaemun-gu, Seoul 130-722, Republic of Korea

E-mail address: harada@kias.re.kr