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HASSE-WEIL ZETA FUNCTION OF ABSOLUTELY IRREDUCIBLE SL_2-REPRESENTATIONS OF THE FIGURE 8 KNOT GROUP

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ABSTRACT. Weil-type zeta functions defined by the numbers of absolutely irreducible SL_2-representations of the figure 8 knot group over finite fields are computed explicitly. They are expressed in terms of the congruence zeta functions of reductions of a certain elliptic curve defined over the rational number field. Then the Hasse-Weil type zeta function of the figure 8 knot group is also studied. Its central value is written in terms of the Mahler measures of the Alexander polynomial of the figure 8 knot and a certain family of elliptic curves.

0. INTRODUCTION

The character variety of SL_2-representations of a 3-manifold group is an important object in the study of 3-dimensional topology ([5]). It is known that a character variety for a finitely generated group is an algebraic variety defined by a finite number of polynomials with coefficients in the ring of integers $\mathbb{Z}$. Hence we are naturally led to study the Hasse-Weil type zeta function of the character variety for a 3-manifold group as an arithmetic invariant of the manifold and expect that it encodes interesting information which reflects the topology of the 3-manifold.

In this paper, as a first step in this direction, we compute the *Weil-type zeta functions*

$$Z_d(G_{\mathcal{K}}, q, T) := \exp\left(\sum_{n=1}^{\infty} \frac{A_{d,n}}{n} T^n\right)$$

and a *Hasse-Weil type zeta function* for the knot group $G_{\mathcal{K}} := \pi_1(S^3 \smallsetminus \mathcal{K})$ of the figure 8 knot $\mathcal{K}$ (the fundamental group of the complement of $\mathcal{K}$ in the 3-sphere S^3) defined by the numbers $(A_{d,n})_{n\geq 1}$ for the degree $d = 2$ case. Here, for a finitely generated group G and a power q of a prime number p, $A_{d,n}$ is the number of $\mathrm{GL}_d(\mathbb{F}_{q^n})$-conjugacy classes of absolutely irreducible representations $\rho : G \to \mathrm{SL}_d(\mathbb{F}_{q^n})$, where $\mathbb{F}_{q^n}$ is the finite field of q^n elements.

We note that the figure 8 knot is a unique arithmetic knot among hyperbolic knots ([14]). In particular, the holonomy representation associated with the complete hyperbolic structure on $S^3 \smallsetminus \mathcal{K}$ gives a representation $\rho : G_{\mathcal{K}} \to \mathrm{SL}_2(\mathcal{O}_{-3})$,

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where $\mathcal{O}_{-3}$ is the ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{-3})$. Hence the figure 8 knot is the most appropriate one as a first example for our arithmetic study of the character variety and we may expect that the Weil-type zeta function $Z_d(G_{\mathcal{K}}, q, T)$ for the figure 8 knot $\mathcal{K}$ has more interesting information among all the hyperbolic knots.

In Section 1 we prove that Weil-type zeta functions $Z_d(G, q, T)$ are rational functions in T for any finitely generated group G and degree d (Theorem 1.1). In Section 2 we prove that $Z_2(G_{\mathcal{K}}, q, T)$ are expressed in terms of congruence zeta functions of reductions of a certain elliptic curve over $\mathbb{Q}$ (Theorem 2.8). In Section 3 we calculate the *Hasse-Weil zeta function* of absolutely irreducible SL_2-representations of the figure 8 knot group over $\mathbb{Q}$,

$$\zeta_A(s) := \prod_p Z_2(G_{\mathcal{K}}, p, p^{-s}),$$

where p runs through all the prime numbers. Our main result in this paper is the following:

Theorem (Theorem 3.1)**.** *The function $\zeta_A(s)$ and its completion*

$$\xi_A(s) := \frac{4\pi^{(3s/2)+1}}{(10\sqrt{2})^s \Gamma(s/2)^3} \times \zeta_A(s)$$

are continued meromorphically to the complex plane $\mathbb{C}$ and $\xi_A(s)$ has a functional equation

$$\xi_A(2-s)\xi_{\mathbb{Q}}(2-s)\xi_{\mathbb{Q}(\sqrt{5})}(2-s) = -\xi_A(s)\xi_{\mathbb{Q}}(s)\xi_{\mathbb{Q}(\sqrt{5})}(s),$$

where $\xi_{\mathbb{Q}}(s)$ and $\xi_{\mathbb{Q}(\sqrt{5})}(s)$ are the completed Riemann zeta function and the completed Dedekind zeta function of $\mathbb{Q}(\sqrt{5})$ respectively. The function $\xi_A(s)$ has a simple zero at $s = 1$, and the central value $\xi_A(1)$ is given by

$$\lim_{s\to 1} \frac{\xi_A(s)}{s-1} = -\frac{\mathrm{AGM}(\varphi, \varphi - 1)}{\sqrt{10}\log(\varphi)},$$

where $\varphi = (\sqrt{5}+1)/2$ and $\mathrm{AGM}(\varphi, \varphi-1)$ is the arithmetic-geometric mean of φ and $\varphi - 1$.

Note that φ and $\varphi - 1$ are square roots of the Alexander polynomial of the figure 8 knot. Hence the central value at $s = 1$ is expressed in terms of topological invariants of the figure 8 knot. There is also another description of the central value of the Hasse-Weil zeta function in terms of the Mahler measures of the Alexander polynomial of the figure 8 knot and certain polynomials which define a family of elliptic curves (see Remarks 3.3, 3.4).

1. Rationality of $Z_d(G, q, T)$

Let G be a finitely generated group and $\mathbb{F}_q$ the finite field with q elements. Let $\mathrm{Rep}_d(G)(k)$ be the set of absolutely irreducible representations of G into $\mathrm{SL}_d(k)$ for any field k. Here we say that a representation $\rho : G \to \mathrm{SL}_d(k)$ is *absolutely irreducible* if the composition of homomorphisms $G \xrightarrow{\rho} \mathrm{SL}_d(k) \hookrightarrow \mathrm{GL}_d(\overline{k})$ is irreducible, where $\overline{k}$ is an algebraic closure of the field k. Then the projective linear group $\mathrm{PGL}_d(k)$ acts on $\mathrm{Rep}_d(G)(k)$ by conjugation. Let $A_{d,n} := \#(\mathrm{Rep}_d(G)(\mathbb{F}_{q^n})/\mathrm{PGL}_d(\mathbb{F}_{q^n}))$ be the number of $\mathrm{PGL}_d(\mathbb{F}_{q^n})$-conjugacy classes of absolutely irreducible representations $\rho : G \to \mathrm{SL}_d(\mathbb{F}_{q^n})$. In this section we prove the following:

Theorem 1.1. *$Z_d(G, q, T)$ is a rational function in T.*

In [8] we have proved an analogous result for absolutely irreducible GL_d-representations. It is proved by employing the moduli theory of Procesi ([13]) on absolutely irreducible representations of a non-commutative ring into Azumaya algebras. We can prove Theorem 1.1 by the same argument, but here we give a proof of Theorem 1.1 by the theory of the character variety of a group over $\mathbb{Z}$ by Nakamoto and Saito studied in [12].

Proof. By [12], Theorem 6.18, there exists a separated scheme $\mathrm{Ch}_d(G)$ over $\mathbb{Z}$ such that for any algebraically closed field Ω there is a bijection between $\mathrm{Rep}_d(G)(\Omega)/\mathrm{PGL}_d(\Omega)$ and $\mathrm{Ch}_d(G)(\Omega)$, where $\mathrm{Ch}_d(G)(\Omega)$ is the set of Ω-rational points of the scheme $\mathrm{Ch}_d(G)$. In fact, this induces a bijection between $\mathrm{Rep}_d(G)(k)/\mathrm{PGL}_d(k)$ and $\mathrm{Ch}_d(G)(k)$ for any finite field k by the same argument as in the proof of the GL_d case ([7], Lemma 2.3.1) as follows. Since we consider only absolutely irreducible representations, the natural map

$$\mathrm{Rep}_d(G)(k)/\mathrm{PGL}_d(k) \to \mathrm{Rep}_d(G)(\overline{k})/\mathrm{PGL}_d(\overline{k})$$

is an injection by the Skolem-Noether theorem, where $\overline{k}$ is an algebraic closure of a finite field k. Hence it is sufficient to prove the surjectivity of the following map:

$$\mathrm{Rep}_d(G)(k)/\mathrm{PGL}_d(k) \to \left(\mathrm{Rep}_d(G)(\overline{k})/\mathrm{PGL}_d(\overline{k})\right)^{\mathrm{Gal}\,(\overline{k}/k)} \tilde{\to}\ \mathrm{Ch}_d(G)(k).$$

Here, for any element σ of $\mathrm{Gal}\,(\overline{k}/k)$ the group actions on both sides are defined by $(\sigma\rho)(g) := \sigma(\rho(g))$ and $\sigma x := x \circ \tilde{\sigma}$ respectively for each $\rho \in \mathrm{Rep}_d(G)(\overline{k})$ and $x \in \mathrm{Ch}_d(G)(\overline{k})$, where $\tilde{\sigma} : \mathrm{Spec}\,\overline{k} \to \mathrm{Spec}\,\overline{k}$ is the morphism over k associated with the k-automorphism σ. Since $\mathrm{Rep}_d(G)(\overline{k})/\mathrm{PGL}_d(\overline{k}) \tilde{\to} \mathrm{Ch}_d(G)(\overline{k})$, for any point x of $\mathrm{Ch}_d(G)(k)$ there is a representation $\rho' \in \mathrm{Rep}_d(G)(\overline{k})$ such that ρ' corresponds to x. For any $\sigma \in \mathrm{Gal}\,(\overline{k}/k)$, both of ρ' and $\sigma\rho'$ correspond to x. Hence there is a unique element $s(\sigma)$ of $\mathrm{PGL}_d(\overline{k})$ such that $\sigma\rho' = s(\sigma) \cdot \rho'$, where $s(\sigma) \cdot \rho'$ is the conjugate of ρ' by $s(\sigma)$. This correspondence $\sigma \mapsto s(\sigma)$ defines a 1-cocycle. Since $H^1(k, \mathrm{PGL}_d(\overline{k})) = 1$, there is an element A of $\mathrm{PGL}_d(\overline{k})$ such that $\sigma(A)A^{-1} = s(\sigma)$ for any $\sigma \in \mathrm{Gal}\,(\overline{k}/k)$. Put $\rho := A^{-1} \cdot \rho'$. Then ρ is $\mathrm{Gal}\,(\overline{k}/k)$-invariant; i.e., ρ is contained in $\mathrm{Rep}_d(G)(k)$ and the image of ρ in $\mathrm{Ch}_d(G)(k)$ is x. Thus the zeta function $Z_d(G, q, T)$ is equal to the congruence zeta function $Z(\mathrm{Ch}_d(G), T)$ of $\mathrm{Ch}_d(G)$. Since $Z(\mathrm{Ch}_d(G), T)$ is a rational function in T by the work [6] of Dwork, $Z_d(G, q, T)$ is a rational function in T. □

2. Weil-type zeta functions of the figure 8 knot group

Let $\mathcal{K}$ be the figure 8 knot and $G_{\mathcal{K}}$ the knot group $\pi_1(S^3 \setminus \mathcal{K})$. Here we calculate the Weil-type zeta function $Z(G_{\mathcal{K}}, q, T) := Z_2(G_{\mathcal{K}}, q, T)$ of the figure 8 knot group for the degree 2 case. It is well known that G_K has a group presentation such as

$$G_{\mathcal{K}} = \langle a, b \mid R(a, b) := b^{-1}a^{-1}bab^{-1}aba^{-1}b^{-1}a = 1\rangle.$$

Note that the elements a and b are conjugate. In fact, we have $b = dad^{-1}$ for $d := a^{-1}bab^{-1}$. First we prepare some lemmas.

Lemma 2.1 (Trace Identity, cf. [21], §2)**.** *Let $\mathbf{A}$ be a commutative ring and let A, B, C be elements of $\mathrm{SL}_2(\mathbf{A})$. Then the following equalities hold:*

(1) $\mathrm{Tr}\,(A) = \mathrm{Tr}\,(A^{-1})$.
(2) $\mathrm{Tr}\,(AB) = \mathrm{Tr}\,(A)\mathrm{Tr}\,(B) - \mathrm{Tr}\,(AB^{-1})$.

(3) $\mathrm{Tr}\,(ABC) = \mathrm{Tr}\,(A)\mathrm{Tr}\,(BC) + \mathrm{Tr}\,(B)\mathrm{Tr}\,(CA) + \mathrm{Tr}\,(C)\mathrm{Tr}\,(AB) - \mathrm{Tr}\,(A)\mathrm{Tr}\,(B)\mathrm{Tr}\,(C) - \mathrm{Tr}\,(ACB)$.

We can prove the following lemmas in the same way as in [21], Lemma 1, Lemma 2, for the complex number field case.

Lemma 2.2 (cf. [21], Lemma 1)**.** *Let $\mathbf{A}$ be a commutative ring. Let $A, B \in \mathrm{SL}_2(\mathbf{A})$ and $R := R(A, B)$. Suppose that $x := \mathrm{Tr}\,(A) = \mathrm{Tr}\,(B)$ and $z := \mathrm{Tr}\,(AB)$. Then the following hold:*

(1) $\mathrm{Tr}\,(RB) = x$.
(2) $\mathrm{Tr}\,(R) - 2 = (x^2 - z - 2)(z^2 - (1 + x^2)z + 2x^2 - 1)^2$.
(3) $\mathrm{Tr}\,(RA^{-1}) - x = x(2 - z)(x^2 - z - 2)(z^2 - (1 + x^2)z + 2x^2 - 1)$.

Lemma 2.3 (cf. [21], Lemma 2)**.** *Let $\mathbf{k}$ be an algebraically closed field of arbitrary characteristic. Let $A, B, R \in \mathrm{SL}_2(\mathbf{k})$ and put $x := \mathrm{Tr}\,(A) = \mathrm{Tr}\,(B)$. Suppose that $\mathrm{Tr}\,(R) = 2$ and $\mathrm{Tr}\,(RA) = \mathrm{Tr}\,(RB) = x$. If the matrix AB satisfies $\mathrm{Tr}\,(AB) \neq 2, x^2 - 2$, then R is the identity matrix.*

By Lemma 2.2 and Lemma 2.3 Whittemore has essentially obtained a description of the $\mathrm{SL}_2(\mathbb{C})$-character variety of the figure 8 knot group (cf. [9], Proposition 4.1). In the following we show that it still holds for an arbitrary algebraically closed field $\mathbf{k}$, at least on the irreducible representation part.

Lemma 2.4 (cf. [11], Lemma 2.5 or [5], Lemma 1.5.5)**.** *Let $\mathbf{k}$ be an algebraically closed field of arbitrary characteristic. Let $\rho : G_{\mathcal{K}} \to \mathrm{SL}_2(\mathbf{k})$ be a representation. Then ρ is reducible if and only if $\mathrm{Tr}\,([\rho(a), \rho(b)]) = 2$.*

Proposition 2.5. *Let $\mathbf{k}$ be an algebraically closed field of arbitrary characteristic. Then there is a bijection between $\mathrm{Rep}_2(G_{\mathcal{K}})(\mathbf{k})/\mathrm{PGL}_2(\mathbf{k})$ and the set*

$$E'(\mathbf{k}) := \left\{(x, z) \in \mathbf{k}^2 \;\mid\; z^2 - (1 + x^2)z + 2x^2 - 1 = 0\right\} \smallsetminus \left\{(\pm\sqrt{5}, 3)\right\}.$$

Proof. Let $\rho\colon G_{\mathcal{K}} \to \mathrm{SL}_2(\mathbf{k})$ be a representation. Put $x := \mathrm{Tr}\,(\rho(a))$ and $z := \mathrm{Tr}\,(\rho(ab))$. Since $\mathrm{Tr}\,(\rho(aba^{-1}b^{-1})) = 2x^2 + z^2 - x^2 z - 2$, the representation ρ is reducible if and only if $z = x^2 - 2$ or $z = 2$ by Lemma 2.4. Hence the set of irreducible representations $\rho\colon G_{\mathcal{K}} \to \mathrm{SL}_2(\mathbf{k})$ is equal to the set of representations $\left\{\rho\colon G_{\mathcal{K}} \to \mathrm{SL}_2(\mathbf{k})\colon z \neq x^2 - 2, 2\right\}$. By Lemma 2.2 (2) we have a map

$$\left\{\rho\colon G_{\mathcal{K}} \to \mathrm{SL}_2(\mathbf{k})\colon z \neq x^2 - 2, 2\right\} / \sim \quad \longrightarrow \quad E'(\mathbf{k})$$

which maps a representation ρ to the pair $(\mathrm{Tr}\,(a), \mathrm{Tr}\,(ab))$. Now we prove that the above map is bijective. Since we consider only irreducible representations, we see that it is injective (cf. [12], Theorem 6.12). For any point (x, z) of $E'(\mathbf{k})$, put

$$A = \begin{pmatrix} \alpha & z - x^2 + 2 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} \alpha & 0 \\ 1 & \alpha^{-1} \end{pmatrix},$$

where α is an element of $\mathbf{k}$ such that $\alpha + \alpha^{-1} = x$. Then we see that $\mathrm{Tr}\,(A) = \mathrm{Tr}\,(B) = x$ and $\mathrm{Tr}\,(AB) = z$, and that $\mathrm{Tr}\,(R(A, B)) = 2$ and $\mathrm{Tr}\,(RA) = x$ by Lemma 2.1 (2) and Lemma 2.2 (2), (3). By Lemma 2.3 we have a representation $\rho\colon G_{\mathcal{K}} \to \mathrm{SL}_2(\mathbf{k})$ which maps a, b to A, B respectively. Note that it is irreducible by Lemma 2.4. Hence we have a bijection between $\mathrm{Rep}_2(G_{\mathcal{K}})(\mathbf{k})/\mathrm{PGL}_2(\mathbf{k})$ and $E'(\mathbf{k})$. $\square$

If we take $\overline{\mathbb{F}}_p$ as $\mathbf{k}$, then the bijection in Proposition 2.5 is an isomorphism between $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$-sets. Since $H^1(\mathbb{F}_p, \mathrm{PGL}_2(\overline{\mathbb{F}}_p)) = 1$, we have

$$\left(\mathrm{Rep}_2(G_{\mathcal{K}})(\overline{\mathbb{F}}_p)/\mathrm{PGL}_2(\overline{\mathbb{F}}_p)\right)^{\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^n})} \tilde{\to} \mathrm{Rep}_2(G_{\mathcal{K}})(\mathbb{F}_{p^n})/\mathrm{PGL}_2(\mathbb{F}_{p^n}).$$

Hence, by taking the $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^n})$-fixed parts on both sides in Proposition 2.5, we have the following:

Corollary 2.6. *Let k be a finite field. Then there is a bijection between $\mathrm{Rep}_2(G_{\mathcal{K}})(k)/\mathrm{PGL}_2(k)$ and the set*

$$E'(k) = \left\{(x,z) \in k^2 \;\mid\; z^2 - (1+x^2)z + 2x^2 - 1 = 0\right\} \smallsetminus \left\{(\pm\sqrt{5}, 3)\right\}.$$

Lemma 2.7. *The algebraic curve E defined by the equation*

$$X^2 - (1+Y^2)X + 2Y^2 - 1 = 0 \tag{1}$$

is an elliptic curve over $\mathbb{Q}$. The minimal Weierstrass equation is given by

$$Y^2 = X^3 - 2X + 1$$

and its conductor $N_{E/\mathbb{Q}}$ is $40 = 2^3 5$. Moreover, mod p reductions of these curves are isomorphic for any prime p.

Proof. It is clear that the curve (1) is an elliptic curve over $\mathbb{Q}$ (for instance $(1, \pm 1)$ are $\mathbb{Q}$-rational points). Next we calculate a Weierstrass form of the equation (1). In the equation

$$X^2 - (1+Y^2)X + 2Y^2 - 1 = X^2 - X - 1 - Y^2(X-2),$$

change the variable $X - 2$ by X'. Then we have

$$X'^2 + 3X' + 1 - Y^2X' = 0.$$

Multiplying X' on each side and replacing $X'Y$ by Y', we have

$$X'^3 + 3X'^2 + X' - Y'^2 = 0.$$

Finally by changing the variable $X' + 1$ by X (and $Y' = Y$), we have the minimal Weierstrass form

$$Y^2 = X^3 - 2X + 1.$$

Hence we also know that mod p reductions of the curves $X^2-(1+Y^2)X+2Y^2-1 = 0$ and $Y^2 - (X^3 - 2X + 1) = 0$ are isomorphic over $\mathbb{F}_p$.

By the Tate Algorithm (cf. [16], Chapter IV, §9) (we used the software PARI-GP for the calculation), we know that its conductor is 40. $\square$

By Corollary 2.6 we have

$$\#\left(\mathrm{Rep}_2(G_{\mathcal{K}})(\mathbb{F}_q)/\mathrm{PGL}_2(\mathbb{F}_q)\right) = \#\overline{E}_p(\mathbb{F}_q) - 1 - (*)_q,$$

where $q = p^r$ and $\overline{E}_p$ is the mod p reduction of the elliptic curve E over $\mathbb{F}_q$ as a projective curve (in the above equation, -1 means to remove the point at infinity). The value $(*)_q$ is

$$(*)_q := \begin{cases} 1 & \text{if } p = 2, 5. \\ 2 & \text{if } p \neq 2, 5, \quad \left(\dfrac{5}{p}\right) = 1 \text{ and } r \geq 1. \\ 1 + (-1)^r & \text{if } p \neq 2, 5, \quad \left(\dfrac{5}{p}\right) = -1 \text{ and } r \geq 1. \end{cases}$$

Hence we have

$$\exp\left(\sum_{n=1}^{\infty}\frac{-(*)_{q^n}}{n}T^n\right)=(1-T)(1-\left(\frac{20}{q}\right)T)=\begin{cases}(1-T) & \text{if } p=2,5.\\ (1-T)^2 & \text{if } p\neq 2,5,\ \left(\dfrac{5}{p}\right)=1 \text{ and } r\geq 1.\\ (1-T)^2 & \text{if } p\neq 2,5,\ \left(\dfrac{5}{p}\right)=-1 \text{ and } 2\mid r.\\ (1-T^2) & \text{if } p\neq 2,5,\ \left(\dfrac{5}{p}\right)=-1 \text{ and } 2\nmid r.\end{cases}$$

Here $\left(\dfrac{20}{q}\right)$ is the Jacobi symbol. Thus we have the following:

Theorem 2.8. *Let $G_{\mathcal{K}}=\pi_1(S^3\setminus\mathcal{K})$ be the knot group of the figure 8 knot and let q be a power of a prime number p. We have*

$$Z(G_{\mathcal{K}},q,T):=Z_2(G_{\mathcal{K}},q,T)=Z_{\overline{E}_p}(q,T)(1-T)^2(1-\left(\frac{20}{q}\right)T).$$

Here, E is an elliptic curve over $\mathbb{Q}$ defined by

$$E:Y^2=X^3-2X+1,$$

$Z_{\overline{E}_p}(q,T)$ is the congruence zeta function

$$Z_{\overline{E}_p}(q,T):=\exp\left(\sum_{n=1}^{\infty}\frac{\#\overline{E}_p(\mathbb{F}_{q^n})}{n}T^n\right)$$

of the mod p reduction $\overline{E}_p$ of the elliptic curve E over $\mathbb{F}_q$, and $\left(\dfrac{20}{q}\right)$ is the Jacobi symbol.

Now we calculate the congruence zeta functions $Z_{\overline{E}_p}(q,T)$ of curves $\overline{E}_p$ over $\mathbb{F}_q$ at the bad primes $2,5$ of E.

Lemma 2.9. *Let E be the elliptic curve $Y^2=X^3-2X+1$. Then we have*

$$\begin{aligned}Z_{\overline{E}_2}(2^r,T) &= \frac{1}{(1-T)(1-2^rT)}.\\ Z_{\overline{E}_5}(5^r,T) &= Z_{\tilde{E}_5}(5^r,T)(1-T)=\frac{1}{1-5^rT}.\end{aligned}$$

Here, $\tilde{E}_5$ is the normalization of $\overline{E}_5$ which is isomorphic to $\mathbb{P}^1$ over $\mathbb{F}_5$.

Proof. Since $\overline{E}_2:(Y-1)^2=X^3$ and the 2-power homomorphism on $\mathbb{F}_{2^{rn}}$ is an isomorphism, we have $\#\overline{E}_2(\mathbb{F}_{2^{rn}})=2^{rn}+1$. Thus $Z_{\overline{E}_2}(2^r,T)=1/(1-T)(1-2^rT)$. For the curve $\overline{E}_5$, changing the coordinate X by $X+2$, we have

$$\overline{E}_5:Y^2=X^2(X+1).$$

It is well known that the curve $\overline{E}_5$ has a unique singular point $(0,0)$, which is an ordinary double point of $\overline{E}_5$ (cf. [10], 7.5, Example 5.14). The normalization of $\operatorname{Spec} A$, where $A:=\mathbb{F}_{5^r}[X,Y]/(Y^2-X^2(X+1))$ is $\operatorname{Spec}\overline{A}=\operatorname{Spec}\mathbb{F}_{5^r}[u]$, $u=Y/X$, $u^2=1+X$ (cf. [10], 8.2, Example 2.42). If $\pi:\operatorname{Spec}\overline{A}\to\operatorname{Spec} A$ is the normalization morphism, the fiber $\pi^{-1}(P)$, where $P=(x,y)\in\operatorname{Spec} A$ is the point corresponding to $(0,0)$, consists of two points $\{(u-1),(u+1)\}$. Hence we have $Z_{\overline{E}_5}(5^r,T)=Z_{\tilde{E}_5}(5^r,T)(1-T)$. Since $\overline{E}_5(\mathbb{F}_5)\neq\{P,O\}$, we have $\tilde{E}_5(\mathbb{F}_5)\neq\{O\}$. Hence $\tilde{E}_5$ is isomorphic to $\mathbb{P}^1$ over $\mathbb{F}_5$ since the arithmetic genus of $\tilde{E}_5$ is $p_a(\tilde{E}_5) = p_a(\overline{E}_5) - 1 = 0$ (cf. [10], 7.5, Proposition 5.4). $\square$

Remark 2.10. There is a related work of Sink ([17]) on Weil-type zeta functions. As an attempt to define a mod p analogue of the Casson invariant, Sink has studied the Weil-type zeta function defined by the numbers of $\mathrm{SL}_2(\mathbb{F}_{p^n})$-conjugacy classes of non-diagonal $\mathrm{SL}_2(\mathbb{F}_{p^n})$-representations of a knot group. In particular he has calculated the Weil-type zeta functions for $(a, 2)$-torus knot groups.

3. Hasse-Weil zeta function of the figure 8 knot group

By the description of the Weil type zeta function obtained in Theorem 2.8, here we study the *Hasse-Weil zeta function* of SL_2-absolutely irreducible representations of $G_{\mathcal{K}}$ over $\mathbb{Q}$,

$$\zeta_A(s) := \prod_p Z(G_{\mathcal{K}}, p, p^{-s}),$$

where p runs through all the prime numbers. We shall prove that the Hasse-Weil zeta function $\zeta_A(s)$ over $\mathbb{Q}$ is continued meromorphically to the whole complex plane $\mathbb{C}$. Moreover, we have a functional equation and calculate its central value at $s = 1$.

3.1. **L-series of $E/\mathbb{Q}$.** For the calculation of $\zeta_A(s)$ it is essential to compute the L-series $L_{E/\mathbb{Q}}(s)$ of the elliptic curve E over $\mathbb{Q}$. Here we review some properties of the L-series of the elliptic curve E over $\mathbb{Q}$. Since E has additive reduction at 2 and split multiplicative reduction at 5, the Hasse-Weil L-series $L_{E/\mathbb{Q}}(s)$ of $E/\mathbb{Q}$ is defined (cf. [15]) by $L_{E/\mathbb{Q}}(s) = \prod_p L_p(p^{-s})^{-1}$, where if $p \neq 2, 5$,

$$L_p(T) := Z_{\overline{E}_p}(p, T)(1 - T)(1 - pT);$$

and if $p = 2, 5$, put

$$L_2(T) := 1, \quad L_5(T) := 1 - T.$$

By the Modularity Theorem ([22], [18] and [2]), for any elliptic curve over $\mathbb{Q}$ there is a Hecke eigenform of weight 2 whose L-function is equal to the Hasse-Weil L-series of the elliptic curve. In particular, the L-series of an elliptic curve over $\mathbb{Q}$ is continued meromorphically to the whole complex plane. By Lemma 2.9 we have

$$L_{E/\mathbb{Q}}(s)\zeta_E(s) = \zeta_{\mathbb{Q}}(s)\zeta_{\mathbb{Q}}(s-1),$$

where $\zeta_E(s) := \prod_p Z_{\overline{E}_p}(p, p^{-s})$. Let $\mathcal{L}_{E/\mathbb{Q}}(s) := N_{E/\mathbb{Q}}^{s/2}(2\pi)^{-s}\Gamma(s)L_{E/\mathbb{Q}}(s)$ be the completed Hasse-Weil L-series, where $\Gamma(s)$ is the Gamma function. The function $\mathcal{L}_{E/\mathbb{Q}}(s)$ has a functional equation $\mathcal{L}_{E/\mathbb{Q}}(s) = w\mathcal{L}_{E/\mathbb{Q}}(2-s)$, where $w = \pm 1$. Since the Mordell-Weil rank and the analytic rank of E are both zero (see Cremona's database on elliptic curves) we have $w = 1$. By the duplication formula of the Gamma function $\Gamma(s)$,

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = 2^{1-s}\sqrt{\pi}\Gamma(s),$$

we have

$$\mathcal{L}_{E/\mathbb{Q}}(s)\zeta_E(s) = \xi_{\mathbb{Q}}(s)\xi_{\mathbb{Q}}(s-1) \times \frac{(s-1)(\sqrt{40})^s}{4\pi},$$

where $\xi_{\mathbb{Q}}(s) := (\sqrt{\pi})^{-s}\Gamma(s/2)\zeta(s)$ is the completed Riemann zeta function, which has a functional equation $\xi_{\mathbb{Q}}(s) = \xi_{\mathbb{Q}}(1-s)$.

3.2. **Calculation of the Hasse-Weil zeta function.** Now we calculate the Hasse-Weil zeta function $\zeta_A(s)$ of absolutely irreducible SL_2-representations of the figure 8 knot group. Let $\zeta_{\mathbb{Q}(\sqrt{5})}(s)$ be the Dedekind zeta function of $\mathbb{Q}(\sqrt{5})$ and $\xi_{\mathbb{Q}(\sqrt{5})}(s) := \sqrt{5}^s \pi^{-s}\Gamma(s/2)^2\zeta_{\mathbb{Q}(\sqrt{5})}(s)$ the completed Dedekind zeta function. It is well known that $\zeta_{\mathbb{Q}(\sqrt{5})}(s)$ is written as $\zeta_{\mathbb{Q}(\sqrt{5})}(s) = \zeta(s)L(\left(\frac{5}{\cdot}\right), s)$, where $L(\left(\frac{5}{\cdot}\right), s)$ is the Dirichlet L-function defined by the character $\left(\frac{5}{\cdot}\right)$. Hence we have

$$\zeta_A(s) = \prod_p Z(G_{\mathcal{K}}, p, p^{-s}) = \frac{\zeta_E(s)}{\zeta_{\mathbb{Q}}(s)\zeta_{\mathbb{Q}(\sqrt{5})}(s)}$$

by Theorem 2.8. Thus $\zeta_A(s)$ is also continued meromorphically to the whole complex plane, and the completion

$$\xi_A(s) := \frac{4\pi^{(3s/2)+1}}{(10\sqrt{2})^s\Gamma(s/2)^3} \times \zeta_A(s) = \frac{(s-1)\xi_{\mathbb{Q}}(s-1)}{\mathcal{L}_{E/\mathbb{Q}}(s)\xi_{\mathbb{Q}(\sqrt{5})}(s)}$$

has a functional equation $\xi_A(2-s)\xi_{\mathbb{Q}}(2-s)\xi_{\mathbb{Q}(\sqrt{5})}(2-s) = -\xi_A(s)\xi_{\mathbb{Q}}(s)\xi_{\mathbb{Q}(\sqrt{5})}(s)$. Now we calculate the central value of $\xi_A(s)$. In the expression of $\xi_A(s)$, we know that $\mathcal{L}_{E/\mathbb{Q}}(1) \neq 0$ since E is an elliptic curve of rank zero. Since $\lim_{s\to 1}(s-1)\xi_{\mathbb{Q}}(s-1) = -1$ and

$$\lim_{s\to 1}(s-1)\zeta_{\mathbb{Q}(\sqrt{5})}(s) = \lim_{s\to 1}(s-1)\zeta(s)L(\left(\frac{5}{\cdot}\right), s) = \frac{\log(\varphi^2)}{\sqrt{5}},$$

where $\varphi := (1+\sqrt{5})/2$, we have

$$\lim_{s\to 1}(s-1)\xi_{\mathbb{Q}(\sqrt{5})}(s) = \log(\varphi^2).$$

It is known that the Birch and Swinnerton-Dyer conjecture holds for the elliptic curve E since E has rank zero. Thus we also have $L_{E/\mathbb{Q}}(1) = \frac{1}{4}\Omega(E)$, where $\Omega(E)$ is the real period of the elliptic curve E. By [3], Algorithm 7.4.7, we have $\Omega(E) = \dfrac{2\pi}{\mathrm{AGM}(\varphi, \varphi-1)}$, where $\mathrm{AGM}(\varphi, \varphi-1)$ is the arithmetic-geometric mean of $\varphi = (\sqrt{5}+1)/2$ and $\varphi - 1 = (\sqrt{5}-1)/2$. Hence we have

$$\mathcal{L}_{E/\mathbb{Q}}(1) = \frac{\sqrt{10}}{4\pi}\Omega(E) = \frac{\sqrt{10}}{2\mathrm{AGM}(\varphi, \varphi-1)}.$$

Thus we have

$$\begin{aligned} \lim_{s\to 1}\frac{\xi_A(s)}{s-1} &= \lim_{s\to 1}\frac{\xi_{\mathbb{Q}}(s-1)}{\mathcal{L}_{E/\mathbb{Q}}(s)\xi_{\mathbb{Q}(\sqrt{5})}(s)} \\ &= -\frac{\mathrm{AGM}(\varphi, \varphi-1)}{\sqrt{10}\log(\varphi)}. \end{aligned}$$

Theorem 3.1. *The function $\zeta_A(s)$ and its completion $\xi_A(s)$ are continued meromorphically to the complex plane $\mathbb{C}$ and $\xi_A(s)$ has a functional equation*

$$\xi_A(2-s)\xi_{\mathbb{Q}}(2-s)\xi_{\mathbb{Q}(\sqrt{5})}(2-s) = -\xi_A(s)\xi_{\mathbb{Q}}(s)\xi_{\mathbb{Q}(\sqrt{5})}(s).$$

The function $\xi_A(s)$ has a simple zero at $s = 1$ and the central value $\xi_A(1)$ is given by

$$\lim_{s\to 1}\frac{\xi_A(s)}{s-1} = -\frac{\mathrm{AGM}(\varphi,\varphi-1)}{\sqrt{10}\log(\varphi)},$$

where $\varphi = (\sqrt{5}+1)/2$.

Remark 3.2. The Alexander polynomial of the figure 8 knot is $\Delta_{\mathcal{K}}(T) = T^2 - 3T + 1$. Thus φ, $\varphi - 1$ are square roots of the roots of $\Delta_{\mathcal{K}}(T)$. We also note that (at least for representations over $\mathbb{C}$) by [4], Proposition 6.2 (see also [9], Corollary 4.3) φ, $\varphi - 1$ are eigenvalues at the meridian $a \in G_{\mathcal{K}}$ of the (non-abelian) reducible SL_2-representations of $G_{\mathcal{K}}$ corresponding to the points $(\pm\sqrt{5}, 3)$ of the SL_2-character variety of $G_{\mathcal{K}}$.

Remark 3.3. It is known that $\log(\varphi^2)$ is written as

$$\begin{aligned}\log(\varphi^2) &= m(\Delta_{\mathcal{K}}(T))\\ &= \lim_{n\to\infty}\frac{\log(\#H_1(M_n,\mathbb{Z}))}{n},\end{aligned}$$

where $m(\Delta_{\mathcal{K}}(T))$ is the logarithmic Mahler measure of $\Delta_{\mathcal{K}}(T)$ and M_n is the n-fold cyclic covering of S^3 branched over the figure 8 knot $\mathcal{K}$. Here, for a given Laurent polynomial $P = P(t_1,\dots,t_n) \in \mathbb{C}[t_1^{\pm 1},\dots,t_n^{\pm 1}]$, the *logarithmic Mahler measure* of P is defined by

$$m(P) := \int_0^1\cdots\int_0^1 \log|P(e^{2\pi t_1\sqrt{-1}},\dots,e^{2\pi t_n\sqrt{-1}})|dt_1\cdots dt_n.$$

Remark 3.4. The arithmetic-geometric mean of φ, $\varphi - 1$ is also written as

$$\mathrm{AGM}(\varphi,\varphi-1) = \frac{1}{2}\left(\frac{d}{dk}m(P_k)(\sqrt{5})\right)^{-1}.$$

Here P_k is a polynomial,

$$x + \frac{1}{x} + y + \frac{1}{y} - 4k,$$

which defines an elliptic curve for any complex number $4k \neq 0, 1$. This follows from the following equality ([20], Lemma):

$$m(P_k) = -\frac{1}{2}\Phi(1/k^2) + \log(4)$$

and the relations

$$x\frac{d}{dx}\Phi(x) = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x\right), \quad \mathrm{AGM}(a,b) = \frac{(a+b)\pi}{4K((a-b)/(a+b))},$$

$$K(x) = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x^2\right),$$

where $\Phi(x)$ is defined by

$$\Phi(x) = \log x + \sum_{n=1}^{\infty}\binom{1/2}{n}^2\frac{x^n}{n},$$

and ${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x\right)$ and $K(x)$ are the classical hypergeometric series and the complete elliptic integral of the first kind respectively. The Weierstrass equations of P_k are given by

$$E_m : y^2 = x^3 + 2m(2m-1)x^2 + m^2x,$$

where m is defined by $m = k^2$. There is a relation between the logarithmic Mahler measure of the A-polynomial of the figure 8 knot and the hyperbolic volume of the figure 8 knot complement ([1]). It would be interesting to find the relation between the figure 8 knot and this family of elliptic curves, especially the meaning of the value $\sqrt{5}$.

Remark 3.5. Can we also have a description of the value

$$\lim_{s \to 2}(s-2)\xi_A(s) = \frac{75}{2\sqrt{5}\pi^2 \mathcal{L}_{E/\mathbb{Q}}(2)}$$

in terms of the Mahler measure of a family of polynomials P_k? It is numerically shown by R. Villegas ([19], Table 4) that $\mathcal{L}_{E/\mathbb{Q}}(2) = L'_{E/\mathbb{Q}}(0) \doteq m(P_{\sqrt{-4}/4})$. We note that the elliptic curve defined by $P_{\sqrt{-4}/4}$ is isomorphic to E.

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