SEPARABLE SUBGROUPS HAVE BOUNDED PACKING

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Abstract. In this paper, we prove that separable subgroups have bounded packing in ambient groups. The notion bounded packing was introduced by Hruska and Wise, and, in particular, our result confirms a conjecture of theirs which states that each subgroup of a virtually polycyclic group has the bounded packing property.

1. Introduction

Bounded packing was introduced for a subgroup of a countable group in Hruska-Wise [3]. Roughly speaking, this property gives a finite upper bound on the number of left cosets of the subgroup that are pairwise close in $G$. Precisely,

**Definition.** Let $G$ be a countable group with a left invariant proper metric $d$. A subgroup $H$ has *bounded packing* in $G$ (with respect to $d$) if for each positive constant $D$, there is a natural number $N = N(G, H, D)$, such that for any collection $\mathcal{C}$ of $N$ left $H$-cosets in $G$, there exist at least two $H$-cosets $gH, g'H \in \mathcal{C}$ satisfying $d(gH, g'H) > D$.

**Remark.** Bounded packing of a subgroup is independent of the choice of the left invariant proper metric $d$. Equivalently, bounded packing says that for each positive constant $D$, every collection of left $H$-cosets in $G$ with pairwise distance at most $D$ has a uniform bound $N = N(G, H, D)$ on their cardinality.

This paper aims to give a proof of the following.

**Theorem.** If $H$ is a separable subgroup of a countable group $G$, then $H$ has bounded packing in $G$.

A subgroup $H$ of a group $G$ is *separable* if $H$ is an intersection of finite index subgroups of $G$. A group is called *subgroup separable* or *LERF* if every finitely generated subgroup is separable. For example, Hall showed that free groups are LERF in [1]. It follows from a theorem of Mal’cev [4] that polycyclic (and in particular finitely generated nilpotent) groups are LERF. A group is called *slender* if every subgroup is finitely generated. Polycyclic groups are also slender by a result

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of Hirsch [2]. Therefore we have the following corollary, which gives a positive answer to [3, Conjecture 2.14].

**Corollary.** Let $P$ be virtually polycyclic. Then each subgroup of $P$ has bounded packing in $P$.

2. **Proof of the Theorem**

We define the norm $|g|_d$ of an element $g \in G$ as the distance $d(1, g)$.

**Proof of the Theorem.** By the definition of bounded packing, it suffices to show, for each positive constant $D$, that there is a uniform bound on the cardinality of every collection of left $H$-cosets in $G$ with pairwise distance at most $D$.

Given such a collection $\mathcal{A}$ satisfying $d(gH, g'H) < D$ for any $gH, g' H \in \mathcal{A}$. Without loss of generality, we can assume $H$ belongs to $\mathcal{A}$, up to a translation of $\mathcal{A}$ by an appropriate element of $G$. Since $d(H, gH) < D$ for each $gH \in \mathcal{A}$, there exists an element $h$ in $H$ such that $d(1, h gH) < D$. Hence we conclude that the collection $\mathcal{A} \setminus \{H\}$ lies in the finite union of double cosets $HgH$ with $|g|_d < D$ and $g \in G \setminus H$.

Since $d$ is a left invariant proper metric on $G$, the set $F = \{g \in G \setminus H : |g|_d < D\}$ is finite. Since $H$ is separable in $G$, we can take a finite index subgroup $K$ of $G$ such that $H < K$ and $F \subset G \setminus K$.

We claim that no two different left $H$-cosets of $\mathcal{A}$ lie in the same left $K$-coset. By way of contradiction, we suppose that there are two $H$-cosets $gkH, g'k'H \in \mathcal{A}$ in the same coset $gK$ such that $d(gkH, g'k'H) < D$. By a similar argument as above, we get that $k^{-1}k'H$ belongs to a double coset $Hg_0H$ with $|g_0|_d < D$. Moreover, we note that $g_0 \in F$. Since we have $k^{-1}k'H = h g_0H$ for some $h \in H$, it is easy to see that $g_0$ belongs to $K$. But by the choice of $K$, we know that $g_0$ belongs to $G \setminus K$. This is a contradiction. Our claim is proved.

Since $K$ is of finite index in $G$, the cardinality of each $A$ is upper bounded by $[G : K]$. Thus for each $D$, we have obtained a uniform bound on every $\mathcal{A}$. Hence $H$ has bounded packing in $G$. \qed

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**References**


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