ROOTS OF EHRHART POLYNOMIALS
OF GORENSTEIN FANO POLYTOPES

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Abstract. Given arbitrary integers $k$ and $d$ with $0 \leq 2k \leq d$, we construct a Gorenstein Fano polytope $P \subset \mathbb{R}^d$ of dimension $d$ such that (i) its Ehrhart polynomial $i(P, n)$ possesses $d$ distinct roots; (ii) $i(P, n)$ possesses exactly $2k$ non-real roots and $d - 2k$ real roots; (iii) the real part of each of the non-real roots is equal to $-1/2$; (iv) all of the real roots belong to the open interval $(-1, 0)$.

Recently, many research papers on convex polytopes, including [2], [3], [4], [5], [8], [9] and [18], discuss roots of Ehrhart polynomials. One of the fascinating topics is the study on roots of Ehrhart polynomials of Gorenstein Fano polytopes.

Let $P \subset \mathbb{R}^N$ be an integral convex polytope of dimension $d$ and $\partial P$ its boundary. (An integral convex polytope is a convex polytope all of whose vertices have integer coordinates.) Given integers $n = 1, 2, \ldots$, we write $i(P, n)$ for the number of integer points belonging to $nP$, where $nP = \{n \alpha : \alpha \in P\}$. In other words,

$$i(P, n) = |nP \cap \mathbb{Z}^N|, \quad n = 1, 2, \ldots$$

In the late 1950’s Ehrhart succeeded in proving that $i(P, n)$ is a polynomial in $n$ of degree $d$ with $i(P, 0) = 1$. We call $i(P, n)$ the Ehrhart polynomial of $P$. Ehrhart’s “loi de réciprocité” guarantees that

$$(-1)^d i(P, -n) = i^*(P, n), \quad n = 1, 2, \ldots,$$

where $i^*(P, n) = |n(P \setminus \partial P) \cap \mathbb{Z}^N|$.

We define the sequence $\delta_0, \delta_1, \delta_2, \ldots$ of integers by the formula

$$(1 - \lambda)^{d+1} \left[ 1 + \sum_{n=1}^{\infty} i(P, n) \lambda^n \right] = \sum_{j=0}^{\infty} \delta_j \lambda^j.$$

Since $i(P, n)$ is a polynomial in $n$ of degree $d$ with $i(P, 0) = 1$, a fundamental fact on generating functions ([21 Corollary 4.3.1]) guarantees that $\delta_j = 0$ for every $j > d$. The sequence

$$\delta(P) = (\delta_0, \delta_1, \ldots, \delta_d)$$

is called the $\delta$-vector of $P$. Thus $\delta_0 = 1$ and $\delta_1 = |P \cap \mathbb{Z}^N| - (d + 1)$. Each $\delta_j$ is non-negative (Stanley [20]). If $\delta_d \neq 0$, then $\delta_1 \leq \delta_j$ for every $1 \leq j < d$ ([12]).
It follows from loi de réciprocité that

\[ \sum_{n=1}^{\infty} i^*(P, n)\lambda^n = \frac{\sum_{j=0}^{d} \delta_{d-j} \lambda^{j+1}}{(1 - \lambda)^{d+1}}. \]

In particular, \( \delta_d = |(P \setminus \partial P) \cap \mathbb{Z}^N| \). Moreover, it follows from (1) that

\[ \max\{i : \delta_i \neq 0\} + \min\{i : i(P \setminus \partial P) \cap \mathbb{Z}^N \neq \emptyset\} = d + 1. \]

We refer the reader to [6], [10], [21], [22], [23] and [24] for further information on Ehrhart polynomials and \( \delta \)-vectors.

A Fano polytope is an integral convex polytope \( P \subset \mathbb{R}^d \) of dimension \( d \) such that the origin of \( \mathbb{R}^d \) is a unique integer point belonging to the interior \( P \setminus \partial P \) of \( P \). A Fano polytope is called Gorenstein if its dual polytope is integral. (Recall that the dual polytope \( P^\vee \) of a Fano polytope \( P \) is the convex polytope which consists of those \( x \in \mathbb{R}^d \) such that \( \langle x, y \rangle \leq 1 \) for all \( y \in P \), where \( \langle x, y \rangle \) is the usual inner product of \( \mathbb{R}^d \).) A Gorenstein Fano polytope is often said to be a reflexive polytope. Gorenstein Fano polytopes are classified when \( d \leq 4 \) in [14] and [15], and the relevance of Gorenstein Fano polytopes to Mirror Symmetry is studied in [1]. We refer the reader to [1], [13], [14], [15] and [17] for related works on toric Fano varieties or Gorenstein toric Fano varieties.

Let \( P \subset \mathbb{R}^d \) be a Fano polytope with \( \delta(P) = (\delta_0, \delta_1, \ldots, \delta_d) \) its \( \delta \)-vector. It follows from [1] and [11] that the following conditions are equivalent:

- \( P \) is Gorenstein;
- \( \delta(P) \) is symmetric, i.e., \( \delta_j = \delta_{d-j} \) for every \( 0 \leq j \leq d \);
- \( i(P, n) = (-1)^d i(P, -n - 1) \).

Let \( P \subset \mathbb{R}^N \) be an integral convex polytope of dimension \( d \) and \( i(P, n) \) its Ehrhart polynomial. A complex number \( a \in \mathbb{C} \) is called a root of \( i(P, n) \) if \( i(P, a) = 0 \). Let \( \Re(a) \) denote the real part of \( a \in \mathbb{C} \). An outstanding conjecture given in [2] says that every root \( a \in \mathbb{C} \) of \( i(P, n) \) satisfies \( -d \leq \Re(a) \leq d - 1 \).

When \( P \subset \mathbb{R}^d \) is a Gorenstein Fano polytope, since \( i(P, n) = (-1)^d i(P, -n - 1) \), the roots of \( i(P, n) \) are distributed symmetrically in the complex plane with respect to the line \( \Re(z) = -1/2 \). Thus, in particular, if \( d \) is odd, then \( -1/2 \) is a root of \( i(P, n) \).

It is well-known that the regular unit crosspolytope is a Gorenstein Fano polytope and the \( d \) roots of its Ehrhart polynomial have real part \( -1/2 \) for any dimension \( d \). It is also known [3, Proposition 1.8] that if all roots \( a \in \mathbb{C} \) of \( i(P, n) \) of an integral convex polytope \( P \subset \mathbb{R}^d \) of dimension \( d \) satisfy \( \Re(a) = -1/2 \), then \( P \) is unimodular isomorphic to a Gorenstein Fano polytope whose volume is at most \( 2^d \). In a recent work [8], the roots of the Ehrhart polynomials of smooth Fano polytopes with small dimensions are completely determined.

In [17], there is a question whether there is a Fano variety the roots \( \alpha \) of whose Hilbert polynomial do not satisfy \( -1 < \Re(\alpha) < 0 \). Moreover, the vertical line \( \Re(\alpha) = -1/2 \) is the bisector of the vertical strip \( -1 < \Re(\alpha) < 0 \). By taking these into consideration, we prove the following:

**Theorem 0.1.** Given arbitrary non-negative integers \( k \) and \( d \) with \( 0 \leq 2k \leq d \), there exists a Gorenstein Fano polytope \( P \subset \mathbb{R}^d \) of dimension \( d \) such that

1. \( i(P, n) \) possesses \( d \) distinct roots;
2. \( i(P, n) \) possesses exactly \( 2k \) non-real roots and \( d - 2k \) real roots;
(iii) the real part of each of the non-real roots is equal to $-1/2$;
(iv) all of the real roots belong to the open interval $(-1, 0)$.

Proof. Let $e_1, \ldots, e_d$ denote the canonical unit vectors of $\mathbb{R}^d$. Let $Q \subset \mathbb{R}^d$ be the convex polytope that is the convex hull of $e_1, \ldots, e_{2k}$ and $-(e_1 + \cdots + e_{2k})$. Then $Q$ is an integral convex polytope of dimension $2k$ with $\delta(Q) = (1, 1, \ldots, 1) \in \mathbb{Z}^{2k+1}$.

In general, when $F \subset \mathbb{R}^N$ is a $d$-dimensional integral convex polytope, if we define $F' \subset \mathbb{R}^{N+1}$ by setting the convex hull of $F \cup \{e_{N+1}\}$, then one has

$$i(F', n) = 1 + \sum_{k=1}^{n} i(F, k).$$

It then follows that

$$\delta(F') = (\delta(F), 0) \in \mathbb{Z}^{d+2}.$$

Let $Q_c \subset \mathbb{R}^d$ be the convex polytope that is the convex hull of $Q \cup \{e_{2k+1}, \ldots, e_d\}$. Then $\delta(Q_c) = (\delta(Q), 0, \ldots, 0) \in \mathbb{Z}^{d+1}$. Hence, by (2), the convex polytope $(d-2k+1)Q_c$ possesses a unique integer point $a$ in its interior. Now, write $P \subset \mathbb{R}^d$ for the integral convex polytope $(d-2k+1)Q_c - a$. Then $P$ is a Fano polytope.

Since

$$\sum_{n=0}^{\infty} i(Q_c, n) \lambda^n = \frac{1 + \lambda + \lambda^2 + \cdots + \lambda^{2k}}{(1-\lambda)^{d+1}},$$

one has

$$i(Q_c, n) = \sum_{i=n-2k}^{n} \binom{d+i}{d} = \sum_{i=0}^{2k} \binom{d + (n - 2k) + i}{d} = \sum_{i=0}^{2k} \binom{d + n - (2k - i)}{d} = \sum_{i=0}^{2k} \binom{n + d - i + 1}{d + 1} - \binom{n + d - i}{d + 1} = \frac{1}{(d+1)!} \prod_{i=1}^{d-2k} (n + i) \left( \prod_{i=0}^{2k} (n + d + 1 - i) - \prod_{i=0}^{2k} (n - i) \right).$$

Since

$$i(P, n) = i((d-2k+1)Q_c, n) = i(Q_c, (d-2k+1)n),$$

one has

$$i(P, n) = \frac{(d-2k+1)^{d+1}}{(d+1)!} \prod_{i=1}^{d-2k} \left( n + \frac{i}{d-2k+1} \right) F(n),$$

where

$$F(n) = \prod_{i=0}^{2k} \left( n + \frac{d + 1 - i}{d-2k+1} \right) - \prod_{i=0}^{2k} \left( n - \frac{i}{d-2k+1} \right) = \prod_{i=0}^{2k} \left( n + \frac{d + 1 - (2k - i)}{d-2k+1} \right) - \prod_{i=0}^{2k} \left( n - \frac{i}{d-2k+1} \right).$$
Thus we obtain the following equalities:

\[
\prod_{i=1}^{d-2k} \left( -n - 1 + \frac{i}{d-2k+1} \right) = (-1)^{d-2k} \prod_{i=1}^{d-2k} \left( n + \frac{d-2k+1-i}{d-2k+1} \right) = (-1)^{d-2k} \prod_{i=1}^{d-2k} \left( n + \frac{i}{d-2k+1} \right);
\]

\[
F(-n-1) = \prod_{i=0}^{2k} \left( -n - 1 + \frac{d+1-i}{d-2k+1} \right) - \prod_{i=0}^{2k} \left( -n - 1 - \frac{i}{d-2k+1} \right) = (-1)^{2k+1} \prod_{i=0}^{2k} \left( n + \frac{d-2k+1+i}{d-2k+1} \right) - (-1)^{2k+1} \prod_{i=0}^{2k} \left( n + \frac{2k-i}{d-2k+1} \right) = (-1)^{2k} F(n).
\]

It then follows that

\[(-1)^{d} i(\mathcal{P}, -n-1) = i(\mathcal{P}, n),\]

which implies that \( \mathcal{P} \) is Gorenstein. Hence our work is to show that \( \mathcal{P} \) enjoys the required properties (i)–(iv).

Now, since

\[-\frac{d+1-(2k-i)}{d-2k+1} < -\frac{1}{2} < \frac{i}{d-2k+1}\]

and since

\[-\frac{d+1-(2k-i)}{d-2k+1} + \frac{i}{d-2k+1} = -1,\]

Lemma 0.2 below guarantees that \( F(n) \) possesses \( 2k \) distinct roots, and each of them is a non-real root with \(-1/2\) its real part. Finally, the real roots of \( i(\mathcal{P}, n) \) are

\[-\frac{i}{d-2k+1}, \quad 1 \leq i \leq d-2k.\]

Each of those roots belongs to the open interval \((-1, 0)\), as desired. \(\square\)

**Lemma 0.2.** Let \( \alpha_0, \alpha_1, \ldots, \alpha_{2k} \) and \( \beta_0, \beta_1, \ldots, \beta_{2k} \) be rational numbers satisfying \( \alpha_i < -1/2 < \beta_i \) and \( \alpha_i + \beta_i = -1 \) for all \( i \). Let

\[f(x) = \prod_{i=0}^{2k} (x - \alpha_i) - \prod_{i=0}^{2k} (x - \beta_i)\]

be a polynomial in \( x \) of degree \( 2k \). Then \( f(x) \) possesses \( 2k \) distinct roots, and each of them is a non-real root with \(-1/2\) its real part.
Proof. We employ a basis technique appearing in [19]. Let $a \in \mathbb{C}$ with $\Re(a) > -1/2$. Since $\alpha_i < \beta_i$ and $\alpha_i + \beta_i = -1$, it follows that

$$|a - \alpha_i|^2 - |a - \beta_i|^2 = (\Re(a) - \alpha_i)^2 - (\Re(a) - \beta_i)^2$$

$$= (2\Re(a) - \alpha_i - \beta_i)(\beta_i - \alpha_i)$$

$$= (2\Re(a) + 1)(\beta_i - \alpha_i)$$

$$> 0.$$  

Hence we have $|a - \alpha_i| > |a - \beta_i|$. Thus $\prod_{i=0}^{2k} |a - \alpha_i| > \prod_{i=0}^{2k} |a - \beta_i|$. Hence $f(a) \neq 0$. Similarly, if $a \in \mathbb{C}$ with $\Re(a) < -1/2$, then $|a - \alpha_i| < |a - \beta_i|$ for all $i$. Thus $\prod_{i=0}^{2k} |a - \alpha_i| < \prod_{i=0}^{2k} |a - \beta_i|$. Hence $f(a) \neq 0$. Consequently, all roots $a \in \mathbb{C}$ of $f(x)$ satisfy $\Re(a) = -1/2$.

Substituting $y = x + 1/2$ and $\gamma_i = \beta_i + 1/2$ in $f(x)$, it follows that each of the roots $a \in \mathbb{C}$ of the polynomial

$$g(y) = \prod_{i=0}^{2k} (\gamma_i + y) + \prod_{i=0}^{2k} (\gamma_i - y)$$

in $y$ of degree $2k$ satisfies $\Re(a) = 0$. Since $\gamma_i > 0$, one has $g(0) \neq 0$. Hence $g(y)$ possesses no real root. Thus all roots of $f(x)$ are non-real roots.

What we must prove is that $g(y)$ possesses $2k$ distinct roots. Let $b \in \mathbb{R}$ and let $\theta_i(b)$ be the argument of $\gamma_i + b\sqrt{-1}$, where $-\pi/2 < \theta_i(b) < \pi/2$. Then $b\sqrt{-1}$ is a root of $g(y)$ if and only if

$$\prod_{i=0}^{2k} e^{\sqrt{-1}\theta_i(b)} = \prod_{i=0}^{2k} e^{-\sqrt{-1}\theta_i(b)}.$$  

In other words, $b\sqrt{-1}$ is a root of $g(y)$ if and only if

$$\prod_{i=0}^{2k} e^{2\sqrt{-1}\theta_i(b)} = -1,$$

which is equivalent to saying that

$$\sum_{i=0}^{2k} \theta_i(b) \equiv \frac{\pi}{2} \pmod{\pi}.$$  

Now, we study the function $h(y) = \sum_{i=0}^{2k} \theta_i(y)$ with $y \in \mathbb{R}$. Since $\gamma_i > 0$, it follows that $h(y)$ is strictly increasing with

$$\lim_{y \to \infty} h(y) = k\pi + \pi/2, \quad \lim_{y \to -\infty} h(y) = -(k + 1)\pi + \pi/2.$$  

Hence the equation

$$h(y) \equiv \frac{\pi}{2} \pmod{\pi}$$

possesses $2k$ distinct real roots, as desired.

Here is an example of Theorem 0.1.

Example 0.3. Let $k = 1$ and $d = 4$. Then there exists a 4-dimensional Gorenstein Fano polytope $P \subset \mathbb{R}^4$ such that $\iota(P, n)$ satisfies the properties (i)–(iv) of Theorem 0.1. In fact, we define $Q^n$ by setting the convex hull of

$$\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (0, 0, 0, 0), (-1, 0, 0, 0), (0, 0, 0, 1), (0, 0, 0, 1)\}.$$
Then $3Q^c$ contains a unique integer point $(0,0,1,1)$ in its interior. Thus $P := 3Q^c - (0,0,1,1)$ is a Gorenstein Fano polytope, which is the convex hull of

$$
\{ (3,0,-1,-1), (0,3,-1,-1), (-3,-3,-1,-1), (0,0,2,-1), (0,0,-1,2) \}.
$$

It can be computed easily that the Ehrhart polynomial of $P$ is equal to

$$
\frac{81}{8} n^4 + \frac{81}{4} n^3 + \frac{135}{8} n^2 + \frac{27}{4} n + 1
$$

and its roots are

$$
-\frac{1}{3}, -\frac{2}{3}, -\frac{1}{2} + \frac{\sqrt{7}}{6} \text{ and } -\frac{1}{2} - \frac{\sqrt{7}}{6}.
$$

**Remark 0.4.** (a) It is disproved in [8] that all of the roots $\alpha$ of the Hilbert polynomial of any Fano variety satisfy $-1 < \Re(\alpha) < 0$, the so-called canonical strip hypothesis, which is stated in [2]. On Theorem 0.1, however, all of the roots of Ehrhart polynomials of our Gorenstein Fano polytopes satisfy this condition. In more detail, they satisfy the narrowed canonical strip hypothesis, which is the condition $-1 + 1/(d+1) \leq \Re(\alpha) \leq -1/(d+1)$. Moreover, if we set $2k = d$ when $d$ is even or $2k = d - 1$ when $d$ is odd, then they also satisfy the canonical line hypothesis, which is the condition $\Re(\alpha) = -1/2$.

(b) We should consider the connections of the Ehrhart polynomials of our Gorenstein Fano polytopes with $L$-functions. Let $i(P,s)$ be the Ehrhart polynomial of our Gorenstein Fano polytope $P$ with $2k = d$ when $d$ is even or with $2k = d - 1$ when $d$ is odd. Then we set $z(s) = i(P,-s)$. Then the function equation

$$
z(1-s) = (-1)^d z(s)
$$

holds and all of its roots $\alpha$ satisfy $\Re(\alpha) = 1/2$, which is, of course, the Riemann zeta function.

**Example 0.5.** Let $G$ be a finite connected graph on the vertex set $V(G) = \{1,\ldots,d\}$ with $E(G)$ its edge set. We assume that $G$ possesses no loop and no multiple edge. Let $e_1,\ldots,e_d$ denote the canonical unit vectors of $\mathbb{R}^d$. For an edge $e = \{i,j\}$ of $G$ with $i < j$, we define $\rho(e)$ and $\mu(e) \in \mathbb{R}^d$ by setting $\rho(e) = e_i - e_j$ and $\mu(e) = e_j - e_i$. Write $P_G^\pm \subset \mathbb{R}^d$ for the convex polytope which is the convex hull of $\{\rho(e) : e \in E(G)\} \cup \{\mu(e) : e \in E(G)\}$. Let $H \subset \mathbb{R}^d$ denote the hyperplane defined by the equation $\sum_{i=1}^d x_i = 0$. Then $P_G^\pm \subset H$. Identifying $H$ with $\mathbb{R}^{d-1}$, it turns out that $P_G^\pm \subset \mathbb{R}^{d-1}$ is a Gorenstein Fano polytope of dimension $d - 1$.

(In detail, see [19] Proposition 3.2.) One of the research problems is to find a combinatorial characterization of the finite graphs $G$ for which all roots $a \in \mathbb{C}$ of

$$
i(P_G^\pm, n)
$$

satisfy $\Re(\alpha) = -1/2$.

For example, if $C$ is a cycle of length 6, then all roots $a \in \mathbb{C}$ of $i(P_C^\pm, n)$ satisfy $\Re(\alpha) = -1/2$. However, if $C$ is a cycle of length 7, then there is a root $a \in \mathbb{C}$ of $i(P_C^\pm, n)$ with $\Re(\alpha) \neq -1/2$.

If $G$ is a tree, then $P_G^\pm$ is unimodular isomorphic to the regular unit crosspolytope of dimension $d - 1$, which is the convex hull of $\{\pm e_1,\ldots,\pm e_{d-1}\}$ in $\mathbb{R}^{d-1}$. Hence the $\delta$-vector of $P_G^\pm$ is $\delta(P_G^\pm) = ((d-1), (d-1), \ldots, (d-1))$. Thus, by using [19] again, all roots $a \in \mathbb{C}$ of $i(P_G^\pm, n)$ satisfy $\Re(\alpha) = -1/2$. 

Let $G$ be a complete bipartite graph of type $(2, d-2)$. Thus the edges of $G$ are either $\{1, j\}$ or $\{2, j\}$ with $3 \leq j \leq d$. Let $\delta(P^+_G) = (\delta_0, \delta_1, \ldots, \delta_{d-1})$. Then

$$
\sum_{k=0}^{d-1} \delta_k x^k = (1 + x)^{d-3}(1 + 2(d-2)x + x^2).
$$

It has been conjectured that all roots $a \in \mathbb{C}$ of $i(P^+_G, n)$ satisfy $\Re(a) = -1/2$ in [16].

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