PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 139, Number 10, October 2011, Pages 3727–3734
S 0002-9939(2011)11013-X
Article electronically published on March 30, 2011

ROOTS OF EHRHART POLYNOMIALS OF GORENSTEIN FANO POLYTOPES

TAKAYUKI HIBI, AKIHIRO HIGASHITANI, AND HIDEFUMI OHSUGI

(Communicated by Jim Haglund)

ABSTRACT. Given arbitrary integers k and d with $0 \leq 2k \leq d$, we construct a Gorenstein Fano polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d such that (i) its Ehrhart polynomial $i(\mathcal{P}, n)$ possesses d distinct roots; (ii) $i(\mathcal{P}, n)$ possesses exactly $2k$ non-real roots and $d - 2k$ real roots; (iii) the real part of each of the non-real roots is equal to $-1/2$; (iv) all of the real roots belong to the open interval $(-1, 0)$.

Recently, many research papers on convex polytopes, including [2], [3], [4], [5], [8], [9] and [18], discuss roots of Ehrhart polynomials. One of the fascinating topics is the study on roots of Ehrhart polynomials of Gorenstein Fano polytopes.

Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d and $\partial\mathcal{P}$ its boundary. (An integral convex polytope is a convex polytope all of whose vertices have integer coordinates.) Given integers $n = 1, 2, \ldots$, we write $i(\mathcal{P}, n)$ for the number of integer points belonging to $n\mathcal{P}$, where $n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}$. In other words,

$$i(\mathcal{P}, n) = |n\mathcal{P} \cap \mathbb{Z}^N|, \qquad n = 1, 2, \ldots.$$

In the late 1950's Ehrhart succeeded in proving that $i(\mathcal{P}, n)$ is a polynomial in n of degree d with $i(\mathcal{P}, 0) = 1$. We call $i(\mathcal{P}, n)$ the *Ehrhart polynomial* of $\mathcal{P}$. Ehrhart's "loi de réciprocité" guarantees that

$$(-1)^d i(\mathcal{P}, -n) = i^*(\mathcal{P}, n), \qquad n = 1, 2, \ldots,$$

where $i^*(\mathcal{P}, n) = |n(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N|$.

We define the sequence $\delta_0, \delta_1, \delta_2, \ldots$ of integers by the formula

$$(1-\lambda)^{d+1}\left[1 + \sum_{n=1}^{\infty} i(\mathcal{P}, n)\lambda^n\right] = \sum_{j=0}^{\infty} \delta_j \lambda^j.$$

Since $i(\mathcal{P}, n)$ is a polynomial in n of degree d with $i(\mathcal{P}, 0) = 1$, a fundamental fact on generating functions ([21, Corollary 4.3.1]) guarantees that $\delta_j = 0$ for every $j > d$. The sequence

$$\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$$

is called the *δ-vector* of $\mathcal{P}$. Thus $\delta_0 = 1$ and $\delta_1 = |\mathcal{P} \cap \mathbb{Z}^N| - (d + 1)$. Each δ_j is non-negative (Stanley [20]). If $\delta_d \neq 0$, then $\delta_1 \leq \delta_j$ for every $1 \leq j < d$ ([12]).

Received by the editors September 3, 2010.
2010 *Mathematics Subject Classification.* Primary 52B20; Secondary 52B12.
Key words and phrases. Ehrhart polynomial, δ-vector, Gorenstein Fano polytope.
This research was supported by JST, CREST.

It follows from loi de réciprocité that

$$\sum_{n=1}^{\infty} i^*(\mathcal{P},n)\lambda^n = \frac{\sum_{j=0}^{d}\delta_{d-j}\lambda^{j+1}}{(1-\lambda)^{d+1}}. \tag{1}$$

In particular, $\delta_d = |(\mathcal{P}\setminus\partial\mathcal{P})\cap\mathbb{Z}^N|$. Moreover, it follows from (1) that

$$\max\{i : \delta_i \neq 0\} + \min\{i : i(\mathcal{P}\setminus\partial\mathcal{P})\cap\mathbb{Z}^N \neq \emptyset\} = d+1. \tag{2}$$

We refer the reader to [6], [10], [21], [22], [23] and [24] for further information on Ehrhart polynomials and δ-vectors.

A *Fano polytope* is an integral convex polytope $\mathcal{P}\subset\mathbb{R}^d$ of dimension d such that the origin of $\mathbb{R}^d$ is a unique integer point belonging to the interior $\mathcal{P}\setminus\partial\mathcal{P}$ of $\mathcal{P}$. A Fano polytope is called *Gorenstein* if its dual polytope is integral. (Recall that the dual polytope $\mathcal{P}^\vee$ of a Fano polytope $\mathcal{P}$ is the convex polytope which consists of those $x\in\mathbb{R}^d$ such that $\langle x,y\rangle\leq 1$ for all $y\in\mathcal{P}$, where $\langle x,y\rangle$ is the usual inner product of $\mathbb{R}^d$.) A Gorenstein Fano polytope is often said to be a *reflexive polytope.* Gorenstein Fano polytopes are classified when $d\leq 4$ in [14] and [15], and the relevance of Gorenstein Fano polytopes to Mirror Symmetry is studied in [1]. We refer the reader to [1], [13], [14], [15] and [17] for related works on toric Fano varieties or Gorenstein toric Fano varieties.

Let $\mathcal{P}\subset\mathbb{R}^d$ be a Fano polytope with $\delta(\mathcal{P})=(\delta_0,\delta_1,\ldots,\delta_d)$ its δ-vector. It follows from [1] and [11] that the following conditions are equivalent:

- $\mathcal{P}$ is Gorenstein;
- $\delta(\mathcal{P})$ is symmetric, i.e., $\delta_j=\delta_{d-j}$ for every $0\leq j\leq d$;
- $i(\mathcal{P},n)=(-1)^d i(\mathcal{P},-n-1)$.

Let $\mathcal{P}\subset\mathbb{R}^N$ be an integral convex polytope of dimension d and $i(\mathcal{P},n)$ its Ehrhart polynomial. A complex number $a\in\mathbb{C}$ is called a *root* of $i(\mathcal{P},n)$ if $i(\mathcal{P},a)=0$. Let $\Re(a)$ denote the real part of $a\in\mathbb{C}$. An outstanding conjecture given in [2] says that every root $a\in\mathbb{C}$ of $i(\mathcal{P},n)$ satisfies $-d\leq\Re(a)\leq d-1$.

When $\mathcal{P}\subset\mathbb{R}^d$ is a Gorenstein Fano polytope, since $i(\mathcal{P},n)=(-1)^d i(\mathcal{P},-n-1)$, the roots of $i(\mathcal{P},n)$ are distributed symmetrically in the complex plane with respect to the line $\Re(z)=-1/2$. Thus, in particular, if d is odd, then $-1/2$ is a root of $i(\mathcal{P},n)$.

It is well-known that the regular unit crosspolytope is a Gorenstein Fano polytope and the d roots of its Ehrhart polynomial have real part $-1/2$ for any dimension d. It is also known [3, Proposition 1.8] that if all roots $a\in\mathbb{C}$ of $i(\mathcal{P},n)$ of an integral convex polytope $\mathcal{P}\subset\mathbb{R}^d$ of dimension d satisfy $\Re(a)=-1/2$, then $\mathcal{P}$ is unimodular isomorphic to a Gorenstein Fano polytope whose volume is at most 2^d. In a recent work [8], the roots of the Ehrhart polynomials of smooth Fano polytopes with small dimensions are completely determined.

In [7], there is a question whether there is a Fano variety the roots α of whose Hilbert polynomial do not satisfy $-1<\Re(\alpha)<0$. Moreover, the vertical line $\Re(\alpha)=-1/2$ is the bisector of the vertical strip $-1<\Re(\alpha)<0$. By taking these into consideration, we prove the following:

Theorem 0.1. *Given arbitrary non-negative integers k and d with $0\leq 2k\leq d$, there exists a Gorenstein Fano polytope $\mathcal{P}\subset\mathbb{R}^d$ of dimension d such that*

(i) *$i(\mathcal{P},n)$ possesses d distinct roots;*
(ii) *$i(\mathcal{P},n)$ possesses exactly $2k$ non-real roots and $d-2k$ real roots;*

(iii) *the real part of each of the non-real roots is equal to* $-1/2$*;*
(iv) *all of the real roots belong to the open interval* $(-1, 0)$.

Proof. Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ denote the canonical unit vectors of $\mathbb{R}^d$. Let $\mathcal{Q} \subset \mathbb{R}^d$ be the convex polytope that is the convex hull of $\mathbf{e}_1, \ldots, \mathbf{e}_{2k}$ and $-(\mathbf{e}_1 + \cdots + \mathbf{e}_{2k})$. Then $\mathcal{Q}$ is an integral convex polytope of dimension $2k$ with $\delta(\mathcal{Q}) = (1, 1, \ldots, 1) \in \mathbb{Z}^{2k+1}$.

In general, when $\mathcal{F} \subset \mathbb{R}^N$ is a d-dimensional integral convex polytope, if we define $\mathcal{F}' \subset \mathbb{R}^{N+1}$ by setting the convex hull of $\mathcal{F} \cup \{\mathbf{e}_{N+1}\}$, then one has

$$i(\mathcal{F}', n) = 1 + \sum_{k=1}^{n} i(\mathcal{F}, k).$$

It then follows that

$$\delta(\mathcal{F}') = (\delta(\mathcal{F}), 0) \in \mathbb{Z}^{d+2}.$$

Let $\mathcal{Q}^c \subset \mathbb{R}^d$ be the convex polytope that is the convex hull of $\mathcal{Q} \cup \{\mathbf{e}_{2k+1}, \ldots, \mathbf{e}_d\}$. Then $\delta(\mathcal{Q}^c) = (\delta(\mathcal{Q}), 0, \ldots, 0) \in \mathbb{Z}^{d+1}$. Hence, by (2), the convex polytope $(d - 2k + 1)\mathcal{Q}^c$ possesses a unique integer point $\mathbf{a}$ in its interior. Now, write $\mathcal{P} \subset \mathbb{R}^d$ for the integral convex polytope $(d - 2k + 1)\mathcal{Q}^c - \mathbf{a}$. Then $\mathcal{P}$ is a Fano polytope.

Since

$$\sum_{n=0}^{\infty} i(\mathcal{Q}^c, n)\lambda^n = \frac{1 + \lambda + \lambda^2 + \cdots + \lambda^{2k}}{(1-\lambda)^{d+1}},$$

one has

$$\begin{aligned} i(\mathcal{Q}^c, n) &= \sum_{i=n-2k}^{n} \binom{d+i}{d} = \sum_{i=0}^{2k} \binom{d + (n-2k) + i}{d} \\ &= \sum_{i=0}^{2k} \binom{d + n - (2k-i)}{d} = \sum_{i=0}^{2k} \binom{n+d-i}{d} \\ &= \sum_{i=0}^{2k} \left(\binom{n+d-i+1}{d+1} - \binom{n+d-i}{d+1} \right) \\ &= \binom{n+d+1}{d+1} - \binom{n+d-2k}{d+1} \\ &= \frac{1}{(d+1)!} \prod_{i=1}^{d-2k} (n+i) \left(\prod_{i=0}^{2k} (n+d+1-i) - \prod_{i=0}^{2k} (n-i) \right). \end{aligned}$$

Since

$$i(\mathcal{P}, n) = i((d-2k+1)\mathcal{Q}^c, n) = i(\mathcal{Q}^c, (d-2k+1)n),$$

one has

$$i(\mathcal{P}, n) = \frac{(d-2k+1)^{d+1}}{(d+1)!} \prod_{i=1}^{d-2k} \left(n + \frac{i}{d-2k+1} \right) F(n),$$

where

$$\begin{aligned} F(n) &= \prod_{i=0}^{2k} \left(n + \frac{d+1-i}{d-2k+1} \right) - \prod_{i=0}^{2k} \left(n - \frac{i}{d-2k+1} \right) \\ &= \prod_{i=0}^{2k} \left(n + \frac{d+1-(2k-i)}{d-2k+1} \right) - \prod_{i=0}^{2k} \left(n - \frac{i}{d-2k+1} \right). \end{aligned}$$

Thus we obtain the following equalities:

$$\begin{aligned}\prod_{i=1}^{d-2k}\left(-n-1+\frac{i}{d-2k+1}\right) &= (-1)^{d-2k}\prod_{i=1}^{d-2k}\left(n+\frac{d-2k+1-i}{d-2k+1}\right)\\ &= (-1)^{d-2k}\prod_{i=1}^{d-2k}\left(n+\frac{i}{d-2k+1}\right);\end{aligned}$$

$$\begin{aligned}F(-n-1) &= \prod_{i=0}^{2k}\left(-n-1+\frac{d+1-i}{d-2k+1}\right)-\prod_{i=0}^{2k}\left(-n-1-\frac{i}{d-2k+1}\right)\\ &= (-1)^{2k+1}\prod_{i=0}^{2k}\left(n+\frac{d-2k+1-d-1+i}{d-2k+1}\right)\\ &\qquad -(-1)^{2k+1}\prod_{i=0}^{2k}\left(n+\frac{d-2k+1+i}{d-2k+1}\right)\\ &= (-1)^{2k}\prod_{i=0}^{2k}\left(n+\frac{d-2k+1+i}{d-2k+1}\right)-(-1)^{2k}\prod_{i=0}^{2k}\left(n-\frac{2k-i}{d-2k+1}\right)\\ &= (-1)^{2k}\prod_{i=0}^{2k}\left(n+\frac{d+1-i}{d-2k+1}\right)-(-1)^{2k}\prod_{i=0}^{2k}\left(n-\frac{i}{d-2k+1}\right)\\ &= (-1)^{2k}F(n).\end{aligned}$$

It then follows that

$$(-1)^d i(\mathcal{P},-n-1)=i(\mathcal{P},n),$$

which implies that $\mathcal{P}$ is Gorenstein. Hence our work is to show that $\mathcal{P}$ enjoys the required properties (i)–(iv).

Now, since

$$-\frac{d+1-(2k-i)}{d-2k+1}<-\frac{1}{2}<\frac{i}{d-2k+1}$$

and since

$$-\frac{d+1-(2k-i)}{d-2k+1}+\frac{i}{d-2k+1}=-1,$$

Lemma 0.2 below guarantees that $F(n)$ possesses $2k$ distinct roots, and each of them is a non-real root with $-1/2$ its real part. Finally, the real roots of $i(\mathcal{P},n)$ are

$$-\frac{i}{d-2k+1},\qquad 1\le i\le d-2k.$$

Each of those roots belongs to the open interval $(-1,0)$, as desired. □

Lemma 0.2. *Let $\alpha_0,\alpha_1,\ldots,\alpha_{2k}$ and $\beta_0,\beta_1,\ldots,\beta_{2k}$ be rational numbers satisfying $\alpha_i<-1/2<\beta_i$ and $\alpha_i+\beta_i=-1$ for all i. Let*

$$f(x)=\prod_{i=0}^{2k}(x-\alpha_i)-\prod_{i=0}^{2k}(x-\beta_i)$$

be a polynomial in x of degree $2k$. Then $f(x)$ possesses $2k$ distinct roots, and each of them is a non-real root with $-1/2$ its real part.

Proof. We employ a basis technique appearing in [19]. Let $a \in \mathbb{C}$ with $\Re(a) > -1/2$. Since $\alpha_i < \beta_i$ and $\alpha_i + \beta_i = -1$, it follows that

$$\begin{aligned} |a-\alpha_i|^2 - |a-\beta_i|^2 &= (\Re(a)-\alpha_i)^2 - (\Re(a)-\beta_i)^2 \\ &= (2\Re(a)-\alpha_i-\beta_i)(\beta_i-\alpha_i) \\ &= (2\Re(a)+1)(\beta_i-\alpha_i) \\ &> 0. \end{aligned}$$

Hence we have $|a-\alpha_i| > |a-\beta_i|$. Thus $\prod_{i=0}^{2k}|a-\alpha_i| > \prod_{i=0}^{2k}|a-\beta_i|$. Hence $f(a) \neq 0$. Similarly, if $a \in \mathbb{C}$ with $\Re(a) < -1/2$, then $|a-\alpha_i| < |a-\beta_i|$ for all i. Thus $\prod_{i=0}^{2k}|a-\alpha_i| < \prod_{i=0}^{2k}|a-\beta_i|$. Hence $f(a) \neq 0$. Consequently, all roots $a \in \mathbb{C}$ of $f(x)$ satisfy $\Re(a) = -1/2$.

Substituting $y = x + 1/2$ and $\gamma_i = \beta_i + 1/2$ in $f(x)$, it follows that each of the roots $a \in \mathbb{C}$ of the polynomial

$$g(y) = \prod_{i=0}^{2k}(\gamma_i + y) + \prod_{i=0}^{2k}(\gamma_i - y)$$

in y of degree $2k$ satisfies $\Re(a) = 0$. Since $\gamma_i > 0$, one has $g(0) \neq 0$. Hence $g(y)$ possesses no real root. Thus all roots of $f(x)$ are non-real roots.

What we must prove is that $g(y)$ possesses $2k$ distinct roots. Let $b \in \mathbb{R}$ and let $\theta_i(b)$ be the argument of $\gamma_i + b\sqrt{-1}$, where $-\pi/2 < \theta_i(b) < \pi/2$. Then $b\sqrt{-1}$ is a root of $g(y)$ if and only if

$$\prod_{i=0}^{2k} e^{\sqrt{-1}\,\theta_i(b)} = -\prod_{i=0}^{2k} e^{-\sqrt{-1}\,\theta_i(b)}.$$

In other words, $b\sqrt{-1}$ is a root of $g(y)$ if and only if

$$\prod_{i=0}^{2k} e^{2\sqrt{-1}\,\theta_i(b)} = -1,$$

which is equivalent to saying that

$$\sum_{i=0}^{2k}\theta_i(b) \equiv \frac{\pi}{2} \pmod{\pi}.$$

Now, we study the function $h(y) = \sum_{i=0}^{2k}\theta_i(y)$ with $y \in \mathbb{R}$. Since $\gamma_i > 0$, it follows that $h(y)$ is strictly increasing with

$$\lim_{y\to\infty} h(y) = k\pi + \pi/2, \quad \lim_{y\to-\infty} h(y) = -(k+1)\pi + \pi/2.$$

Hence the equation

$$h(y) \equiv \frac{\pi}{2} \pmod{\pi}$$

possesses $2k$ distinct real roots, as desired. □

Here is an example of Theorem 0.1.

Example 0.3. Let $k = 1$ and $d = 4$. Then there exists a 4-dimensional Gorenstein Fano polytope $\mathcal{P} \subset \mathbb{R}^4$ such that $i(\mathcal{P}, n)$ satisfies the properties (i)–(iv) of Theorem 0.1. In fact, we define $\mathcal{Q}^c$ by setting the convex hull of

$$\{(1,0,0,0),(0,1,0,0),(-1,-1,0,0),(0,0,1,0),(0,0,0,1)\}.$$

Then $3\mathcal{Q}^c$ contains a unique integer point $(0,0,1,1)$ in its interior. Thus $\mathcal{P} := 3\mathcal{Q}^c - (0,0,1,1)$ is a Gorenstein Fano polytope, which is the convex hull of

$$\{(3,0,-1,-1),(0,3,-1,-1),(-3,-3,-1,-1),(0,0,2,-1),(0,0,-1,2)\}.$$

It can be computed easily that the Ehrhart polynomial of $\mathcal{P}$ is equal to

$$\frac{81}{8}n^4 + \frac{81}{4}n^3 + \frac{135}{8}n^2 + \frac{27}{4}n + 1$$

and its roots are

$$-\frac{1}{3}, -\frac{2}{3}, -\frac{1}{2} + \frac{\sqrt{-7}}{6} \text{ and } -\frac{1}{2} - \frac{\sqrt{-7}}{6}.$$

Remark 0.4. (a) It is disproved in [8] that all of the roots α of the Hilbert polynomial of any Fano variety satisfy $-1 < \Re(\alpha) < 0$, the so-called *canonical strip hypothesis*, which is stated in [7]. On Theorem 0.1, however, all of the roots of Ehrhart polynomials of our Gorenstein Fano polytopes satisfy this condition. In more detail, they satisfy *the narrowed canonical strip hypothesis*, which is the condition $-1 + 1/(d+1) \leq \Re(\alpha) \leq -1/(d+1)$. Moreover, if we set $2k = d$ when d is even or $2k = d-1$ when d is odd, then they also satisfy *the canonical line hypothesis*, which is the condition $\Re(\alpha) = -1/2$.

(b) We should consider the connections of the Ehrhart polynomials of our Gorenstein Fano polytopes with L-functions. Let $i(\mathcal{P}, s)$ be the Ehrhart polynomial of our Gorenstein Fano polytope $\mathcal{P}$ with $2k = d$ when d is even or with $2k = d-1$ when d is odd. Then we set $z(s) = i(\mathcal{P}, -s)$. Then the function equation

$$z(1-s) = (-1)^d z(s)$$

holds and all of its roots α satisfy $\Re(\alpha) = 1/2$, which is, of course, the Riemann zeta function.

Example 0.5. Let G be a finite connected graph on the vertex set $V(G) = \{1, \ldots, d\}$ with $E(G)$ its edge set. We assume that G possesses no loop and no multiple edge. Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ denote the canonical unit vectors of $\mathbb{R}^d$. For an edge $e = \{i, j\}$ of G with $i < j$, we define $\rho(e)$ and $\mu(e)$ of $\mathbb{R}^d$ by setting $\rho(e) = \mathbf{e}_i - \mathbf{e}_j$ and $\mu(e) = \mathbf{e}_j - \mathbf{e}_i$. Write $\mathcal{P}_G^{\pm} \subset \mathbb{R}^d$ for the convex polytope which is the convex hull of $\{\rho(e) : e \in E(G)\} \cup \{\mu(e) : e \in E(G)\}$. Let $\mathcal{H} \subset \mathbb{R}^d$ denote the hyperplane defined by the equation $\sum_{i=1}^d x_i = 0$. Then $\mathcal{P}_G^{\pm} \subset \mathcal{H}$. Identifying $\mathcal{H}$ with $\mathbb{R}^{d-1}$, it turns out that $\mathcal{P}_G^{\pm} \subset \mathbb{R}^{d-1}$ is a Gorenstein Fano polytope of dimension $d-1$. (In detail, see [16, Proposition 3.2].) One of the research problems is to find a combinatorial characterization of the finite graphs G for which all roots $a \in \mathbb{C}$ of $i(\mathcal{P}_G^{\pm}, n)$ satisfy $\Re(a) = -1/2$.

For example, if C is a cycle of length 6, then all roots $a \in \mathbb{C}$ of $i(\mathcal{P}_C^{\pm}, n)$ satisfy $\Re(a) = -1/2$. However, if C is a cycle of length 7, then there is a root $a \in \mathbb{C}$ of $i(\mathcal{P}_C^{\pm}, n)$ with $\Re(a) \neq -1/2$.

If G is a tree, then $\mathcal{P}_G^{\pm}$ is unimodular isomorphic to the regular unit crosspolytope of dimension $d-1$, which is the convex hull of $\{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_{d-1}\}$ in $\mathbb{R}^{d-1}$. Hence the δ-vector of $\mathcal{P}_G^{\pm}$ is $\delta(\mathcal{P}_G^{\pm}) = (\binom{d-1}{0}, \binom{d-1}{1}, \ldots, \binom{d-1}{d-1})$. Thus, by using [19] again, all roots $a \in \mathbb{C}$ of $i(\mathcal{P}_G^{\pm}, n)$ satisfy $\Re(a) = -1/2$.

Let G be a complete bipartite graph of type $(2, d-2)$. Thus the edges of G are either $\{1, j\}$ or $\{2, j\}$ with $3 \leq j \leq d$. Let $\delta(\mathcal{P}_G^{\pm}) = (\delta_0, \delta_1, \ldots, \delta_{d-1})$. Then

$$\sum_{k=0}^{d-1} \delta_k x^k = (1+x)^{d-3}(1 + 2(d-2)x + x^2).$$

It has been conjectured that all roots $a \in \mathbb{C}$ of $i(\mathcal{P}_G^{\pm}, n)$ satisfy $\Re(a) = -1/2$ in [16].

References

[1] V. Batyrev, Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties, *J. Algebraic Geom.* **3** (1994), 493–535. MR1269718 (95c:14046)

[2] M. Beck, J. A. De Loera, M. Develin, J. Pfeifle and R. P. Stanley, Coefficients and roots of Ehrhart polynomials, Contemp. Math. **374**, Amer. Math. Soc., 2005, 15–36. MR2134759 (2006e:52019)

[3] C. Bey, M. Henk and J. M. Wills, Notes on the roots of Ehrhart polynomials, *Discrete Comput. Geom.* **38** (2007), 81–98. MR2322117 (2008k:52027)

[4] B. Braun, Norm bounds for Ehrhart polynomial roots, *Discrete Comput. Geom.* **39** (2008), 191–193. MR2383758 (2009d:52019)

[5] B. Braun and M. Develin, Ehrhart polynomial roots and Stanley's non-negativity theorem, Contemp. Math. **452**, Amer. Math. Soc., 2008, 67–78. MR2405764 (2009f:52040)

[6] E. Ehrhart, "Polynômes Arithmétiques et Méthode des Polyèdres en Combinatoire," Birkhäuser, Boston/Basel/Stuttgart, 1977. MR0432556 (55:5544)

[7] V. V. Golyshev, On the canonical strip, *Russ. Math. Surv.* **64** (2009), no. 1, 145–147. MR2503098 (2010g:14060)

[8] G. Hegedüs and A. M. Kasprzyk, Roots of Ehrhart polynomials of smooth Fano polytopes, *Discrete Comput. Geom.* (2010).

[9] M. Henk, A. Schürmann and J. M. Wills, Ehrhart polynomials and successive minima, *Mathematika* **52** (2005), 1–16. MR2261838 (2007h:52013)

[10] T. Hibi, "Algebraic Combinatorics on Convex Polytopes," Carslaw Publications, Glebe, N.S.W., Australia, 1992.

[11] T. Hibi, Dual polytopes of rational convex polytopes, *Combinatorica* **12** (1992), 237–240. MR1179260 (93f:52018)

[12] T. Hibi, A lower bound theorem for Ehrhart polynomials of convex polytopes, *Adv. in Math.* **105** (1994), 162–165. MR1275662 (95b:52018)

[13] A. M. Kasprzyk, Toric Fano threefolds with terminal singularities, *Tohoku Math. J. (2)* **58** (2006), no. 1, 101–121. MR2221794 (2007b:14089)

[14] M. Kreuzer and H. Skarke, On the classification of reflexive polyhedra, *Comm. Math. Phys.* **185** (1997), no. 2, 495–508. MR1463052 (98f:32029)

[15] M. Kreuzer and H. Skarke, Complete classification of reflexive polyhedra in four dimensions, *Adv. Theor. Math. Phys.* **4** (2000), no. 6, 1209–1230. MR1894855 (2003e:52020)

[16] T. Matsui, A. Higashitani, Y. Nagazawa, H. Ohsugi and T. Hibi, Roots of Ehrhart polynomials arising from graphs, *Journal of Algebraic Combinatorics*, to appear.

[17] B. Nill, Gorenstein toric Fano varieties, *Manuscripta Math.* **116** (2005), no. 2, 183–210. MR2122419 (2005k:14110)

[18] J. Pfeifle, Gale duality bounds for roots of polynomials with nonnegative coefficients, *J. Comb. Theory Ser. A* **117** (2010), no. 3, 248–271. MR2592900

[19] F. Rodriguez-Villegas, On the zeros of certain polynomials, *Proc. Amer. Math. Soc.* **130** (2002), 2251–2254. MR1896405 (2003d:12002)

[20] R. P. Stanley, Decompositions of rational convex polytopes, *Annals of Discrete Math.* **6** (1980), 333–342. MR593545 (82a:52007)

[21] R. P. Stanley, "Enumerative Combinatorics, Volume 1," Wadsworth & Brooks/Cole, Monterey, Calif., 1986. MR0847717 (87j:05003)

[22] R. P. Stanley, On the Hilbert function of a graded Cohen–Macaulay domain, *J. Pure and Appl. Algebra* **73** (1991), 307–314. MR1124790 (92f:13017)

[23] R. P. Stanley, A monotonicity property of h-vectors and h^*-vectors, *Europ. J. Combinatorics* **14** (1993), 251–258. MR1215335 (94f:52016)

[24] R. P. Stanley, "Combinatorics and Commutative Algebra," Second Ed., Birkhäuser, 1996. MR1453579 (98h:05001)

Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Toyonaka, Osaka 560-0043, Japan
E-mail address: hibi@math.sci.osaka-u.ac.jp

Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Toyonaka, Osaka 560-0043, Japan
E-mail address: sm5037ha@ecs.cmc.osaka-u.ac.jp

Department of Mathematics, College of Science, Rikkyo University, Toshima-ku, Tokyo 171-8501, Japan
E-mail address: ohsugi@rikkyo.ac.jp