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ESSENTIALLY FINITE VECTOR BUNDLES ON VARIETIES WITH TRIVIAL TANGENT BUNDLE

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Abstract. Let X be a smooth projective variety, defined over an algebraically closed field of positive characteristic, such that the tangent bundle TX is trivial. Let $F_X : X \longrightarrow X$ be the absolute Frobenius morphism of X. We prove that for any $n \geq 1$, the n–fold composition F^n_X is a torsor over X for a finite group–scheme that depends on n. For any vector bundle $E \longrightarrow X$, we show that the direct image $(F^n_X)_*E$ is essentially finite (respectively, F–trivial) if and only if E is essentially finite (respectively, F–trivial).

1. Introduction

For a smooth projective variety X over a field of characteristic zero, the tangent bundle TX is trivial if and only if X is an abelian variety. This is not true for fields of characteristic $p > 0$. Examples of varieties, different from abelian varieties, with trivial tangent bundle can be found in [4], [5] (see also [6]).

Let X be a smooth projective variety, defined over an algebraically closed field of positive characteristic, with the property that TX is trivial. Let $F_X : X \longrightarrow X$ be the absolute Frobenius morphism. For any $n \geq 1$, let F^n_X be the n–fold composition of the self–map F_X.

Nori introduced the fundamental group–scheme [9], [10]. We recall that after introducing the essentially finite vector bundles and then showing that they form a neutral Tannakian category, Nori defined the fundamental group–scheme to be the one given by this neutral Tannakian category. We also recall that a vector bundle V on a smooth projective variety is essentially finite if and only if the pullback of V by some projective surjective morphism is trivial. A special class of essentially finite vector bundles is the F–trivial vector bundle; a vector bundle is called F–trivial if its pullback by some power of the Frobenius morphism is trivial. The F–trivial vector bundles also define a neutral Tannakian category. The corresponding group–scheme is called the local fundamental group–scheme.

We prove the following proposition (see Proposition 2.5):

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Proposition 1.1. *Let E be a vector bundle on a smooth projective variety X with trivial tangent bundle. If E is essentially finite, then the direct image $(F_X^n)_*E$ is essentially finite for every n.*

If $(F_X^n)_*E$ is essentially finite for some n, then E is essentially finite.

We also prove the following theorem (see Theorem 2.6):

Theorem 1.2. *Let X and E be as in Proposition* 1.1. *If E is F–trivial, then $(F_X^n)_*E$ is F–trivial for every n.*

If $(F_X^n)_*E$ is F–trivial for some n, then E is F–trivial.

The condition that TX is trivial is used in the following three ways: All nonzero vector fields on X are nowhere vanishing; the cotangent bundle Ω_X^1 is a subbundle of a trivial vector bundle (equivalently, TX is globally generated), and for a vector bundle E on X, the Chern class $c_1(E)$ is numerically trivial if and only if $c_1((F_X^n)_*E)$ is numerically trivial. More precisely, the crucial Lemma 2.1 is proved using the fact that all nonzero vector fields on X are nowhere vanishing and that TX is globally generated. In the proofs of Proposition 2.5 and Theorem 2.6, we use the fact that $c_1(E)$ is numerically trivial if and only if $c_1((F_X^n)_*E)$ is numerically trivial.

2. Varieties with trivial tangent bundle

Let k be an algebraically closed field of characteristic p, with $p > 0$. Let X be an irreducible smooth projective variety defined over k such that the tangent bundle TX is trivial. Let d be the dimension of X.

The d–dimensional vector space $H^0(X, TX)$ is equipped with the Lie bracket operation. Note that $H^0(X, TX)$ is a p–Lie algebra. There is a natural bijective correspondence between the p–Lie algebras over k and the local group–schemes over k of height one [8, p. 139]. Let

$$G \tag{1}$$

be the local group–scheme of height one corresponding to the p–Lie algebra $H^0(X, TX)$.

Let

$$F_X : X \longrightarrow X \tag{2}$$

be the absolute Frobenius morphism. For any integer $n \geq 1$, let

$$F_X^n := \overbrace{F_X \circ \cdots \circ F_X}^{n\text{-times}} : X \longrightarrow X$$

be the n–fold iteration of F_X. By F_X^0 we will denote the identity morphism of X.

Lemma 2.1. *The absolute Frobenius morphism F_X defines a G–torsor over X, where G is defined in* (1).

Proof. Since G corresponds to a Lie algebra of derivations on X, the group–scheme G has a tautological action on X. Let

$$q : X \longrightarrow Z := X/G$$

be the quotient morphism. We note that the action of G on X is free, because all the nonzero vector fields on X are nowhere vanishing. Therefore, the above morphism q defines a G–torsor on Z.

It can be shown that the Frobenius morphism F_X factors as

$$\begin{array}{ccc} X & \stackrel{q}{\longrightarrow} & Z = X/G \\ \| & & \big\downarrow \phi \\ X & \stackrel{F_X}{\longrightarrow} & X \end{array} \tag{3}$$

To prove this, let $\mathcal{E}$ denote the above principal G–bundle $q : X \longrightarrow Z$. Let $F_Z : Z \longrightarrow Z$ be the Frobenius morphism of Z. The pullback $F_Z^*\mathcal{E}$ is identified with the principal G–bundle obtained by extending the structure group of $\mathcal{E}$ using the Frobenius homomorphism $F_G : G \longrightarrow G$. We recall that G is of height one, which means that the image of F_G is the identity. Hence $F_Z^*\mathcal{E}$ is the trivial principal G–bundle $Z \times G \longrightarrow Z$. Therefore, we have a commutative diagram:

$$\begin{array}{ccc} Z \times G & \stackrel{\psi}{\longrightarrow} & X \\ \big\downarrow & & \big\downarrow \\ Z & \stackrel{F_Z}{\longrightarrow} & Z \end{array}$$

The morphism ϕ in (3) is the restriction of ψ to the Cartesian product of Z with the identity of G.

Now considering the scheme theoretic fibers of q and F_X we conclude that ϕ is an isomorphism. $\square$

Essentially finite vector bundles were defined in [9], [10]. A vector bundle E on a smooth projective variety Y is essentially finite if and only if there is a projective variety Z and a surjective morphism $\psi : Z \longrightarrow Y$ such that the pullback ψ^*E is trivial [2, Theorem 1.1].

Corollary 2.2. *The vector bundle $(F_X)^*(F_X)_*\mathcal{O}_X \longrightarrow X$ is trivial. In particular, the direct image $(F_X)_*\mathcal{O}_X$ is essentially finite.*

Proof. From Lemma 2.1 it follows that $(F_X)^*(F_X)_*\mathcal{O}_X$ is the trivial vector bundle over X with fiber $k[G]$ (see [8, p. 120, Corollary 2]). Hence $(F_X)_*\mathcal{O}_X$ is essentially finite by the above criterion. $\square$

Lemma 2.3. *Let $\gamma : Y \longrightarrow X$ be an étale cover. Then the tangent bundle of Y is trivial.*

Proof. Let $d\gamma : TY \longrightarrow \gamma^*TX$ be the differential of the morphism γ. Since γ is étale, we know that $d\gamma$ is an isomorphism. The vector bundle γ^*TX is trivial because TX is trivial. Hence the isomorphic vector bundle TY is trivial. $\square$

Lemma 2.4. *Let E be an essentially finite vector bundle over X. Then the direct image $(F_X)_*E$ is also essentially finite.*

Proof. There is an étale Galois covering

$$\gamma : Y \longrightarrow X \tag{4}$$

and a positive integer m such that $(F_Y^m)^*\gamma^*E$ is trivial, where

$$F_Y : Y \longrightarrow Y$$

is the absolute Frobenius morphism of Y [10, Chapter II, Proposition 7] (see also [1, p. 557]). Since γ is étale, the following diagram is Cartesian:

$$\begin{array}{ccc} Y & \xrightarrow{\gamma\circ F_Y^m} & X \\ \Big\downarrow F_Y & & \Big\downarrow F_X \\ Y & \xrightarrow{F_X^m\circ\gamma} & X \end{array}$$

Hence

$$F_{Y*}(F_Y^m)^*\gamma^*E = F_{Y*}(\gamma\circ F_Y^m)^*E = (F_X^m\circ\gamma)^*((F_X)_*E) \tag{5}$$

(see [3, p. 255, Proposition 9.3]).

Since $(F_Y^m)^*\gamma^*E$ is trivial, from Corollary 2.2 we know that $F_{Y*}(F_Y^m)^*\gamma^*E$ is essentially finite (note that Lemma 2.3 implies that Corollary 2.2 applies). Hence $(F_X^m\circ\gamma)^*((F_X)_*E)$ is essentially finite by (5). This implies that the direct image $(F_X)_*E$ is essentially finite. □

Proposition 2.5. *For any essentially finite vector bundle $E \longrightarrow X$, and any $n \geq 1$, the direct image $(F_X^n)_*E$ is essentially finite. In particular, $(F_X^n)_*\mathcal{O}_X$ is essentially finite.*

A vector bundle E on X is essentially finite if $(F_X^n)_*E$ is essentially finite for some n.

Proof. Since $(F_X^{m+1})_*E = (F_X^1)_*(F_X^m)_*E$, from Lemma 2.4 we conclude that $(F_X^n)_*E$ is essentially finite if E is essentially finite.

To prove the second part, let $E \longrightarrow X$ be a vector bundle, and let n be a positive integer such that the direct image $(F_X^n)_*E$ is essentially finite. Since TX is trivial and $(F_X^n)_*E$ is essentially finite, it follows that $c_1(E)$ is numerically equivalent to zero.

The pullback $(F_X^n)^*(F_X^n)_*E$ is essentially finite because $(F_X^n)_*E$ is as well. Since F_X^n is a finite morphism, the natural homomorphism

$$(F_X^n)^*(F_X^n)_*E \longrightarrow E$$

is surjective; hence E is a quotient of the vector bundle $(F_X^n)^*(F_X^n)_*E$. As E is a quotient of an essentially finite vector bundle and $c_1(E)$ is numerically equivalent to zero, we conclude that E is essentially finite; see Remark 2.7. □

We recall that a vector bundle E over X is called F*–trivial* if the vector bundle $(F_X^{n_0})^*E$ is trivial for some n_0.

Theorem 2.6. *Let $E \longrightarrow X$ be an F–trivial vector bundle. Then for every n, the direct image $(F_X^n)_*E$ is F–trivial.*

If E is a vector bundle on X such that $(F_X^n)_*E$ is F–trivial for some n, then E is F–trivial.

Proof. To prove the first part, it suffices to show that $(F_X)_*E$ is F–trivial.

Let n_0 be an integer such that the vector bundle $(F_X^{n_0})^*E$ is trivial. In (4), take $Y = X$ and set γ to be the identity morphism of X. Now, from (5),

$$F_{X*}(F_X^{n_0})^*E = (F_X^{n_0})^*((F_X)_*E)\,. \tag{6}$$

To prove that $(F_X)_*E$ is F–trivial, it is enough to show that $(F_X^{n_0})^*((F_X)_*E)$ is F–trivial.

On the other hand, $F_{X*}(F_X^{n_0})^*E$ is F–trivial if $F_{X*}\mathcal{O}_X$ is F–trivial, because $(F_X^{n_0})^*E$ is a trivial vector bundle. Hence from (6) we conclude that $(F_X^{n_0})^*((F_X)_*E)$ is F–trivial if $F_{X*}\mathcal{O}_X$ is F–trivial. But $F_{X*}\mathcal{O}_X$ is F–trivial by Corollary 2.2. This completes the proof of the first part.

To prove the second part, let $E \longrightarrow X$ be a vector bundle of rank r, and let n be a positive integer such that the direct image $(F_X^n)_*E$ is F–trivial. So there is a positive integer m such that the pullback $(F_X^m)^*(F_X^n)_*E$ is a trivial vector bundle. Any pullback of a trivial vector bundle is trivial. Hence we may – and we will – assume that $m \geq n$.

We saw in the proof of Proposition 2.5 that E is a quotient of $(F_X^n)^*(F_X^n)_*E$. Hence the vector bundle $(F_X^{m-n})^*E$ is a quotient of the trivial vector bundle

$$(F_X^{m-n})^*(F_X^n)^*(F_X^n)_*E \,=\, (F_X^m)^*(F_X^n)_*E\,.$$

Define $V_0 := H^0(X,\,(F_X^m)^*(F_X^n)_*E)$, and let $\mathrm{Gr}(V_0, r)$ be the Grassmannian parametrizing quotients of V_0 of dimension r. Note that the evaluation of sections identifies $(F_X^m)^*(F_X^n)_*E$ with the trivial vector bundle $X \times V_0$. The quotient map

$$(F_X^m)^*(F_X^n)_*E \twoheadrightarrow (F_X^{m-n})^*E$$

produces a quotient map $X \times V_0 \twoheadrightarrow (F_X^{m-n})^*E$. Therefore, there is a morphism

$$f_E \,:\, X \longrightarrow \mathrm{Gr}(V_0, r)$$

such that $(F_X^{m-n})^*E$ is the pullback f_E^*Q, where $Q \longrightarrow \mathrm{Gr}(V_0, r)$ is the tautological quotient bundle.

Since TX is trivial and $(F_X^n)_*E$ is F–trivial, it follows that $c_1(E)$ is numerically trivial. Hence $c_1((F_X^{m-n})^*E)$ is numerically trivial (this also uses the assumption that TX is trivial). Since the line bundle $\det Q \longrightarrow \mathrm{Gr}(V_0, r)$ is ample and $c_1((F_X^{m-n})^*E) \,=\, c_1(f_E^*Q)$ is numerically trivial, it follows that f_E is a constant morphism. Consequently, the vector bundle $(F_X^{m-n})^*E \,=\, f_E^*Q$ is trivial. In particular, E is F–trivial. □

Remark 2.7. Let E be an essentially finite vector bundle over a smooth projective variety M, and let Q be a quotient bundle of E such that $c_1(Q)$ is numerically trivial. Then Q is also essentially finite. This can be derived from the definition of semistable bundles given in [10, p. 81] (note that this definition differs from the usual definition of semistability) and the definition of an essentially finite vector bundle given in [10, p. 82]. Here we have used the following characterization of an essentially finite vector bundle: There is an étale Galois covering

$$\gamma \,:\, \widetilde{M} \longrightarrow M$$

such that $(F_{\widetilde{M}}^n)^*\gamma^*E$ is trivial, where n is some positive integer and $F_{\widetilde{M}}$ is the absolute Frobenius morphism of $\widetilde{M}$. The Chern class $c_1((F_{\widetilde{M}}^n)^*\gamma^*Q) \,=\, (F_{\widetilde{M}}^n)^*\gamma^*c_1(Q)$ is numerically trivial because $c_1(Q)$ is as well. Since $(F_{\widetilde{M}}^n)^*\gamma^*Q$ is a quotient of the trivial vector bundle $(F_{\widetilde{M}}^n)^*\gamma^*E$ and $c_1((F_{\widetilde{M}}^n)^*\gamma^*Q)$ is numerically trivial, we know that the vector bundle $(F_{\widetilde{M}}^n)^*\gamma^*Q$ is trivial; see the proof of the second part of Theorem 2.6. Consequently, Q is essentially finite [10], [2, Theorem 1.1].

3. The local fundamental group–scheme

We continue with the set-up of the previous section. Let X be ordinary.

Let $\{e_1, \cdots, e_d\}$ be a basis of the k–vector space $H^0(X, TX)$. Recall that $H^0(X, TX)$ is the Lie algebra of G.

Given a vector field $\theta \in H^0(X, TX)$, we have a vector field θ^p defined by

$$\theta^p(f) := \overbrace{\theta \circ \cdots \circ \theta}^{p\text{–times}}(f)$$

for all locally defined functions on X. Let

$$\mu : H^0(X, TX) \longrightarrow H^0(X, TX) \tag{7}$$

be the additive group homomorphism defined by $\theta \longmapsto \theta^p$. Note that

$$\mu(c\theta) = c^p \mu(\theta) \tag{8}$$

for all $c \in k$. Since μ is an additive group homomorphism and $\mu^{-1}(0) = 0$, we conclude that μ is injective. Since

$$\mu(\sum_{i=1}^{d} c_i \cdot e_i) = \sum_{i=1}^{d} c_i^p \cdot \mu(e_i), \tag{9}$$

it follows that $\{\mu(e_1), \cdots, \mu(e_d)\}$ is a basis for the k–vector space $H^0(X, TX)$. Now from (9) it follows that μ is surjective. Therefore, μ is an isomorphism.

Recall the above basis $\{e_1, \cdots, e_d\}$ of $H^0(X, TX)$. Let $\{e_1^*, \cdots, e_d^*\}$ be the dual basis of $H^0(X, TX)^*$. For any $i \in [1, d]$, the element of the coordinate ring $k[G]$ of G given by e_i^* will be denoted by x_i. So

$$k[G] = \frac{k[x_1, \cdots, x_d]}{(x_1^p, \cdots, x_d^p)}.$$

Let

$$x_i \longmapsto \sum_j f_j^i \otimes g_j^i \tag{10}$$

be the co–multiplication structure of $k[G]$.

Define $G_1 := G$. Let G_2 be the group–scheme defined by the rule

$$k[G_2] := \frac{k[x_1^{1/p}, \cdots, x_d^{1/p}]}{(x_1^p, \cdots, x_d^p)}$$

with the co–multiplication rule

$$x_i^{1/p} \longmapsto \sum_j (f_j^i \otimes g_j^i)^{1/p},$$

where f_j^i and g_j^i are defined in (10). We note that G_2 is a local group–scheme of height two.

Let $F_{G_2} : G_2 \longrightarrow G_2$ be the absolute Frobenius morphism. Note that the image $F_{G_2}(G_2)$ coincides with the image of the natural homomorphism $G_1 := G \longrightarrow G_2$. More precisely, we have a short exact sequence of group–schemes:

$$e \longrightarrow G_1 \longrightarrow G_2 \xrightarrow{F_{G_2}} G_1 \longrightarrow e. \tag{11}$$

More generally, for any positive integer n, define the group–scheme G_{n+1} as follows:

$$k[G_{n+1}] := \frac{k[x_1^{1/p^n}, \cdots, x_d^{1/p^n}]}{(x_1^p, \cdots, x_d^p)}$$

with co–multiplication rule

$$x_i^{1/p^n} \longmapsto \sum_j (f_j^i \otimes g_j^i)^{1/p^n}.$$

Note that G_n is a local group–scheme of height n. Let

$$F_{G_n} : G_n \longrightarrow G_n$$

be the absolute Frobenius morphism. The image $F_{G_n}(G_n)$ coincides with the image of the natural homomorphism $G_{n-1} \longrightarrow G_n$. As in (11), we have a short exact sequence of group–schemes:

$$e \longrightarrow G_1 \longrightarrow G_n \xrightarrow{F_{G_n}} G_{n-1} \longrightarrow e. \tag{12}$$

We will show that G_{n+1} acts on X.

Given a vector field D on X and any positive integer n, let D^{1/p^n} be the vector field on X defined by

$$D^{1/p^n}(f) := D((f)^{1/p^n})^{p^n} \tag{13}$$

for all locally defined functions f on X. So $D^{1/p^n} = (\mu^n)^{-1}(D)$, where μ is the isomorphism in (7).

Let $S = \operatorname{Spec} k[t]/t^2$ be the Artin local k–algebra. Consider $G_{n+1}(S)$. We will construct an action of $G_{n+1}(S)$ on $X(S)$. A point of $G_{n+1}(S)$ is a k–algebra homomorphism:

$$\frac{k[x_1^{1/p^n}, \cdots, x_d^{1/p^n}]}{(x_1^p, \cdots, x_d^p)} \longrightarrow k[t]/t^2.$$

Consider the homomorphism

$$\frac{k[x_1^{1/p^n}, \cdots, x_d^{1/p^n}]}{(x_1^p, \cdots, x_d^p)} \longrightarrow k[t]/t^2$$

defined by $x_i^{1/p^n} \longmapsto t$. The action of this point of $G_{n+1}(S)$ on $X(S)$ is constructed as follows: The action of $(e_i)^{1/p^n}$ sends a tangent vector v to $v + (e_i)^{1/p^n}$. (The tangent vector $(e_i)^{1/p^n}$ is defined in (13); the tangent vectors $\{e_i\}$ are dual to x_i.)

The following diagram is commutative:

$$\begin{array}{ccc} G_n \times X & \longrightarrow & X \\ {\scriptstyle F_{G_n} \times F_X}\Big\downarrow & & {\scriptstyle F_X}\Big\downarrow \\ G_{n-1} \times X & \longrightarrow & X \end{array} \tag{14}$$

where the morphisms $G_n \times X \longrightarrow X$ and $G_{n-1} \times X \longrightarrow X$ are the actions on X of G_n and G_{n-1} respectively.

The quotient X/G_n is identified with X, and the quotient morphism

$$X \longrightarrow X/G_n$$

coincides with the morphism

$$F_X^n : X \longrightarrow X.$$

This is already proved for $n = 1$ (see the proof of Lemma 2.1). To prove the general case, use induction on n. The quotient by the subgroup–scheme $G = G_1$ in (12) is the Frobenius morphism F_X (the case of $n = 1$); hence from (12) it follows that the quotient by G_n is the composition F_X^n.

Thus the morphism F_X^n defines a principal G_n–bundle over X.

Fix a k–rational point $x_0 \in X$. The *local fundamental group–scheme* $\varpi_1^{\mathrm{loc}}(X, x_0)$ is defined to be the group–scheme associated to the neutral Tannakian category given by the F–trivial vector bundles on X (see [7]); the definition of F–trivial vector bundles is recalled in Section 2. There is a tautological universal principal $\varpi_1^{\mathrm{loc}}(X, x_0)$–bundle

$$\widehat{X} \longrightarrow X\,. \tag{15}$$

Theorem 3.1. *The local fundamental group–scheme $\varpi_1^{\mathrm{loc}}(X, x_0)$ is the inverse limit of the group–schemes $\{G_n\}_{n\geq 1}$ constructed using the homomorphisms in* (12)*. The tautological principal $\varpi_1^{\mathrm{loc}}(X, x_0)$–bundle $\widehat{X}$ in* (15) *is the inverse limit of the morphisms $F_X^n : X \longrightarrow X$.*

Proof. Let $\mathcal{G}$ denote the inverse limit of the group–schemes $\{G_n\}_{n\geq 1}$ constructed using the homomorphisms in (12). Let

$$E_{\mathcal{G}} \longrightarrow X$$

be the principal $\mathcal{G}$–bundle defined by the inverse limit of the morphisms $F_X^n : X \longrightarrow X$. From the commutativity of the diagram in (14) it follows that the inverse limit of the morphisms F_X^n is a principal $\mathcal{G}$–bundle. Take any rational representation V of $\mathcal{G}$. Therefore V is a rational representation of G_{n_0} for some n_0. Let

$$E_V^{n_0} \longrightarrow X$$

be the vector bundle associated to the principal G_{n_0}–bundle

$$F_X^{n_0} : X \longrightarrow X$$

for the G_{n_0}–module V. So the vector bundle $(F_X^{n_0})^* E_V^{n_0}$ is trivializable. Thus $E_V^{n_0}$ is F–trivial. Consequently, we obtain a homomorphism,

$$\rho : \varpi_1^{\mathrm{loc}}(X, x_0) \longrightarrow \mathcal{G}\,. \tag{16}$$

This also produces an isomorphism of the principal $\mathcal{G}$–bundle $E_{\mathcal{G}}$ with the principal $\mathcal{G}$–bundle $\widehat{X} \times^{\varpi_1^{\mathrm{loc}}(X,x_0)} \mathcal{G}$ obtained by extending the structure group of the principal $\varpi_1^{\mathrm{loc}}(X, x_0)$–bundle $\widehat{X}$ using the homomorphism ρ in (16).

For the converse direction, let $E \longrightarrow X$ be a F–trivial vector bundle of rank r. Let n_0 be an integer such that the pullback $(F_X^{n_0})^* E$ is trivializable. Fix an isomorphism of $(F_X^{n_0})^* E$ with the trivial vector bundle $X \times k^{\oplus r}$. Using this trivialization, the natural action of G_{n_0} on the fiber $((F_X^{n_0})^* E)|_{(F_X^{n_0})^{-1}(x_0)}$ defines a linear action of G_{n_0} on $k^{\oplus r}$. Therefore, we obtain a homomorphism,

$$\eta : \mathcal{G} \longrightarrow \varpi_1^{\mathrm{loc}}(X, x_0),$$

which is the inverse of ρ. □

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