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HECKE OPERATORS FOR NON-CONGRUENCE SUBGROUPS OF BIANCHI GROUPS

SAEID HAMZEH ZARGHANI

(Communicated by Kathrin Bringmann)

Before the first draft of this work was completed, Fritz Grunewald tragically passed away. Indeed, without his guidance and support this work would never have been done. I dedicate this paper to his memory with admiration, gratitude and love.

Abstract. We prove that the action of the Hecke operators on the cohomology of a finite index non-congruence subgroup Γ of a Bianchi group is essentially the same as the action of Hecke operators on the cohomology groups of $\hat{\Gamma}$, the congruence closure of Γ. This is a generalization of Atkin's conjecture, first confirmed in a special case by Serre in 1987 and proved in general by Berger in 1994.

Introduction

For every finite index subgroup H of $\mathrm{PSL}(2,\mathbb{Z})$ ($H \leq_f \mathrm{PSL}(2,\mathbb{Z})$ for short) and every $k, p \in \mathbb{N}$, p prime, let $M_k(H)$ denote the space of the H-modular forms of weight k and recall that the Hecke operator $T_p^H : M_k(H) \to M_k(H)$ is defined by

$$T_p^H(f) := \sum_i (f \mid_k \tilde{p}) \mid_k g_i,$$

where $\tilde{p} := \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $H = \bigcup_i H_p g_i$, and $H_p := H \cap \tilde{p}^{-1} H \tilde{p}$. A conjecture of Atkin (now a theorem, first confirmed in a special case by Serre in 1987 (see Appendix of [15]) and finally proved in general by Berger in 1994 (see [4])) states that the action of the Hecke operators T_p^H on the space of the modular forms of any given weight k associated to a non-congruence subgroup $H \leq \mathrm{PSL}(2,\mathbb{Z})$ is essentially the same as the action of the Hecke operators $T_p^{\hat{H}}$ on $M_k(\hat{H})$, where $\hat{H}$ is the congruence closure of H. More precisely,

Theorem (Serre, Berger)**.** *For every prime p (not dividing the level of H), we have $T_p^H = T_p^{\hat{H}} \circ Tr_H^{\hat{H}}$, where Tr is the trace map. That is, the following diagram*

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commutes:

$$\begin{array}{ccc} M_k(H) & \xrightarrow{Tr_H^{\hat{H}}} & M_k(\hat{H}) \\ {\scriptstyle T_p^H}\downarrow & & \downarrow{\scriptstyle T_p^{\hat{H}}} \\ M_k(H) & \xleftarrow[incl.]{} & M_k(\hat{H}) \end{array}$$

In this paper, motivated by the recent interest in the classical modular forms for non-congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$ ([2]) and in the arithmetic of Bianchi modular forms ([5] and [7]), we will prove the following generalization of Atkin's conjecture to the cohomology of subgroups of Bianchi groups. The main idea, which is a generalization of Berger's idea in [4], is as follows: starting with pure group theory, let G be an arbitrary group, $H \leq_f K \leq G$ and $g \in G$ be such that $K = (K_g)H$ and $[K_g : H_g] = [K : H]^2$, where $K_g := K \cap g^{-1}Kg$, etc. We show that:

Theorem (Theorem 16)**.** *Under the above assumptions, for every G-module M and every $q \geq 1$ the following diagram commutes:*

$$\begin{array}{ccc} H^q(H,M) & \xrightarrow{tr_H^K} & H^q(K,M) \\ {\scriptstyle T_g^H}\downarrow & & \downarrow{\scriptstyle T_g^K} \\ H^q(H,M) & \xleftarrow[res_H^K]{} & H^q(K,M) \end{array}$$

where tr_H^K (res_H^K resp.) denotes the trace (restriction resp.) map.

Now let d be any square-free negative integer and $H \leq_f \Gamma_d = \mathrm{PSL}(2,\mathcal{O}_d)$ be of level $\mathfrak{a}$ (see below), and let $\hat{H}$ be its congruence closure (see section 1). Suppose that $p \in \mathcal{O}_d$ is prime and $\mathfrak{a} + p\mathcal{O}_d = \mathcal{O}_d$. Define $\tilde{p} := \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{PGL}(2,\mathbb{C})$. We show that H and $\hat{H}$ satisfy the above conditions, and hence

Theorem (Theorem 23)**.** *Under the above assumptions, we have for every Γ_d-module M and every $q \geq 1$ that the following diagram commutes:*

$$\begin{array}{ccc} H^q(H,M) & \xrightarrow{tr_H^{\hat{H}}} & H^q(\hat{H},M) \\ {\scriptstyle T_{\tilde{p}}^H}\downarrow & & \downarrow{\scriptstyle T_{\tilde{p}}^{\hat{H}}} \\ H^q(H,M) & \xleftarrow[res_H^{\hat{H}}]{} & H^q(\hat{H},M) \end{array}$$

By the term "level" of H here we mean the (unique) ideal $\mathfrak{a}$ of $\mathcal{O}_d$ which is maximal with the property that the normal closure of $\{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{PSL}(2,\mathcal{O}_d) \mid a \in \mathfrak{a}\}$ is included in H (see [8] and section 1).

In this paper, only Hecke operators at the places of $\mathcal{O}_d$ corresponding to the principal prime ideals are considered.

This work consists of five sections. Section 1 is a brief review of basic definitions and facts about the extended notion of level and congruence closure. The goal of Section 2 is to prove the congruence subgroup property (CSP) for $\mathrm{PSL}(2,\mathcal{O}_d[1/p])$ and show that for any ideal $I \lhd \mathcal{O}_d$ and for certain elements $g \in \mathrm{PGL}(2,\mathbb{Q}(\sqrt{d}))$, the amalgamated product $\Gamma(I) *_{\Gamma(I)\cap\Gamma(I)^g} \Gamma(I)^g$ is isomorphic to a finite index subgroup of $PSL(2, \mathcal{O}_d[1/p])$ (Theorem 14). This will be used in Section 4 in order to show that H and $\hat{H}$ satisfy the conditions of the aforementioned pure group-theoretic result. In Section 3 we prove the group-theoretic background for our generalization of Atkin's conjecture (Theorem 16). Finally, in Section 4, we use the results of the previous sections and prove our generalization of Atkin's conjecture in Theorem 23.

1. An extended notion of level

Let R be a commutative ring with unit and $\mathfrak{a}$ a non-zero ideal of R. Consider the map $res_{\mathfrak{a}} : \mathrm{SL}(n, R) \to \mathrm{SL}(n, R/\mathfrak{a})$ obtained by restriction mod $\mathfrak{a}$. The kernel of $res_{\mathfrak{a}}$ is called the **principal** or **full congruence subgroup** of **level** $\mathfrak{a}$ and is denoted by $\mathrm{SL}(n, R, \mathfrak{a})$.

Let $\pi : \mathrm{SL}(n, R) \to \mathrm{PSL}(n, R)$ be the canonical surjection and define

$$\Gamma(n, R, \mathfrak{a}) := \mathrm{PSL}(n, R, \mathfrak{a}) := \pi(\mathrm{SL}(n, R, \mathfrak{a}))$$

and call it the **principal congruence subgroup** of PSL of **level** $\mathfrak{a}$. It is a normal subgroup of $\mathrm{PSL}(n, R)$ since π is surjective.

A **congruence subgroup** H of $\mathrm{SL}(n, R)$ or $\mathrm{PSL}(n, R)$ is a subgroup which contains a full congruence subgroup $\mathrm{SL}(n, R, \mathfrak{a})$ or $\mathrm{PSL}(n, R, \mathfrak{a})$ respectively for some *non-zero* ideal $\mathfrak{a}$ of R. If $\mathfrak{a}$ is maximal among the ideals having this property, we say that H is congruence of **level** $\mathfrak{a}$.

In particular, when $R = \mathcal{O}_d$, d any square-free negative integer, every congruence subgroup is of finite index. In this work we are particularly interested in the case $R = \mathcal{O}_d, n = 2$ and the groups PSL. Following an idea of Fricke the concept of level of a full congruence subgroup was extended to arbitrary subgroups of $\mathrm{SL}(2, \mathbb{Z})$ of finite index by Wohlfahrt [16], [17]. This has been generalized to $\mathrm{PSL}(n, \mathcal{O}_d)$ by Grunewald and Schwermer [8]. Let us start with some terminology and prove some elementary results about it. We use the following notation: $\boldsymbol{H^g}:= g^{-1}Hg$ and $\boldsymbol{{}^gH}:= gHg^{-1}$, for an element g and a subgroup H of an arbitrary group G as well as $\boldsymbol{H_G}$ which is the normal core of H in G and $\boldsymbol{H^G}$ which denotes the normal closure of H in G. R will be a commutative ring with unit element.

Definition 1. Suppose $\mathfrak{a}$ is a non-zero ideal of R. We define the subgroup $\boldsymbol{M(\mathfrak{a})}$ of *unipotent* elements of $\mathrm{PSL}(2, R)$ as follows:

$$M(\mathfrak{a}) := \left\{ \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix} \right) \in \mathrm{PSL}(2, R) \;\mid\; a \in \mathfrak{a} \right\}.$$

We denote the normal closure of $M(\mathfrak{a})$ in $\mathrm{PSL}(2, R)$ by $\boldsymbol{Q(\mathfrak{a})}$. Clearly $M(0) = Q(0) = 0$ and if R is a local or Euclidean ring, then $Q(R) = \mathrm{PSL}(2, R)$ (see [3], 5.9.2 and [9], 2.4). Moreover, by [6], Theorem 6.1, $Q(\mathcal{O}_d) = \mathrm{PSL}(2, \mathcal{O}_d)$ if and only if $d \in \{-1, -2, -3, -7, -11\}$. For simplicity put $\boldsymbol{\Gamma(\mathfrak{a})}:= \Gamma(2, R, \mathfrak{a})$ when R is clear from the context.

Consider an arbitrary subgroup H of $\mathrm{PSL}(2, R)$ of finite index. The set $X = \{I \trianglelefteq R \mid \; Q(I) \subseteq H\}$, partially ordered by inclusion, has the maximum $\Sigma X := \Sigma\{I \mid \; I \in X\}$, since $Q(\Sigma I_\alpha) = \bigvee Q(I_\alpha)$, for every family $\{I_\alpha \lhd R \mid \; \alpha \in A\}$, where $\bigvee Q(I_\alpha)$ denotes the subgroup generated by $Q(I_\alpha)$, $\alpha \in A$. Now we define

Definition 2 (see [8])**.** Let H be an arbitrary subgroup of $\mathrm{PSL}(2, R)$ of finite index. Then we say that H is a subgroup of **level** $\mathfrak{a}_H$ if $\mathfrak{a}_H = \Sigma\{I \trianglelefteq R \mid \; Q(I) \subseteq H\}$. Clearly $\mathfrak{a}_{\Gamma(I)} = I$ for every ideal I of R.

Proposition 3. *Let H, K, and H_α ($\alpha \in A$) be subgroups of* $\mathrm{PSL}(2, R)$ *of finite index.*

(1) *If $H \leq K$, then $\mathfrak{a}_H \subseteq \mathfrak{a}_K$.*
(2) *For any family H_α of subgroups of Γ_d, $\mathfrak{a}_{\bigcap H_\alpha} = \bigcap \mathfrak{a}_{H_\alpha}$.*
(3) *$\mathfrak{a}_{H^g} = \mathfrak{a}_{{}^gH} = \mathfrak{a}_H$ for every $g \in \Gamma_d$. In particular, $\mathfrak{a}_{H_G} = \mathfrak{a}_H$.*
(4) *For any subgroup $N \leq H$ we have $\mathfrak{a}_{N \cap \Gamma(\mathfrak{a}_H)} = \mathfrak{a}_N$.*

Proof. Straightforward. □

Corollary 4. *The intersection of any family of congruence subgroups H_α, of level $\mathfrak{a}_\alpha$, is congruence if and only if $\bigcap \mathfrak{a}_\alpha$ is non-zero. In this case, $\mathfrak{a}_{\bigcap H_\alpha} = \bigcap \mathfrak{a}_\alpha$.*

Proof. If $\bigcap \mathfrak{a}_\alpha$ is non-zero, then clearly $\bigcap H_\alpha$ is congruence. Conversely, suppose that $\bigcap H_\alpha$ is congruence. So $\mathfrak{a}_{\bigcap H_\alpha} \neq 0$ and by Proposition 3 part (2), $\mathfrak{a}_{\bigcap H_\alpha} = \bigcap \mathfrak{a}_\alpha$. Therefore $\bigcap \mathfrak{a}_\alpha$ is non-zero. □

There are examples of finite index subgroups of $\mathrm{PSL}(2, k[x])$, k a finite field, of level zero; see [12]. For Γ_d, however, the situation is different, as we see in the next proposition and its corollaries:

Proposition 5. *Let H be a subgroup of $G = \mathrm{PSL}(2, R)$. If H has finite index in G and $char(R) \nmid [G : H_G]$, then $\mathfrak{a}_H$ is non-zero.*

Proof. We know that the normal core of H in G, H_G, has finite index in G, say m. So for every $g \in G$, $g^m \in H_G$. This implies that $M(mR) \subseteq H_G$. Since H_G is normal in G, we have $Q(mR) \subseteq H_G$. Since $char(R) \nmid m$, $mR \neq 0$. Hence the level of H is not zero either. □

Corollary 6. *Let H be a finite index subgroup of Γ_d, for any square-free negative d. Then $\mathfrak{a}_H$ is non-zero.*

We continue with studying some basic properties of the congruence closure.

Definition 7. Let R be a commutative ring with unit and H be a subgroup of $\mathrm{PSL}(2, R)$. We define the **congruence hull** or **closure** of H in $\mathrm{PSL}(2, R)$ as the smallest congruence subgroup of $\mathrm{PSL}(2, R)$ containing H, when it exists, and denote it by $\hat{H}$. Note that there always exists a congruence subgroup containing H, e.g. $H\Gamma(I)$ for every non-zero ideal I of R.

Remark 8. Let H be a subgroup of $\mathrm{PSL}(2, R)$. If $\mathfrak{a}_H$ is non-zero, then $\hat{H}$ is defined and is equal to $\Gamma(\mathfrak{a}_H)H$. In particular, if $R = \mathcal{O}_d$ and H is of finite index in Γ_d, then $\hat{H}$ is defined.

Proposition 9. *Let H, K be subgroups of* $\mathrm{PSL}(2, R)$ *such that $\hat{H}$ and $\hat{K}$ are defined.*

(1) *For every congruence subgroup N of $\hat{H}$, $\hat{H} = NH$. In particular, $\hat{H} = \Gamma(\mathfrak{a}_H)H$ if $\mathfrak{a}_H$ is non-zero.*
(2) *If $H \subseteq K$, then $\hat{H} \subseteq \hat{K}$.*
(3) *If $\widehat{H \cap K}$ is defined, then $\widehat{H \cap K} \subseteq \hat{H} \cap \hat{K}$.*
(4) *If $\mathfrak{a}_H$ is non-zero, then $H_G \widehat{\cap \Gamma(}\mathfrak{a}_H) = \Gamma(\mathfrak{a}_H)$.*
(5) *If $\mathfrak{a}_H$ is non-zero, then for every $N \leq H$ with $\mathfrak{a}_N \neq 0$ and $N \widehat{\cap \Gamma(\mathfrak{a}_H)} = \hat{N}$ we have $N \subseteq \Gamma(\mathfrak{a}_H)$.*

Proof. (1) Follows from the fact that NH is a congruence subgroup containing H and included in $\hat{H}$. (2), (3), and (4) are consequences of Proposition 3. For (5): If $\widehat{N \cap \Gamma(\mathfrak{a}_H)} = \hat{N}$, then by part (1) and Proposition 3, part (4), $\Gamma(\mathfrak{a}_N)N = \Gamma(\mathfrak{a}_N)(N \cap \Gamma(\mathfrak{a}_H))$. On the other hand, $\Gamma(\mathfrak{a}_N) \cap N = \Gamma(\mathfrak{a}_N) \cap (N \cap \Gamma(\mathfrak{a}_H))$, whence the result. □

2. Congruence subgroup property of $\mathrm{SL}(2, \mathcal{O}[\frac{1}{p}])$

Let $p \in \mathcal{O} := \mathcal{O}_d$ be prime, d any square-free negative integer. In this section we show that the group $\mathrm{SL}(2, \mathcal{O}[\frac{1}{p}])$ (and hence $\mathrm{PSL}(2, \mathcal{O}[\frac{1}{p}])$) has the congruence subgroup property. As a result, we see that the group $\Gamma(I \cdot \mathcal{O}[\frac{1}{p}])$ satisfies the CSP, for every non-zero ideal I of $\mathbb{Q}(\sqrt{d})$ such that $p \notin I$. This result will be used in the proof of Proposition 22. We start by stating an important theorem of Serre, which, applied to quadratic fields, is the key for proving the CSP of $\mathrm{SL}(2, \mathcal{O}[\frac{1}{p}])$. For the proof and details of the theorem, see [13], Theorem 2. Let S_{all} be the set of all places of a number field K and $S \subseteq S_{all}$ be a finite subset of S_{all} containing S_∞, the set of all Archimedean places of K. The ring of **S-integers** of K is by definition:

$$\mathcal{O}_S := \{x \in K \mid v(x) \geq 0, \text{ for every place } v \in S_{all} - S\}.$$

$\mathcal{O}_S$ is a Dedekind domain whose maximal ideals are in 1-1 correspondence with the elements of $S_{all} - S$. We have:

Theorem 10 (Serre). *Let S be a finite subset of the places of the field $\mathbb{Q}(\sqrt{d})$ that contains at least 2 places, at least one of them real or non-Archimedean, and $\mathcal{O}_S$ the subring of S-integers. Then the group $\mathrm{SL}(2, \mathcal{O}_S)$ (and hence $\mathrm{PSL}(2, \mathcal{O}_S)$) satisfies the CSP.*

Let $K := \mathbb{Q}(\sqrt{d})$ and S_p be the set consisting of the Archimedean place of K together with the non-Archimedean place corresponding to the prime ideal $P := p\mathcal{O}$. It is easy to show that $\mathcal{O}_{S_p} = \mathcal{O}[\frac{1}{p}]$, and so by Theorem 10 we have:

Proposition 11. *Let $p \in \mathcal{O}$ be prime. Then the group $\mathrm{SL}(2, \mathcal{O}[\frac{1}{p}])$ (and hence $\mathrm{PSL}(2, \mathcal{O}[\frac{1}{p}])$) has the congruence subgroup property.*

In order to prove the main result of this section, we also need the following two lemmas.

Lemma 12. *Let R be a commutative ring with unit, $p \in R$ with Rp maximal, and $I, J \trianglelefteq R$ coprime to Rp. Then IJ is coprime to Rp^k, for every $k \in \mathbb{N}$.*

Proof. Clearly $IJ + Rp = R$. So there exist $r \in R,\ x \in I,\ and\ y \in J$ such that $1 = rp + xy$, and hence $p^{k-1} = xyp^{k-1} + rp^k \in IJ + Rp^k$, so $IJ + Rp^{k-1} = IJ + Rp^k$ for all k, whence the result follows by induction on k. □

Lemma 13. *Let $I \triangleleft \mathcal{O}$ and $p \in \mathcal{O}$ prime such that $p \notin I$. Set $\tilde{I} := I \cdot \mathcal{O}[\frac{1}{p}]$. Then $\tilde{I}$ is dense in $K = \mathbb{Q}(\sqrt{d})$ with respect to the topology induced by the absolute value $|\ |_{\mathcal{O}p}$ on K.*

Proof. Let $P := \mathcal{O}p$, so $I + P = \mathcal{O}$. Let $x \in K$ and $\epsilon > 0$. Write $x = \frac{a_1}{b_1}$ with $a_1, b_1 \in \mathcal{O}$, $b_1 \neq 0$. Let v_p denote the p-adic valuation on K. Suppose that $v_P(a_1) = \alpha,\ v_P(b_1) = \beta$, and write $a_1 = p^\alpha a,\ b_1 = p^\beta b$ with $a, b \in \mathcal{O}$ and $p \nmid a, b$, so $x = p^{\alpha-\beta}\frac{a}{b}$. Put $s := \alpha - \beta$. Choose $k \in \mathbb{N}$ such that $p^{-k} < \epsilon$. Let $s < 0$. By the previous lemma, $bI + P^{k-s} = \mathcal{O}$. So there exist $g \in I,\ y \in \mathcal{O}$ such that $a = bg + yp^{k-s}$; hence $p^{k-s} \mid a - bg$ and so $|a - bg|_P \leq p^{-k+s}$. Since $|b|_P = 1$, we have $|\frac{a}{b} - g|_P = |a - bg|_P \leq p^{-k+s}$, so $|x - p^s g|_P \leq p^{-s}p^{-k+s} = p^{-k} < \epsilon$ and $p^s g \in \tilde{I}$. The case $s \geq 0$ is handled in a similar way. ☐

Theorem 14. *Let $0 \neq I \lhd \mathcal{O} = \mathcal{O}_d$ and $p \in \mathcal{O}$ be prime such that $p \notin I$. Define $g := \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{PGL}(2, K)$. Then $\Gamma(I) *_{\Gamma(I)\cap\Gamma(I)^g} \Gamma(I)^g \cong \Gamma(I \cdot \mathcal{O}[\frac{1}{p}])$, and the latter satisfies the congruence subgroup property.*

Proof. Let $\tilde{I} := I \cdot \mathcal{O}[\frac{1}{p}]$. Consider the p-adic valuation $v : K \to \mathbb{Z} \cup \{\infty\}$ and let R be its valuation ring. We know from the previous lemma that $\tilde{I}$ is dense in K with respect to the p-adic norm. So clearly the closure of $\Gamma(\tilde{I})$ in $\mathrm{PGL}(2, K)$ contains $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$, and so $\mathrm{PSL}(2, K)$. Recall that (e.g. Theorem 3, chapter 2 of [14]) the group $\mathrm{SL}(2, K)$ ($\mathrm{PSL}(2, K)$ resp.) acts on the set of lattices (on the tree of lattice classes resp.) of $K \oplus K$ in the following way: Let $L = l_1 R + l_2 R$ be a lattice and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, K)$. Then

$$g \cdot L := (al_1 + bl_2)R + (cl_1 + dl_2)R.$$

Now apply Theorem 2 of chapter 2 of [14] with $G = \Gamma(\tilde{I})$, $L' := R \cdot p^2 \oplus R \cdot p \subseteq L := R \cdot p \oplus R \cdot p$, to see that $\Gamma(\tilde{I}) = G_L *_{G_L \cap G_{L'}} G_{L'}$, where $G_L, G_{L'}$ are the stabilizers of L, L' under the G-action respectively. We show that $G_L = \Gamma(I)$ and $G_{L'} = \Gamma(I)^g$:

Lemma 15. *For every $x \in \tilde{I}$, if $x \notin R$ then $x - 1 \notin I$.*

To show that $G_L = \Gamma(I)$, consider $u = \left(\begin{smallmatrix} a+1 & b \\ c & d+1 \end{smallmatrix}\right) \in \Gamma(I)$, with $a, b, c, d \in I$. So

$$\begin{pmatrix} a+1 & b \\ c & d+1 \end{pmatrix}\begin{pmatrix} p \\ 0 \end{pmatrix} = \begin{pmatrix} p(a+1) \\ cp \end{pmatrix},$$
$$\begin{pmatrix} a+1 & b \\ c & d+1 \end{pmatrix}\begin{pmatrix} 0 \\ p \end{pmatrix} = \begin{pmatrix} bp \\ p(d+1) \end{pmatrix}.$$

By Lemma 15, $a + 1, d + 1 \in R$, so $uL \subseteq L$ for every $u \in \Gamma(I)$; that is, $uL = L$ for every $u \in \Gamma(I)$.

To show that $G_{L'} = \Gamma(I)^g$, consider $u \in \Gamma(I)^g$ and write $u = \left(\begin{smallmatrix} a+1 & b/p \\ cp & d+1 \end{smallmatrix}\right)$ with $a, b, c, d \in I$ and argue as above.

Finally, we note that $\Gamma(\tilde{I})$ is of finite index in $\mathrm{PSL}(2, \mathcal{O}[\frac{1}{p}])$, and the congruence subgroup property follows from Proposition 11. ☐

3. The action of Hecke operators on the cohomology groups

Let G be a group. For every $x \in G$ and every $H \leq G$, let H_x denote the intersection $H \cap H^x$ (where as before $H^x = x^{-1}Hx$). It is well known that for every left G-module A and every finite index subgroup H of G, there is a natural action of the Hecke operators (double cosets of H) on the cohomology groups $H^*(H, A)$ defined in the following way: write H as a union of $\mu = \mu(x) := [H : H_x]$ disjoint cosets of H_x in H: $H = \bigsqcup_1^\mu H_x h_i$. It is easy then to check that $HxH = \bigsqcup_1^\mu Hx_i$, where $x_i := xh_i$. Since for every $y \in H$, $HxHy = HxH$, we have $HxH = \bigsqcup_1^\mu Hx_i = \bigsqcup_1^\mu Hx_i y$, so for every $1 \leq i \leq \mu$,

$$x_i y = t_i(y) x_{i(y)} \tag{3.1}$$

for a unique element $t_i(y) \in H$ and a unique index $i(y)$. So $(x_{1(y)} \cdots x_{\mu(y)})$ is a permutation of $(x_1 \cdots x_\mu)$. For each $y, y' \in H$, $(x_i y)y' = t_i(y)(x_{i(y)}y') = t_i(y)t_{i(y)}(y')x_{(i(y))(y')}$. On the other hand, $x_i(yy') = t_i(yy')x_{i(yy')}$, so

$$i(yy') = (i(y))(y'),\ t_i(yy') = t_i(y)t_{i(y)}(y'). \tag{3.2}$$

Given a non-negative integer q we denote the group of all (non-homogeneous) q-co-chains from H to A by $C^q(H, A)$ and we define the action of HxH on a co-chain $f \in C^q(H, A)$ as follows:

$$(f \cdot HxH)(y_1, \dots, y_q) := \sum_1^{\mu} x_i^{-1} f[t_i(y_1), t_{i(y_1)}(y_2), t_{i(y_1 y_2)}(y_3) \dots, t_{i(y_1 \cdots y_{q-1})}(y_q)], \tag{3.3}$$

for all $y_1, \cdots, y_q \in H$. For more details, see [11] and [1].

We also need an explicit description of the transfer (or co-restrtiction) map between cohomology groups. Given a finite index subgroup $H \leq G$ and a G-module A, let $[G : H] = n$ and $\{s_1, \dots, s_n\}$ be a transversal for the left cosets of H in G. For every $x \in G$, let $\bar{x}$ be the unique element s_i with $x \in Hs_i$. So we have $x\bar{x}^{-1} \in H$. Now for every $k > 0$, $f : H^k \to A \in C^k(H, A)$, and $g_1, \dots g_k \in G$, we have

$$tr_H^G(f)[g_1, \dots, g_k] = \sum_1^n s_i^{-1} f[s_i g_1 \overline{(s_i g_1)}^{-1}, \dots, \overline{(s_i g_1 \dots g_{k-1})} g_k \overline{(s_i g_1 \dots g_k)}^{-1}],$$

and it can be shown that the corresponding induced map on the cohomology groups is independent of the transversal's choice. We now come to our main task of this section. The next theorem provides the group-theoretic background for our generalization of Atkin's conjecture. If we apply it to a finite index subgroup of $\mathrm{PSL}(2, \mathcal{O}_d)$ and its congruence closure, then we get a generalization of Atkin's conjecture for Hecke operators acting on the cohomology groups (see the next section).

Theorem 16. *Let $H \leq_f K \leq G$ and $g \in G$ be such that $K = (K_g)H$. Consider the following conditions:*

(1) $[K_g : H_g] = [K : H]^2$.
(2) $K = H(K \cap {}^gH)$, *where* ${}^gH := gHg^{-1}$.
(3) $[H \cap K^g : H_g] = [K : H]$.
(4) *For every G-module A and every $q \geq 1$ the following diagram commutes:*

$$\begin{tikzcd} H^q(H, A) \arrow[r, "tr_H^K"] \arrow[d, "T_g^H"'] & H^q(K, A) \arrow[d, "T_g^K"] \\ H^q(H, A) & H^q(K, A) \arrow[l, "res_H^K"] \end{tikzcd}$$

Then $1 \Leftrightarrow 2 \Leftrightarrow 3$ *and* $3 \Rightarrow 4$.

Proof. The equivalence of $1, 2$ and 3 is easy to prove. We start by proving $3 \Rightarrow 4$. Let $\mu := \mu_H(g) = [H : H_g]$ and write $H = \bigsqcup_1^\mu H_g g_i$. Since $K = (K_g)H$, $K_g g_i \subseteq K$ for every i and $K = \bigcup_1^\mu K_g g_i$. Without loss of generality, assume that $K = \bigsqcup_1^b K_g g_i$, where $b = [K : K_g]$, so

$$KgK = \bigsqcup_1^b Kgg_i. \tag{3.4}$$

For every $y \in K$, define $t_i(y)$ as the unique element of K such that $gg_iy = t_i(y)gg_{i(y)}$ for a unique index $i(y)$ (see equation (3.1)). Write

$$H \cap K^g = \bigsqcup_1^m H_g h_j, \tag{3.5}$$

where $m = [H \cap K^g : H_g]$. By assumption $m = [K : H]$. Since for every j, $gh_jg^{-1} \in K$ and $\{Hgh_jg^{-1} \mid 1 \le j \le m\}$ consists of exactly m disjoint cosets, we see that

$$K = \bigsqcup_1^m Hgh_jg^{-1}. \tag{3.6}$$

For every $y \in K$, define $\bar{y}$ as the unique gh_jg^{-1} such that $y \in Hgh_jg^{-1}$. Since $b = [H : H \cap K^g]$ and $\{(H \cap K^g)g_i \mid 1 \le i \le b\}$ consists of exactly b disjoint cosets, we have $H = \bigsqcup_1^b (H \cap K^g)g_i$, so by equation (3.5) we have $H = \bigsqcup_1^b \bigsqcup_1^m H_g h_j g_i$ and hence

$$HgH = \bigsqcup_1^b \bigsqcup_1^m Hgh_jg_i = \bigsqcup_1^b \bigsqcup_1^m Hz_{(j,i)}, \tag{3.7}$$

where $z_{(j,i)} := gh_jg_i$. For every $x \in H$, define $t_{(j,i)}(x)$ as the unique element of H such that $z_{(j,i)}x = t_{(j,i)}(x)z_{(j,i)(x)}$, for a unique pair of indices $(j,i)(x)$ (see equation (3.1)).

We start with $q = 1$. Consider $[f] \in H^1(H, A)$, where $f \in C^1(H, A)$ is a derivation. We show that $(T_g^H f)(x) = T_g^K(tr_H^K(f))(x)$ for every $x \in H$. For $x \in H$, we compute

$$(T_g^H f)(x) = (f \cdot HgH)(x) = \sum_{i=1}^b \sum_{j=1}^m (gh_jg_i)^{-1} f(t_{(j,i)}(x))$$

and

$$\begin{aligned} T_g^K(tr_H^K(f))(x) = (tr_H^K(f) \cdot KgK)(x) &= \sum_{i=1}^b g_i^{-1}g^{-1}tr_H^K(f)(t_i(x)) \\ = \sum_{i=1}^b g_i^{-1}g^{-1} \sum_{j=1}^m (gh_jg^{-1})^{-1} & f(gh_jg^{-1}t_i(x)(\overline{gh_jg^{-1}t_i(x)})^{-1}) \\ &= \sum_{i=1}^b \sum_{j=1}^m g_i^{-1}h_j^{-1}g^{-1}f(w_{(j,i)}(x)), \end{aligned}$$

where $w_{(j,i)}(x) := gh_jg^{-1}t_i(x)(\overline{gh_jg^{-1}t_i(x)})^{-1}$. We show that $w_{(j,i)}(x) = t_{(j,i)}(x)$. Clearly $w_{(j,i)}(x) \in H$. Note that $t_i(x)$ satisfies $gg_ix = t_i(x)gg_{i(x)}$, and $\overline{gh_jg^{-1}t_i(x)} = gh_kg^{-1}$ for some k; hence

$$\begin{aligned} gh_jg_ix &= gh_jg^{-1}gg_ix = gh_jg^{-1}t_i(x)gg_{i(x)} \\ &= gh_jg^{-1}t_i(x)(gh_kg^{-1})^{-1}(gh_kg^{-1})gg_{i(x)} = w_{(j,i)}(x)gh_kg_{i(x)}, \end{aligned}$$

so by definition of $t_{(j,i)}$,

$$w_{(j,i)}(x) = t_{(j,i)}(x) \text{ for every } x \in H, \tag{3.8}$$

and this finishes the case $q = 1$.

Let $q \geq 2$ and consider $[f] \in H^q(H, A)$, where $f \in C^q(H, A)$. We show that

$$(T_g^H f)(x_1, \cdots, x_q) = T_g^K(tr_H^K(f))(x_1, \cdots, x_q)$$

for every $x_1, \cdots, x_q \in H$. We have

$$\begin{aligned} &(T_g^H f)(x_1, \cdots, x_q) \\ &= \sum_{i=1}^{b}\sum_{j=1}^{m} (gh_j g_i)^{-1} f(t_{(j,i)}(x_1), \cdots, t_{(j,i)(x_1\cdots x_{q-1})}(x_q)) \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &T_g^K(tr_H^K(f))(x_1, \cdots, x_q) = (tr_H^K(f) \cdot KgK)(x_1, \cdots, x_q) \\ &= \sum_{i=1}^{b} g_i^{-1} g^{-1} \sum_{j=1}^{m} (gh_j g^{-1})^{-1} f(gh_j g^{-1} t_i(x_1)(\overline{gh_j g^{-1} t_i(x_1)})^{-1}, \cdots, \\ &\overline{gh_j g^{-1} t_i(x_1) \cdots t_{i(x_1\cdots x_{q-2})}(x_{q-1})}\, t_{i(x_1\cdots x_{q-1})}(x_q) \overline{gh_j g^{-1} t_i(x_1) \cdots t_{i(x_1\cdots x_{q-1})}(x_q)}^{\ -1}). \end{aligned} \tag{3.10}$$

Comparing the corresponding entries of f in equations (3.9) and (3.10) and recalling the equation (3.8), we see that it is enough to show that for every $2 \leq r \leq q$,

$$\begin{aligned} &t_{(j,i)(x_1\cdots x_{r-1})}(x_r) \\ &= \overline{gh_j g^{-1} t_i(x_1) \cdots t_{i(x_1\cdots x_{r-2})}(x_{r-1})}\, t_{i(x_1\cdots x_{r-1})}(x_r) \overline{gh_j g^{-1} t_i(x_1) \cdots t_{i(x_1\cdots x_{r-1})}(x_r)}^{\ -1}, \end{aligned} \tag{3.11}$$

and this equality can be verified by induction on r using properties (3.1) and (3.2) as follows. By equation (3.8),

$$w_{(j,i)}(x_1 \cdots x_r) = t_{(j,i)}(x_1 \cdots x_r).$$

Using equation (3.2), we get

$$t_{(j,i)}(x_1 \cdots x_r) = t_{(j,i)}(x_1) t_{(j,i)(x_1)}(x_2) t_{(j,i)(x_1 x_2)}(x_3) \cdots t_{(j,i)(x_1\cdots x_{r-1})}(x_r),$$

as well as

$$\begin{aligned} w_{(j,i)}(x_1 \cdots x_r) &= gh_j g^{-1} t_i(x_1 \cdots x_r) \overline{(gh_j g^{-1} t_i(x_1 \cdots x_r))}^{-1} \\ &= gh_j g^{-1} t_i(x_1) \cdots t_{i(x_1\cdots x_{r-1})}(x_r) \overline{(gh_j g^{-1} t_i(x_1) \cdots t_{i(x_1\cdots x_{r-1})}(x_r))}^{-1}; \end{aligned}$$

hence

$$\begin{aligned} &t_{(j,i)}(x_1) t_{(j,i)(x_1)}(x_2) t_{(j,i)(x_1 x_2)}(x_3) \cdots t_{(j,i)(x_1\cdots x_{r-1})}(x_r) \\ &= gh_j g^{-1} t_i(x_1) \cdots t_{i(x_1\cdots x_{r-1})}(x_r) \overline{(gh_j g^{-1} t_i(x_1) \cdots t_{i(x_1\cdots x_{r-1})}(x_r))}^{-1}. \end{aligned} \tag{3.12}$$

Now we prove equation (3.11) by induction on $r \geq 2$. For $r = 2$, equation (3.12) reduces to

$$t_{(j,i)}(x_1) t_{(j,i)(x_1)}(x_2) = gh_j g^{-1} t_i(x_1) t_{i(x_1)}(x_2) \overline{(gh_j g^{-1} t_i(x_1) t_{i(x_1)}(x_2))}^{-1}.$$

Since $t_{(j,i)}(x_1)^{-1} gh_j g^{-1} t_i(x_1) = \overline{gh_j g^{-1} t_i(x_1)}$ (by equation (3.8)), we have

$$t_{(j,i)(x_1)}(x_2) = \overline{gh_j g^{-1} t_i(x_1)}\, t_{i(x_1)}(x_2) \overline{(gh_j g^{-1} t_i(x_1) t_{i(x_1)}(x_2))}^{-1}.$$

Now assuming equation (3.11) for any $s \le r-1$, we have

$$\begin{aligned}
&\overline{gh_jg^{-1}t_i(x_1)\cdots t_{i(x_1\cdots x_{r-2})}(x_{r-1})}\\
&= t_{(j,i)(x_1\cdots x_{r-2})}(x_{r-1})^{-1}\overline{gh_jg^{-1}t_i(x_1)\cdots t_{i(x_1\cdots x_{r-3})}(x_{r-2})}t_{i(x_1\cdots x_{r-2})}(x_{r-1})\\
&= \cdots\\
&= t_{(j,i)(x_1\cdots x_{r-2})}(x_{r-1})^{-1}t_{(j,i)(x_1\cdots x_{r-3})}(x_{r-2})^{-1}\cdots\\
&\quad t_{(j,i)}(x_1)^{-1}gh_jg^{-1}t_i(x_1)t_{i(x_1)}(x_2)\cdots t_{i(x_1\cdots x_{r-2})}(x_{r-1}).
\end{aligned}$$

Replacing this in the right-hand side of equation (3.11) and using equation (3.12) we get

$$\begin{aligned}
&\overline{gh_jg^{-1}t_i(x_1)t_{i(x_1)}(x_2)\cdots t_{i(x_1\cdots x_{r-2})}(x_{r-1})}\cdot\\
&\quad t_{i(x_1\cdots x_{r-1})}(x_r)\overline{gh_jg^{-1}t_i(x_1)\cdots t_{i(x_1\cdots x_{r-1})}(x_r)}^{-1}\\
&= t_{(j,i)(x_1\cdots x_{r-2})}(x_{r-1})^{-1}t_{(j,i)(x_1\cdots x_{r-3})}(x_{r-2})^{-1}\cdots\\
&\quad t_{(j,i)}(x_1)^{-1}[gh_jg^{-1}t_i(x_1)t_{i(x_1)}(x_2)\cdots t_{i(x_1\cdots x_{r-2})}(x_{r-1})\cdot\\
&\quad t_{i(x_1\cdots x_{r-1})}(x_r)\overline{gh_jg^{-1}t_i(x_1)\cdots t_{i(x_1\cdots x_{r-1})}(x_r)}^{-1}]\\
&= t_{(j,i)(x_1\cdots x_{r-2})}(x_{r-1})^{-1}\cdots t_{(j,i)}(x_1)^{-1}[t_{(j,i)}(x_1)\cdots t_{(j,i)(x_1\cdots x_{r-1})}(x_r)]\\
&= t_{(j,i)(x_1\cdots x_{r-1})}(x_r),
\end{aligned}$$

proving equation (3.11), which finishes the proof. ☐

4. Hecke operators on non-congruence subgroups

In this section we use Theorem 16 to prove a generalization of Atkin's conjecture for Hecke operators on the cohomology groups of $\Gamma_d = \mathrm{PSL}(2,\mathcal{O}_d)$, with coefficients in any Γ_d-module. We prove, in a sequence of lemmas, that if $H \le_f \Gamma_d$ is of level $\mathfrak{a} := \mathfrak{a}_H$ and $p \in \mathcal{O}_d$ is prime such that $\mathfrak{a} + p\mathcal{O}_d = \mathcal{O}_d$, then for $g := \left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right) \in \mathrm{PGL}(2,\mathbb{Q}(\sqrt{d}))$, the groups $H \le \hat{H} \le \Gamma_d$ satisfy the conditions of Theorem 16. As before, d will be a square-free negative integer. We start with some technical elementary lemmas:

Lemma 17. *Let $I \trianglelefteq \mathcal{O}_d$ and $p \in \mathcal{O}_d$ such that $I + p\mathcal{O}_d = \mathcal{O}_d$. Define $g := \left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right) \in \mathrm{PGL}(2,\mathbb{Q}(\sqrt{d}))$. Then*

$$\Gamma(I) \cap \Gamma(p\mathcal{O}_d) \subseteq \Gamma(I) \cap \Gamma(I)^g \cap {}^g\Gamma(I).$$

Proof. Suppose that $1 = v + o\cdot p$ with $v \in I, o \in \mathcal{O}_d$.

Let $u \in \Gamma(I) \cap \Gamma(p\mathcal{O}_d)$, so we may write $u = \left(\begin{smallmatrix} a+1 & b\\ c & d+1\end{smallmatrix}\right) = \left(\begin{smallmatrix} a'p+1 & b'p\\ c'p & d'p+1\end{smallmatrix}\right)$, with $a,b,c,d \in I$ and $a',b',c',d' \in \mathcal{O}_d$. So $u^g = \left(\begin{smallmatrix} a+1 & b/p\\ cp & d+1\end{smallmatrix}\right) = \left(\begin{smallmatrix} a'p+1 & b'\\ c'p^2 & d'p+1\end{smallmatrix}\right)$. Therefore we have $b/p = b' \in \mathcal{O}_d$, so that $b' = vb' + ob \in I$. This shows that $u^g \in \Gamma(I)$, i.e. $u \in {}^g\Gamma(I)$. The inclusion $\Gamma(I) \cap \Gamma(p\mathcal{O}_d) \subseteq \Gamma(I)^g$ is proved in a similar way. ☐

Lemma 18. *For any element g of a group G and any subgroups $H \le_f K \le G$, we have*

1. $[H \cap K^g : H \cap H^g] \le [K : H]$; *equality holds if and only if $K = H(K \cap {}^gH)$.*
2. *If $K = NH$ for some $N \le K$, then in* (1) *equality holds if and only if $N \subseteq H(K \cap {}^gH)$.*

(3) *Let $g \in G$ and $H, N \leq K \leq S \leq G$ with $K = NH$. Set $H_1 := N \cap H_S$, where H_S denotes the normal core of H in S. If $[H_1 \cap N^g : H_1 \cap H_1^g] = [N : H_1]$, then $[H \cap K^g : H \cap H^g] = [K : H]$.*

Proof. Easy verifications. □

Let $g \in G$ and $H \leq K \leq G$. Define $\pi, \pi_g : K \cap K^g \to H\backslash\backslash K := \{Hx \mid x \in K\}$ by $\pi(x) := Hx$ and $\pi_g(x) := H({}^g x)$, where ${}^g x := gxg^{-1}$. It is clear that $\ker(\pi) = H \cap K^g$ and $\ker(\pi_g) = H^g \cap K$ and that we have the map

$$(\pi, \pi_g) : K \cap K^g \to H\backslash\backslash K \times H\backslash\backslash K,$$

called (π, π_g), for $H \leq K$. If $H \trianglelefteq K$, then (π, π_g) is a homomorphism with $\ker(\pi, \pi_g) = \ker(\pi) \cap \ker(\pi_g) = H \cap H^g$, which is onto if and only if $[K \cap K^g : H \cap H^g] = [K : H]^2$.

Lemma 19. *Let $H \leq_f K \leq G$. For any $g \in G$, if the map (π, π_g) is onto, then $[H \cap K^g : H \cap H^g] = [K : H]$.*

Proof. Let (π, π_g) be onto, and consider an element $x \in K$. So there exists $y \in K \cap K^g$ with $(\pi, \pi_g)(y) = (H, Hx)$, that is, $y \in H$ and $x = h({}^g y)$ for some $h \in H$. So $x \in H(K \cap {}^g H)$. Hence by Lemma 18, part 1, $[H \cap K^g : H \cap H^g] = [K : H]$. □

Lemma 20. *Let $H, N \leq K \leq S \leq G$ with $K = NH$, $[K : H] < \infty$ and $H_1 := N \cap H_S$. For any $g \in G$, if (π, π_g) for (H_1, N) is onto, then $[H \cap K^g : H \cap H^g] = [K : H]$.*

Proof. This is immediate from the previous lemma and part 3 of Lemma 18. □

Finally, we need the following lemma from Serre (see appendix of [15]):

Proposition 21. *Suppose X, X_1, X_2 are arbitrary groups with epimorphisms $X_1 \overset{f_1}{\twoheadleftarrow} X \overset{f_2}{\twoheadrightarrow} X_2$. If $X \xrightarrow{(f_1, f_2)} X_1 \times X_2$ is not surjective, then there exist a group $Y \neq 1$ and epimorphisms $X_1 \overset{h_1}{\twoheadrightarrow} Y \overset{h_2}{\twoheadleftarrow} X_2$ such that $h_1 f_1 = h_2 f_2$.*

Now we can show that for every finite index subgroup H of Γ_d, its congruence closure, and special elements $g \in PGL(2, \mathbb{C})$, the conditions of 16 are satisfied:

Proposition 22. *Let $p \in \mathcal{O}_d$ be prime and define*

$$g := \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \in G := \mathrm{PGL}(2, \mathbb{Q}(\sqrt{d})).$$

Consider $H \leq_f \Gamma_d$ of level $\mathfrak{a} := \mathfrak{a}_H$ and assume $\mathfrak{a} + p\mathcal{O}_d = \mathcal{O}_d$. Let $\hat{H}$ be the congruence closure of H in Γ_d. Then

$$[H \cap \hat{H}^g : H \cap H^g] = [\hat{H} : H].$$

Proof. Set $H_1 := H_{\Gamma_d} \cap \Gamma(\mathfrak{a})$, where H_{Γ_d} is the normal core of H in Γ_d. Then $\hat{H} = \Gamma(\mathfrak{a})H$ (by Proposition 9, part 1) and as $\mathfrak{a}_{H_1} = \mathfrak{a}$, $\hat{H}_1 = \Gamma(\mathfrak{a})$ (Proposition 9, part 2). If the map (π, π_g) for $H_1 \leq N := \Gamma(\mathfrak{a})$ is onto, then by Lemma 20 we are done. First we show that both π and π_g are onto:

Consider the canonical surjections $\pi_1 : N \twoheadrightarrow N/H_1$ and $\pi_2 : N \twoheadrightarrow N/(N \cap \Gamma(p\mathcal{O}_d))$ and let $\psi := (\pi_1, \pi_2) : N \to N/H_1 \times N/(N \cap \Gamma(p\mathcal{O}_d))$. If ψ is not onto, then by Proposition 21 there exist a group $T \neq 1$ and $N/H_1 \overset{k_1}{\twoheadrightarrow} T \overset{k_2}{\twoheadleftarrow} N/(N \cap \Gamma(p\mathcal{O}_d))$ such that $k_1\pi_1 = k_2\pi_2$. Suppose $T = N/W$, where $W := \ker(k_1\pi_1)$. Since $T \neq 1$, $H_1 \subseteq W \subsetneq N$, and N is the congruence closure of H_1, W is not congruence. But $N \cap \Gamma(p\mathcal{O}_d) \subseteq W$, a contradiction. Hence ψ is onto. Now for $s \in N/H_1$, there exists

$t \in N$ such that $\psi(t) = (s, 1) = (H_1t,\ (N \cap \Gamma(p\mathcal{O}_d))t\ )$; that is, $t \in N \cap \Gamma(p\mathcal{O}_d)$ and $s = H_1t$. Now by Lemma 17 we have $N \cap \Gamma(p\mathcal{O}_d) \subseteq N \cap N^g$, which implies that $\pi(t) = H_1t = s$; i.e. π is onto. On the other hand, $N \cap \Gamma(p\mathcal{O}_d) \subseteq N \cap {}^gN$ (again by Lemma 17), implying $t^g \in N \cap N^g$; hence $\pi_g(t^g) = H_1({}^g(t^g)) = H_1t = s$, so π_g is also onto.

Now contrarily assume that (π, π_g) for $H_1 \leq N$ is not onto. Then by Proposition 21 there exist a group $Y \neq 1$ and epimorphisms $N/H_1 \overset{h_1}{\twoheadrightarrow} Y \overset{h_2}{\twoheadleftarrow} N/H_1$ such that $h_1\pi = h_2\pi_g$. Now define $F_1 : N \to Y$ and $F_2 : N^g \to Y$ by $F_1(x) := h_1(H_1x)$ and $F_2(x) := h_2(H_1 \cdot {}^gx)$. Clearly $F_1 |_{N\cap N^g} = F_2 |_{N\cap N^g}$. So we have a map

$$F : N *_{N\cap N^g} N^g \to Y$$

such that $F |_N = F_1$ and $F |_{N^g} = F_2$. Hence $[N *_{N\cap N^g} N^g : \ker(F)] \leq |\, Y \,| \leq [N : H_1] < \infty$ and $[N : N \cap \ker(F)] = |\, Y \,| \gneq 1$. So $H_1 \subseteq N \cap \ker(F) \subsetneq N$, but N is the congruence closure of H_1; therefore $N \cap \ker(F)$ cannot be congruence. Hence by Corollary 4, $\ker(F)$ is a non-congruence subgroup of $N *_{N\cap N^g} N^g$ of finite index, contradicting Theorem 14. □

Now we sum up what we have proved so far in this section in the following:

Theorem 23. *Let $H \leq_f \Gamma_d = \mathrm{PSL}(2, \mathcal{O}_d)$ be of level $\mathfrak{a} := \mathfrak{a}_H$, and let $\hat{H}$ be its congruence closure. Suppose that $p \in \mathcal{O}_d$ is prime and $\mathfrak{a} + p\mathcal{O}_d = \mathcal{O}_d$ and define $g := \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{PGL}(2, \mathbb{Q}(\sqrt{d}))$. Then for every Γ_d-module M and every $q \geq 1$ the following diagram commutes:*

$$\begin{array}{ccc} H^q(H, M) & \xrightarrow{tr_H^{\hat{H}}} & H^q(\hat{H}, M) \\ {\scriptstyle T_g^H}\downarrow & & \downarrow{\scriptstyle T_g^{\hat{H}}} \\ H^q(H, M) & \xleftarrow[res_H^{\hat{H}}]{} & H^q(\hat{H}, M) \end{array}$$

Proof. Since $\hat{H}_g$ is congruence, by Proposition 9 we have $\hat{H} = (\hat{H}_g)H$. Now using Proposition 22 and Theorem 16 we are done. □

Corollary 24. *Under the notation of the above theorem, every eigenvector $u \in H^q(H, M)$ for T_g^H with non-zero eigenvalue is of the form $res(v)$, for some $v \in H^q(\ \hat{H}, M)$.*

Remark 25. Note that since the virtual cohomological dimension of a Bianchi group is two, the most interesting cases of these results are for $q = 1, 2$.

Remark 26. One application of Theorem 23 is in the theory of Bianchi modular forms for imaginary quadratic fields of class number one. The Eichler-Shimura-Harder correspondence (see [10]) allows us to see these forms as classes in the cohomology of finite index subgroups of Bianchi groups. Theorem 23 can be used to deduce that the Hecke action on Bianchi modular forms for a non-congruence subgroup of a Bianchi group is essentially the same as the Hecke action on Bianchi modular forms for its congruence closure.

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Department of Mathematics, Heinrich-Heine University Düsseldorf, Düsseldorf, Germany

E-mail address: zarghani@math.uni-duesseldorf.de, zarghani.s@gmail.com