HECKE OPERATORS FOR NON-CONGRUENCE SUBGROUPS OF BIANCHI GROUPS

SAEID HAMZEH ZARGHANI

(Communicated by Kathrin Bringmann)

Before the first draft of this work was completed, Fritz Grunewald tragically passed away. Indeed, without his guidance and support this work would never have been done. I dedicate this paper to his memory with admiration, gratitude and love.

Abstract. We prove that the action of the Hecke operators on the cohomology of a finite index non-congruence subgroup $\Gamma$ of a Bianchi group is essentially the same as the action of Hecke operators on the cohomology groups of $\hat{\Gamma}$, the congruence closure of $\Gamma$. This is a generalization of Atkin’s conjecture, first confirmed in a special case by Serre in 1987 and proved in general by Berger in 1994.

Introduction

For every finite index subgroup $H$ of $\text{PSL}(2, \mathbb{Z})$ ($H \leq \text{PSL}(2, \mathbb{Z})$ for short) and every $k, p \in \mathbb{N}$, $p$ prime, let $M_k(H)$ denote the space of the $H$-modular forms of weight $k$ and recall that the Hecke operator $T_p^H : M_k(H) \to M_k(H)$ is defined by

$$T_p^H(f) := \sum_i \langle f | k \tilde{p} \rangle | k g_i,$$

where $\tilde{p} := \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, $H = \bigcup_i H_p g_i$, and $H_p := H \cap \tilde{p}^{-1} H \tilde{p}$. A conjecture of Atkin (now a theorem, first confirmed in a special case by Serre in 1987 (see Appendix of [15]) and finally proved in general by Berger in 1994 (see [4])) states that the action of the Hecke operators $T_p^H$ on the space of the modular forms of any given weight $k$ associated to a non-congruence subgroup $H \leq \text{PSL}(2, \mathbb{Z})$ is essentially the same as the action of the Hecke operators $T_p^\hat{H}$ on $M_k(\hat{H})$, where $\hat{H}$ is the congruence closure of $H$. More precisely,

Theorem (Serre, Berger). For every prime $p$ (not dividing the level of $H$), we have $T_p^H = T_p^\hat{H} \circ \text{Tr}^\hat{H}$, where $\text{Tr}$ is the trace map. That is, the following diagram

Received by the editors May 13, 2010 and, in revised form, September 21, 2010.
2010 Mathematics Subject Classification. Primary 11F03, 11F25, 11F75, 20G30; Secondary 19B37.
The author was supported in part by Graduirtenkolleg Homotopie und Kohomologie (GRK1150).
In this paper, motivated by the recent interest in the classical modular forms for non-congruence subgroups of \( \text{SL}(2, \mathbb{Z}) \) (\cite{2}) and in the arithmetic of Bianchi modular forms (\cite{5} and \cite{7}), we will prove the following generalization of Atkin’s conjecture to the cohomology of subgroups of Bianchi groups. The main idea, which is a generalization of Berger’s idea in \cite{4}, is as follows: starting with pure group theory, let \( G \) be an arbitrary group, \( H \leq f K \leq G \) and \( g \in G \) be such that \( K = (Kg)H \) and \( [Kg : Hg] = [K : H]^2 \), where \( K_g := K \cap g^{-1}Kg \), etc. We show that:

**Theorem** (Theorem \cite{15}). Under the above assumptions, for every \( G \)-module \( M \) and every \( q \geq 1 \) the following diagram commutes:

\[
\begin{array}{ccc}
H^q(H,M) & \xrightarrow{tr^K_H} & H^q(K,M) \\
\downarrow^{T^K_p} & & \downarrow^{T^K_p} \\
H^q(H,M) & \xleftarrow{\text{res}^{K}_H} & H^q(K,M)
\end{array}
\]

where \( tr^K_H \) (resp.) denotes the trace map.

Now let \( d \) be any square-free negative integer and \( H \leq_f \Gamma_d = \text{PSL}(2, \mathbb{O}_d) \) be of level \( a \) (see below), and let \( \hat{H} \) be its congruence closure (see section 1). Suppose that \( p \in \mathbb{O}_d \) is prime and \( n + p\mathbb{O}_d = \mathbb{O}_d \). Define \( \hat{p} := \left( \begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix} \right) \in \text{PGL}(2, \mathbb{C}) \). We show that \( H \) and \( \hat{H} \) satisfy the above conditions, and hence

**Theorem** (Theorem \cite{23}). Under the above assumptions, we have for every \( \Gamma_d \)-module \( M \) and every \( q \geq 1 \) that the following diagram commutes:

\[
\begin{array}{ccc}
H^q(H,M) & \xrightarrow{\text{tr}^{\hat{K}}_{\hat{H}}} & H^q(\hat{H},M) \\
\downarrow^{T^\hat{p}} & & \downarrow^{T^\hat{p}} \\
H^q(H,M) & \xleftarrow{\text{res}^{\hat{K}}_{\hat{H}}} & H^q(\hat{H},M)
\end{array}
\]

By the term “level” of \( H \) here we mean the (unique) ideal \( a \) of \( \mathbb{O}_d \) which is maximal with the property that the normal closure of \( \{ \left( \begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix} \right) \in \text{PSL}(2, \mathbb{O}_d) \mid a \in a \} \) is included in \( H \) (see \cite{3} and section \cite{8}).

In this paper, only Hecke operators at the places of \( \mathbb{O}_d \) corresponding to the principal prime ideals are considered.

This work consists of five sections. Section \cite{1} is a brief review of basic definitions and facts about the extended notion of level and congruence closure. The goal of Section \cite{2} is to prove the congruence subgroup property (CSP) for \( \text{PSL}(2, \mathbb{O}_d[1/p]) \) and show that for any ideal \( I \triangleleft \mathbb{O}_d \) and for certain elements \( g \in \text{PGL}(2, \mathbb{Q}(\sqrt{d})) \), the
amalgamated product $\Gamma(I) \ast \Gamma(I) \ast \Gamma(I)$ is isomorphic to a finite index subgroup of $\text{PSL}(2, O_d[1/p])$ (Theorem 13). This will be used in Section 4 in order to show that $H$ and $H'$ satisfy the conditions of the aforementioned pure group-theoretic result. In Section 4, we prove the group-theoretic background for our generalization of Atkin’s conjecture (Theorem 16). Finally, in Section 3, we use the results of the previous sections and prove our generalization of Atkin’s conjecture in Theorem 23.

1. AN EXTENDED NOTION OF LEVEL

Let $R$ be a commutative ring with unit and $a$ a non-zero ideal of $R$. Consider the map $res_a : \text{SL}(n, R) \to \text{SL}(n, R/a)$ obtained by restriction mod $a$. The kernel of $res_a$ is called the principal or full congruence subgroup of level $a$ and is denoted by $\text{SL}(n, R, a)$.

Let $\pi : \text{SL}(n, R) \to \text{PSL}(n, R)$ be the canonical surjection and define

$$\Gamma(n, R, a) := \text{PSL}(n, R, a) := \pi(\text{SL}(n, R, a))$$

and call it the principal congruence subgroup of $\text{PSL}$ of level $a$. It is a normal subgroup of $\text{PSL}(n, R)$ since $\pi$ is surjective.

A congruence subgroup $H$ of $\text{SL}(n, R)$ or $\text{PSL}(n, R)$ is a subgroup which contains a full congruence subgroup $\text{SL}(n, R, a)$ or $\text{PSL}(n, R, a)$ respectively for some non-zero ideal $a$ of $R$. If $a$ is maximal among the ideals having this property, we say that $H$ is congruence of level $a$.

In particular, when $R = O_d$, $d$ any square-free negative integer, every congruence subgroup is of finite index. In this work we are particularly interested in the case $R = O_d$, $n = 2$ and the groups $\text{PSL}$. Following an idea of Fricke the concept of level of a full congruence subgroup was extended to arbitrary subgroups of $\text{SL}(2, \mathbb{Z})$ of finite index by Wohlfahrt [16], [17]. This has been generalized to $\text{PSL}(n, O_d)$ by Grunewald and Schwermer [8]. Let us start with some terminology and prove some elementary results about it. We use the following notation:

- $M_a := \{ (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \in \text{PSL}(2, R) \ | \ a \in a \}$. We denote the normal closure of $M_a$ in $\text{PSL}(2, R)$ by $Q(a)$. Clearly $M(0) = Q(0) = 0$ if $R$ is a local or Euclidean ring, then $Q(R) = \text{PSL}(2, R)$ (see [3], 5.9.2 and [9], 2.4). Moreover, by [6], Theorem 6.1, $Q(O_d) = \text{PSL}(2, O_d)$ if and only if $d \in \{-1, -2, -3, -7, -11\}$. For simplicity put $\Gamma(a) := \Gamma(2, R, a)$ when $R$ is clear from the context.

Consider an arbitrary subgroup $H$ of $\text{PSL}(2, R)$ of finite index. The set $X = \{ I \leq R \ | \ \Sigma Q(I) \leq H \}$, partially ordered by inclusion, has the maximum $\Sigma X := \Sigma \{ I \ | \ I \in X \}$, since $\Sigma Q(I_\alpha) = \bigvee Q(I_\alpha)$, for every family $\{ I_\alpha \leq R \ | \ \alpha \in A \}$, where $\bigvee Q(I_\alpha)$ denotes the subgroup generated by $Q(I_\alpha)$, $\alpha \in A$. Now we define

**Definition 2** (see [8]). Let $H$ be an arbitrary subgroup of $\text{level}$ $a_H$ if $a_H = \Sigma \{ I \leq R \ | \ Q(I) \leq H \}$. Clearly $a_{\Gamma(I)} = I$ for every ideal $I$ of $R$. 

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Proposition 3. Let $H, K$, and $H_\alpha$ $(\alpha \in A)$ be subgroups of $\text{PSL}(2, R)$ of finite index.

1. If $H \leq K$, then $a_H \subseteq a_K$.
2. For any family $H_\alpha$ of subgroups of $\Gamma_d$, $a_\bigcap H_\alpha = \bigcap a_{H_\alpha}$.
3. $a_{H_\Gamma} = a_H$ for every $g \in \Gamma_d$. In particular, $a_{H_G} = a_H$.
4. For any subgroup $N \leq H$ we have $a_{N \cap \Gamma(a_H)} = a_N$.

Proof. Straightforward. □

Corollary 4. The intersection of any family of congruence subgroups $H_\alpha$, of level $a_\alpha$, is congruence if and only if $\bigcap a_\alpha$ is non-zero. In this case, $a_{\bigcap H_\alpha} = \bigcap a_\alpha$.

Proof. If $\bigcap a_\alpha$ is non-zero, then clearly $\bigcap H_\alpha$ is congruence. Conversely, suppose that $\bigcap H_\alpha$ is congruence. So $a_{\bigcap H_\alpha} \neq 0$ and by Proposition 3 part (2), $a_{\bigcap H_\alpha} = \bigcap a_\alpha$. Therefore $\bigcap a_\alpha$ is non-zero. □

There are examples of finite index subgroups of $\text{PSL}(2, k[x])$, $k$ a finite field, of level zero; see [12]. For $\Gamma_d$, however, the situation is different, as we see in the next proposition and its corollaries:

Proposition 5. Let $H$ be a subgroup of $G = \text{PSL}(2, R)$. If $H$ has finite index in $G$ and $\text{char}(R) \nmid [G : H]$, then $a_H$ is non-zero.

Proof. We know that the normal core of $H$ in $G$, $H_G$, has finite index in $G$, say $m$. So for every $g \in G$, $g^m \in H_G$. This implies that $M(mR) \subseteq H_G$. Since $H_G$ is normal in $G$, we have $Q(mR) \subseteq H_G$. Since $\text{char}(R) \nmid m$, $mR \neq 0$. Hence the level of $H$ is not zero either. □

Corollary 6. Let $H$ be a finite index subgroup of $\Gamma_d$, for any square-free negative $d$. Then $a_H$ is non-zero.

We continue with studying some basic properties of the congruence closure.

Definition 7. Let $R$ be a commutative ring with unit and $H$ be a subgroup of $\text{PSL}(2, R)$. We define the congruence hull or closure of $H$ in $\text{PSL}(2, R)$ as the smallest congruence subgroup of $\text{PSL}(2, R)$ containing $H$, when it exists, and denote it by $\hat{H}$. Note that there always exists a congruence subgroup containing $H$, e.g. $H\Gamma(I)$ for every non-zero ideal $I$ of $R$.

Remark 8. Let $H$ be a subgroup of $\text{PSL}(2, R)$. If $a_H$ is non-zero, then $\hat{H}$ is defined and is equal to $\Gamma(a_H)H$. In particular, if $R = O_d$ and $H$ is of finite index in $\Gamma_d$, then $\hat{H}$ is defined.

Proposition 9. Let $H, K$ be subgroups of $\text{PSL}(2, R)$ such that $\hat{H}$ and $\hat{K}$ are defined.

1. For every congruence subgroup $N$ of $\hat{H}$, $\hat{H} = NH$. In particular, $\hat{H} = \Gamma(a_H)H$ if $a_H$ is non-zero.
2. If $H \subseteq K$, then $\hat{H} \subseteq \hat{K}$.
3. If $\hat{H} \cap \hat{K}$ is defined, then $\hat{H} \cap \hat{K} \subseteq \hat{H} \cap \hat{K}$.
4. If $a_H$ is non-zero, then $H_G \cap \Gamma(a_H) = \Gamma(a_H)$.
5. If $a_H$ is non-zero, then for every $N \leq H$ with $a_N \neq 0$ and $N \cap \Gamma(a_H) = \hat{N}$ we have $N \subseteq \Gamma(a_H)$. 

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Proof. (1) Follows from the fact that $NH$ is a congruence subgroup containing $H$ and included in $\hat{H}$. (2), (3), and (4) are consequences of Proposition 22. For (5): If $N \cap \Gamma(a_H) = \hat{N}$, then by part (1) and Proposition 22 part (4), $\Gamma(a_N)N = \Gamma(a_N)(N \cap \Gamma(a_H))$. On the other hand, $\Gamma(a_N) \cap N = \Gamma(a_N) \cap (N \cap \Gamma(a_H))$, whence the result. \hfill $\square$

2. Congruence subgroup property of $\text{SL}(2, \mathcal{O}_K^{1/p})$

Let $p \in \mathcal{O} := \mathcal{O}_d$ be prime, $d$ any square-free negative integer. In this section we show that the group $\text{SL}(2, \mathcal{O}_K^{1/p})$ (and hence $\text{PSL}(2, \mathcal{O}_K^{1/p})$) has the congruence subgroup property. As a result, we see that the group $\Gamma(I \cdot \mathcal{O}_K^{1/p})$ satisfies the CSP, for every non-zero ideal $I$ of $\mathbb{Q}(\sqrt{d})$ such that $p \notin I$. This result will be used in the proof of Proposition 22. We start by stating an important theorem of Serre, which, applied to quadratic fields, is the key for proving the CSP of $\text{SL}(2, \mathcal{O}_K^{1/p})$. For the proof and details of the theorem, see [13], Theorem 2. Let $S_{\text{all}}$ be the set of all places of a number field $K$ and $S \subseteq S_{\text{all}}$ be a finite subset of $S_{\text{all}}$ containing $S_{\infty}$, the set of all Archimedean places of $K$. The ring of $\mathbf{S}$-integers of $K$ is by definition:

$$\mathcal{O}_S := \{ x \in K \mid v(x) \geq 0, \text{ for every place } v \in S_{\text{all}} - S \}.$$ $$\mathcal{O}_S$$ is a Dedekind domain whose maximal ideals are in 1-1 correspondence with the elements of $S_{\text{all}} - S$. We have:

Theorem 10 (Serre). Let $S$ be a finite subset of the places of the field $\mathbb{Q}(\sqrt{d})$ that contains at least 2 places, at least one of them real or non-Archimedean, and $\mathcal{O}_S$ the subring of $\mathbf{S}$-integers. Then the group $\text{SL}(2, \mathcal{O}_S)$ (and hence $\text{PSL}(2, \mathcal{O}_S)$) satisfies the CSP.

Let $K := \mathbb{Q}(\sqrt{d})$ and $\mathcal{O}_p$ be the set consisting of the Archimedean place of $K$ together with the non-Archimedean place corresponding to the prime ideal $P := p\mathcal{O}$. It is easy to show that $\mathcal{O}_S = \mathcal{O}_p^{1/p}$, and so by Theorem 11 we have:

Proposition 11. Let $p \in \mathcal{O}$ be prime. Then the group $\text{SL}(2, \mathcal{O}_p^{1/p})$ (and hence $\text{PSL}(2, \mathcal{O}_p^{1/p})$) has the congruence subgroup property.

In order to prove the main result of this section, we also need the following two lemmas.

Lemma 12. Let $R$ be a commutative ring with unit, $p \in R$ with $Rp$ maximal, and $I, J \subseteq R$ coprime to $Rp$. Then $IJ$ is coprime to $Rp^k$, for every $k \in \mathbb{N}$.

Proof. Clearly $IJ + Rp = R$. So there exist $r \in R$, $x \in I$, and $y \in J$ such that $1 = rp + xy$, and hence $p^{k-1} = xyp^{k-1} + rp^k \in IJ + Rp^k$, so $IJ + Rp^{k-1} = IJ + Rp^k$ for all $k$, whence the result follows by induction on $k$. \hfill $\square$

Lemma 13. Let $I \subset \mathcal{O}$ and $p \in \mathcal{O}$ prime such that $p \notin I$. Set $\hat{I} := I \cdot \mathcal{O}_p^{1/p} \subset K$. Then $\hat{I}$ is dense in $K = \mathbb{Q}(\sqrt{d})$ with respect to the topology induced by the absolute value $| \cdot |_{\mathcal{O}_p}$ on $K$.

Proof. Let $P := \mathcal{O}_p$, so $I + P = \mathcal{O}$. Let $x \in \mathcal{K}$ and $\epsilon > 0$. Write $x = \frac{a}{b}$ with $a_1, b_1 \in \mathcal{O}$, $b_1 \neq 0$. Let $v_p$ denote the $p$-adic valuation on $K$. Suppose that $v_p(a_1) = \alpha$, $v_p(b_1) = \beta$, and write $a_1 = p^\alpha a$, $b_1 = p^\beta b$ with $a, b \in \mathcal{O}$ and $p \nmid a, b,$
Let the group $SL(2)$. Put $s := \frac{\alpha - \beta}{p}$. Choose $k \in \mathbb{N}$ such that $p^{-k} < \epsilon$. Let $s < 0$. By the previous lemma, $bI + P^k - s = \mathcal{O}$. So there exist $g \in I$, $y \in \mathcal{O}$ such that $a = bg + yp^{-k} \cdot b$, hence $p^k - s \mid a - bg$ and so $|a - bg|/p \leq p^{-k+s}$. Since $|b|_p = 1$, we have $\frac{\alpha}{p^k} - g/p = |a - bg|/p \leq p^{-k+s}$, so $|x - p^s g|/p \leq p^{-s+k} = p^{-k} < \epsilon$ and $p^s g \in I$. The case $s \geq 0$ is handled in a similar way. \hfill \Box

**Theorem 14.** Let $0 \neq I \triangleleft \mathcal{O} = \mathcal{O}_d$ and $p \in \mathcal{O}$ be prime such that $p \notin I$. Define $g := \left( \begin{smallmatrix} p & 0 \\ 1 & p \end{smallmatrix} \right) \in \text{PGL}(2, K)$. Then $\Gamma(I) *_{\Gamma(I) \cap \Gamma(I)^g} \Gamma(I)^g \cong \Gamma(I \cdot \mathcal{O}[\frac{1}{p}])$, and the latter satisfies the congruence subgroup property.

**Proof.** Let $\bar{I} := I \cdot \mathcal{O}[\frac{1}{p}]$. Consider the $p$-adic valuation $v : K \to \mathbb{Z} \cup \{\infty\}$ and let $R$ be its valuation ring. We know from the previous lemma that $\bar{I}$ is dense in $K$ with respect to the $p$-adic norm. So clearly the closure of $\Gamma(\bar{I})$ in $\text{PGL}(2, K)$ contains $(\frac{1}{1} 1)$ and $(\frac{1}{s} 1)$, and so $\text{PSL}(2, K)$. Recall that (e.g. Theorem 3, Chapter 2 of [14]) the group $\text{SL}(2, K)$ (PSL$(2, K)$ resp.) acts on the set of lattices (on the tree of lattice classes resp.) of $K \oplus K$ in the following way: Let $L = l_1 R + l_2 R$ be a lattice and $g = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}(2, K)$. Then

$$g \cdot L := (aL_1 + bL_2)R + (cL_1 + dL_2)R.$$ 

Now apply Theorem 2 of Chapter 2 of [14] with $G = \Gamma(\bar{I})$, $L' := R \cdot p^2 \oplus R \cdot p \subseteq L := R \cdot p \oplus R \cdot p$, to see that $\Gamma(\bar{I}) = G_L *_{G_L \cap G_L} G_L'$, where $G_L, G_L'$ are the stabilizers of $L, L'$ under the $G$-action respectively. We show that $G_L = \Gamma(I)$ and $G_L' = \Gamma(I)^g$:

**Lemma 15.** For every $x \in \bar{I}$, if $x \notin R$ then $x - 1 \notin I$.

To show that $G_L = \Gamma(I)$, consider $u = \left( \begin{smallmatrix} a+1 \\ c \\ d+1 \end{smallmatrix} \right) \in \Gamma(I)$, with $a, b, c, d \in I$. So

$$\left( \begin{array}{cc} a+1 & b \\ c & d+1 \end{array} \right) \left( \begin{array}{c} p \\ 0 \end{array} \right) = \left( \begin{array}{c} p(a+1) \\ cp \end{array} \right),$$

$$\left( \begin{array}{cc} a+1 & b \\ c & d+1 \end{array} \right) \left( \begin{array}{c} 0 \\ p \end{array} \right) = \left( \begin{array}{c} bp \\ p(d+1) \end{array} \right).$$

By Lemma 15, $a + 1, d + 1 \in R$, so $uL \subseteq L$ for every $u \in \Gamma(I)$; that is, $uL = L$ for every $u \in \Gamma(I)$.

To show that $G_L' = \Gamma(I)^g$, consider $u \in \Gamma(I)^g$ and write $u = \left( \begin{smallmatrix} a+1 \\ cp \\ d+1 \end{smallmatrix} \right)$ with $a, b, c, d \in I$ and argue as above.

Finally, we note that $\Gamma(\bar{I})$ is of finite index in $\text{PSL}(2, \mathcal{O}[\frac{1}{p}])$, and the congruence subgroup property follows from Proposition [14] \hfill \Box

3. The action of Hecke operators on the cohomology groups

Let $G$ be a group. For every $x \in G$ and every $H \leq G$, let $H_x$ denote the intersection $H \cap H^x$ (where as before $H^x = x^{-1}Hx$). It is well known that for every left $G$-module $A$ and every finite index subgroup $H$ of $G$, there is a natural action of the Hecke operators (double cosets of $H$) on the cohomology groups $H^*(H, A)$ defined in the following way: write $H$ as a union of $\mu = \mu(x) := [H : H_x]$ disjoint cosets of $H_x$ in $H$: $H = \bigsqcup_1^\mu H_x h_i$. It is easy then to check that $H x H = \bigsqcup_1^\mu H_x h_i$, where $x_i := h_i$. Since for every $y \in H$, $H x H y = H x H$, we have $H x H = \bigsqcup_1^\mu H_x h_i y$, so for every $1 \leq i \leq \mu$,

$$x_i y = t_i(y)x_i(y)$$

(3.1)

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for a unique element \( t_i(y) \in H \) and a unique index \( i(y) \). So \( (x_1(y) \cdots x_{\mu(y)}) \) is a permutation of \( (x_1 \cdots x_{\mu}) \). For each \( y, y' \in H \), \( (x_i(y)y') = t_i(y)(x_i(y')y') = t_i(y)t_i(y')(x_i(y)y') \). On the other hand, \( x_i(y') = t_i(y')x_i(y)y' \), so

\[
(i(y')) = (i(y))(y'), \quad t_i(y') = t_i(y)t_i(y').
\]

Given a non-negative integer \( q \) we denote the group of all (non-homogeneous) \( q \)-co-chains from \( H \) to \( A \) by \( C^q(H, A) \) and we define the action of \( HxH \) on a co-chain \( f \in C^q(H, A) \) as follows:

\[
(f : HxH)(y_1, \ldots, y_q) := \sum_{i=1}^{\mu} x_i^{-1} f[t_i(y_1), t_i(y_2), t_i(y_3), \ldots, t_i(y_q)]
\]

for all \( y_1, \ldots, y_q \in H \). For more details, see [11] and [1].

We also need an explicit description of the transfer (or co-restriction) map between cohomology groups. Given a finite index subgroup \( H \leq G \) and a \( G \)-module \( A \), let \( \{G : H\} = n \) and \( \{s_1, \ldots, s_n\} \) be a transversal for the left cosets of \( H \) in \( G \). For every \( x \in G \), let \( \bar{x} \) be the unique element \( s_i \) with \( x \in Hs_i \). So we have \( \bar{x}^{-1} \in H \). Now for every \( k > 0 \), \( f : H^k \to A \in C^k(H, A) \), and \( g_1, \ldots, g_k \in G \), we have

\[
tr^K_{\bar{g}}(f)[g_1, \ldots, g_k] = \sum_{i=1}^n s_i^{-1} f[s_1, s_2, \ldots, s_k, \bar{g}_1, \bar{g}_2, \ldots, \bar{g}_k]^{-1}.
\]

and it can be shown that the corresponding induced map on the cohomology groups is independent of the transversal’s choice. We now come to our main task of this section. The next theorem provides the group-theoretic background for our generalization of Atkin’s conjecture. If we apply it to a finite index subgroup of \( \text{PSL}(2, \mathcal{O}_d) \) and its congruence closure, then we get a generalization of Atkin’s conjecture for Hecke operators acting on the cohomology groups (see the next section).

**Theorem 16.** Let \( H \leq f \text{ K } \leq G \) and \( g \in G \) be such that \( K = (K_g)H \). Consider the following conditions:

1. \( [K_g : H_g] = [K : H]^2 \).
2. \( K = H(K \cap gH) \), where \( gH := gh^{-1} \).
3. \( [H \cap K^g : H_g] = [K : H] \).
4. For every \( G \)-module \( A \) and every \( q \geq 1 \) the following diagram commutes:

\[
\begin{array}{ccc}
H^q(H, A) & \xrightarrow{\text{tr}^K_{\bar{g}}} & H^q(K, A) \\
\downarrow{T^g} & & \downarrow{T^K} \\
H^q(H, A) & \xleftarrow{\text{res}^K_{\bar{g}}} & H^q(K, A)
\end{array}
\]

Then \( 1 \iff 2 \iff 3 \) and \( 3 \implies 4 \).

**Proof.** The equivalence of 1, 2 and 3 is easy to prove. We start by proving 3 \( \implies 4 \). Let \( \mu := \mu_H(g) = [H : H_g] \) and write \( H = \bigsqcup_i^\mu H g_i \). Since \( K = (K_g)H \), \( K_gg_i \leq K \) for every \( i \) and \( K = \bigsqcup_i^\mu K_gg_i \). Without loss of generality, assume that \( K = \bigsqcup_i^b K_gg_i \), where \( b = [K : K_g] \), so

\[
K_gK = \bigsqcup_{i=1}^b K_gg_i.
\]
For every $y \in K$, define $t_i(y)$ as the unique element of $K$ such that $gg_iy = t_i(y)gg_i(y)$ for a unique index $i(y)$ (see equation (3.1)). Write

\[
H \cap K^g = \bigsqcup_{j=1}^m Hgh_j,
\]

where $m = [H \cap K^g : H_g]$. By assumption $m = [K : H]$. Since for every $j$, $gh_jg^{-1} \in K$ and $\{Hgh_jg^{-1} \mid 1 \leq j \leq m\}$ consists of exactly $m$ disjoint cosets, we see that

\[
K = \bigsqcup_{j=1}^m Hgh_jg^{-1}.
\]

For every $y \in K$, define $\bar{g}$ as the unique $gh_jg^{-1}$ such that $y \in Hgh_jg^{-1}$. Since $b = [H : H \cap K^g]$ and $\{\{H \cap K^g\}g_i \mid 1 \leq i \leq b\}$ consists of exactly $b$ disjoint cosets, we have $H = \bigsqcup_{i=1}^b (H \cap K^g)g_i$, so by equation (3.5) we have $H = \bigsqcup_{i=1}^b \bigsqcup_{j=1}^m Hgh_jg_i$, and hence

\[
HgH = \bigsqcup_{i=1}^b \bigsqcup_{j=1}^m Hgh_jg_i = \bigsqcup_{j=1}^m Hz_{(j,i)},
\]

where $z_{(j,i)} := gh_jg_i$. For every $x \in H$, define $t_{(j,i)}(x)$ as the unique element of $H$ such that $z_{(j,i)}x = t_{(j,i)}(x)z_{(j,i)}(x)$, for a unique pair of indices $(j,i)$ (see equation (3.1)).

We start with $q = 1$. Consider $[f] \in H^1(H,A)$, where $f \in C^1(H,A)$ is a derivation. We show that $(T_g^H f)(x) = T_g^K (tr^K_H(f))(x)$ for every $x \in H$. For $x \in H$, we compute

\[
(T_g^H f)(x) = (f \cdot HgH)(x) = \sum_{i=1}^b \sum_{j=1}^m (gh_jg_i)^{-1}f(t_{(j,i)}(x))
\]

and

\[
T_g^K (tr^K_H(f))(x) = (tr^K_H(f) \cdot KgK)(x) = \sum_{i=1}^b g_i^{-1}g^{-1}tr^K_H(f)(t_i(x))
\]

\[
= \sum_{i=1}^b g_i^{-1}g^{-1} \sum_{j=1}^m (gh_jg_i^{-1})^{-1}f(gh_jg^{-1}t_i(x)(gh_jg_i^{-1}t_i(x))^{-1})
\]

\[
= \sum_{i=1}^b \sum_{j=1}^m g_i^{-1}h_j^{-1}g^{-1}f(w_{(j,i)}(x)),
\]

where $w_{(j,i)}(x) := gh_jg^{-1}t_i(x)(gh_jg_i^{-1}t_i(x))^{-1}$. We show that $w_{(j,i)}(x) = t_{(j,i)}(x)$. Clearly $w_{(j,i)}(x) \in H$. Note that $t_i(x)$ satisfies $gg_i x = t_i(x)gg_i(x)$, and $gh_jg^{-1}t_i(x) = gh_kg^{-1}$ for some $k$; hence

\[
gh_jg_i x = gh_jg^{-1}gg_i x = gh_jg^{-1}t_i(x)gg_i(x)
\]

\[
= gh_jg^{-1}t_i(x)(gh_kg^{-1})(gh_kg^{-1})gg_i(x) = w_{(j,i)}(x)gh_kg_i(x),
\]

so by definition of $t_{(j,i)}$,

\[
w_{(j,i)}(x) = t_{(j,i)}(x) \text{ for every } x \in H,
\]
and this finishes the case \( q = 1 \).

Let \( q \geq 2 \) and consider \( [f] \in H^q(H, A) \), where \( f \in C^q(H, A) \). We show that

\[
(T_g^H f)(x_1, \ldots, x_q) = T_g^K (\text{tr}_H^K f)(x_1, \ldots, x_q)
\]

for every \( x_1, \ldots, x_q \in H \). We have

\[
(T_g^H f)(x_1, \ldots, x_q) = \sum_{i=1}^{b} \sum_{j=1}^{m} (gh_j g_i)^{-1} f(t_{(j,i)}(x_1), \ldots, t_{(j,i)(x_1\cdots x_{q-1})(x_q)})
\]

and

\[
T_g^K (\text{tr}_H^K f)(x_1, \ldots, x_q) = (\text{tr}_H^K(f) \cdot KgK)(x_1, \ldots, x_q)
\]

\[
= \sum_{i=1}^{b} g_i^{-1} g_i^{-1} \sum_{j=1}^{m} (gh_j g_i)^{-1} f(gh_j g_i^{-1} t_i(x_1)(gh_j g_i^{-1} t_i(x_1))^{-1}, \ldots,
\]

\[
gh_j g_i^{-1} t_i(x_1) \cdots t_i(x_{1\cdots x_{q-2}})(x_{q-1}) t_i(x_{1\cdots x_{q-1}}(x_q)) gh_j g_i^{-1} t_i(x_1) \cdots t_i(x_{1\cdots x_{q-1}}(x_q))^{-1})
\]

Comparing the corresponding entries of \( f \) in equations (3.9) and (3.10) and recalling the equation (3.8), we see that it is enough to show that for every \( 2 \leq r \leq q \),

\[
t_{(j,i)(x_1\cdots x_{r-1})(x_r)} = gh_j g_i^{-1} t_i(x_1) \cdots t_i(x_{1\cdots x_{r-2}})(x_{r-1}) t_i(x_{1\cdots x_{r-1}}(x_r)) gh_j g_i^{-1} t_i(x_1) \cdots t_i(x_{1\cdots x_{r-1}}(x_r))^{-1},
\]

and this equality can be verified by induction on \( r \) using properties (3.1) and (3.2) as follows. By equation (3.8),

\[
w_{(j,i)}(x_1 \cdots x_r) = t_{(j,i)}(x_1 \cdots x_r).
\]

Using equation (3.2), we get

\[
t_{(j,i)}(x_1 \cdots x_r) = t_{(j,i)}(x_1) t_{(j,i)(x_1)}(x_2) t_{(j,i)(x_1 x_2)}(x_3) \cdots t_{(j,i)(x_1 \cdots x_{r-1})(x_r)},
\]

as well as

\[
w_{(j,i)}(x_1 \cdots x_r) = gh_j g_i^{-1} t_i(x_1 \cdots x_r)(gh_j g_i^{-1} t_i(x_1 \cdots x_r))^{-1}
\]

\[
= gh_j g_i^{-1} t_i(x_1) \cdots t_i(x_{1\cdots x_{r-1}})(x_r)(gh_j g_i^{-1} t_i(x_1) \cdots t_i(x_{1\cdots x_{r-1}})(x_r))^{-1}
\]

hence

\[
t_{(j,i)}(x_1) t_{(j,i)(x_1)}(x_2) t_{(j,i)(x_1 x_2)}(x_3) \cdots t_{(j,i)(x_1 \cdots x_{r-1})(x_r)} = gh_j g_i^{-1} t_i(x_1) \cdots t_i(x_{1\cdots x_{r-1}})(x_r)(gh_j g_i^{-1} t_i(x_1) \cdots t_i(x_{1\cdots x_{r-1}})(x_r))^{-1}
\]

Now we prove equation (3.11) by induction on \( r \geq 2 \). For \( r = 2 \), equation (3.12) reduces to

\[
t_{(j,i)}(x_1) t_{(j,i)(x_1)}(x_2) = gh_j g_i^{-1} t_i(x_1) t_{(j,i)(x_1)}(x_2)(gh_j g_i^{-1} t_i(x_1) t_{(j,i)(x_1)}(x_2))^{-1}
\]

Since \( t_{(j,i)(x_1)} = gh_j g_i^{-1} t_i(x_1) \) (by equation (3.8)), we have

\[
t_{(j,i)(x_1)}(x_2) = gh_j g_i^{-1} t_i(x_1)(gh_j g_i^{-1} t_i(x_1) t_{(j,i)(x_1)}(x_2))^{-1}
\]
Now assuming equation (3.11) for any \( s \leq r - 1 \), we have
\[
gh_j g^{-1} t_i(x_1) \cdots t_i(x_{1 \cdots r-2}) (x_{r-1})
\]
\[
= t_{(j,i)}(x_{1 \cdots r-2}) (x_{r-1})^{-1} gh_j g^{-1} t_i(x_1) \cdots t_i(x_{1 \cdots r-3}) (x_{r-2}) t_i(x_{1 \cdots r-2}) (x_{r-1})
\]
\[
= \cdots
\]
\[
= t_{(j,i)}(x_{1 \cdots r-2}) (x_{r-1})^{-1} t_{(j,i)}(x_{1 \cdots r-3}) (x_{r-2})^{-1} \cdots t_{(j,i)}(x_1) (x_2) \cdots t_{(j,i)}(x_{1 \cdots r-2}) (x_{r-1})
\]
\[
= t_{(j,i)}(x_{1 \cdots r-2}) (x_{r-1})^{-1} \cdots t_{(j,i)}(x_1) (x_2) \cdots t_{(j,i)}(x_{1 \cdots r-1}) (x_r)
\]
\[
= t_{(j,i)}(x_{1 \cdots r-1}) (x_r),
\]
proving equation (3.11), which finishes the proof. \( \square \)

4. Hecke operators on non-congruence subgroups

In this section we use Theorem 16 to prove a generalization of Atkin’s conjecture for Hecke operators on the cohomology groups of \( \Gamma_d \). We prove, in a sequence of lemmas, that if \( H = \text{PSL}(2, \mathbb{O}_d) \), with coefficients in any \( \Gamma_d \)-module, we have
\[
\Gamma \cap \Gamma(p \mathcal{O}_d) \subseteq \Gamma \cap \Gamma(I)^g \cap g \Gamma(I).
\]

Lemma 17. Let \( I \subseteq \mathcal{O}_d \) and \( p \in \mathcal{O}_d \) such that \( I + p \mathcal{O}_d = \mathcal{O}_d \). Define \( g := \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{PGL}(2, \mathbb{Q}(\sqrt{d})) \). Then
\[
\Gamma(I) \cap \Gamma(p \mathcal{O}_d) \subseteq \Gamma(I) \cap \Gamma(I)^g \cap g \Gamma(I).
\]

Proof. Suppose that \( 1 = v + o \cdot p \) with \( v \in I, o \in \mathcal{O}_d \).

Let \( u \in \Gamma(I) \cap \Gamma(p \mathcal{O}_d) \), so we may write \( u = \left( \begin{smallmatrix} a + 1 & b \\ c + 1 & d \end{smallmatrix} \right) = \left( \begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix} \right) \), with \( a, b, c, d \in I \) and \( a', b', c', d' \in \mathcal{O}_d \). So \( u^g = \left( \begin{smallmatrix} a + 1 & b/p \\ c/p & d + 1 \end{smallmatrix} \right) = \left( \begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix} \right) \). Therefore we have \( b/p = b' \in \mathcal{O}_d \), so that \( b' = vb' + ob \in I \). This shows that \( u^g \in \Gamma(I) \), i.e. \( u \in g \Gamma(I) \). The inclusion \( \Gamma(I) \cap \Gamma(p \mathcal{O}_d) \subseteq \Gamma(I)^g \) is proved in a similar way. \( \square \)

Lemma 18. For any element \( g \) of a group \( G \) and any subgroups \( H \leq f \), we have

1. \( [H \cap K^g : H \cap H^g] \leq [K : H] \); equality holds if and only if \( K = H(K \cap H) \).
2. \( H = NH \) for some \( N \leq K \), then in (1) equality holds if and only if \( N \leq H(K \cap H) \).
(3) Let \( g \in G \) and \( H, N \leq K \leq S \leq G \) with \( K = NH \). Set \( H_1 := N \cap H_S \), where \( H_S \) denotes the normal core of \( H \) in \( S \). If \( [H_1 \cap N^g : H_1 \cap H^g] = [N : H_3] \), then \( [H \cap K^g : H \cap H^g] = [K : H] \).

Proof. Easy verifications. \( \square \)

Let \( g \in G \) and \( H \leq K \leq G \). Define \( \pi, \pi_g : K \cap K^g \to H \setminus \set{K} := \{Hx \mid x \in K\} \) by \( \pi(x) := Hx \) and \( \pi_g(x) := H^g(x) \), where \( g x := g x g^{-1} \). It is clear that \( \ker(\pi) = H \cap K^g \) and \( \ker(\pi_g) = H^g \cap K \) and that we have the map

\[
(\pi, \pi_g) : K \cap K^g \to H \setminus K \times H \setminus K,
\]

called \((\pi, \pi_g)\), for \( H \leq K \). If \( H \leq K \), then \((\pi, \pi_g)\) is a homomorphism with \( \ker(\pi, \pi_g) = \ker(\pi) \cap \ker(\pi_g) = H \cap H^g \), which is onto if and only if \([K \cap K^g : H \cap H^g] = [K : H]^2\).

Lemma 19. Let \( H \leq_f K \leq G \). For any \( g \in G \), if the map \((\pi, \pi_g)\) is onto, then

\[
[H \cap K^g : H \cap H^g] = [K : H].
\]

Proof. Let \((\pi, \pi_g)\) be onto, and consider an element \( x \in K \). So there exists \( y \in K \cap K^g \) with \((\pi, \pi_g)(y) = (H, Hx)\), that is, \( y \in H \) and \( x = hy \) for some \( h \in H \). So \( x \in H(K \cap K^g) \). Hence by Lemma 18 part 1, \([H \cap K^g : H \cap H^g] = [K : H]\). \( \square \)

Lemma 20. Let \( H, N \leq K \leq S \leq G \) with \( K = NH \), \([K : H] < \infty \) and \( H_1 := N \cap H_S \). For any \( g \in G \), if \((\pi, \pi_g)\) for \((H_1, N)\) is onto, then \([H \cap K^g : H \cap H^g] = [K : H]\).

Proof. This is immediate from the previouslemma and part 3 of Lemma 18. \( \square \)

Finally, we need the following lemma from Serre (see appendix of [15]):

Proposition 21. Suppose \( X, X_1, X_2 \) are arbitrary groups with epimorphisms \( X_1 \xrightarrow{f_1} X \xrightarrow{f_2} X_2 \). If \( X \xrightarrow{(f_1, f_2)} X_1 \times X_2 \) is not surjective, then there exist a group \( Y \neq 1 \) and epimorphisms \( X_1 \xrightarrow{h_1} Y \xrightarrow{h_2} X_2 \) such that \( h_1 f_1 = h_2 f_2 \).

Now we can show that for every finite index subgroup \( H \) of \( \Gamma_d \), its congruence closure, and special elements \( g \in PGL(2, \mathbb{C}) \), the conditions of [10] are satisfied:

Proposition 22. Let \( p \in \mathcal{O}_d \) be prime and define

\[
g := (p_0 \ 0) \in G := PGL(2, \mathbb{Q}(\sqrt{d})).
\]

Consider \( H \leq_f \Gamma_d \) of level \( \mathfrak{a} := \mathfrak{a}_H \) and assume \( \mathfrak{a} + \mathfrak{p} \mathcal{O}_d = \mathcal{O}_d \). Let \( \hat{H} \) be the congruence closure of \( H \) in \( \Gamma_d \). Then

\[
[H \cap H^g : H \cap H^g] = [\hat{H} : H].
\]

Proof. Set \( H_1 := \Gamma_d \cap \Gamma(\mathfrak{a}) \), where \( H_{\Gamma_d} \) is the normal core of \( H \) in \( \Gamma_d \). Then \( \hat{H} = \Gamma(\mathfrak{a})H \) (by Proposition 9 part 1) and as \( \mathfrak{a}_{H_1} = \mathfrak{a}, \hat{H}_1 = \Gamma(\mathfrak{a}) \) (Proposition 9 part 2). If the map \((\pi, \pi_g)\) for \( H_1 \leq N := \Gamma(\mathfrak{a}) \) is onto, then by Lemma 20 we are done. First we show that both \( \pi \) and \( \pi_g \) are onto:

Consider the canonical surjections \( \pi_1 : N \to N/H_1 \) and \( \pi_2 : N \to N/(N \cap \Gamma(p \mathcal{O}_d)) \) and let \( \psi := (\pi_1, \pi_2) : N \to N/H_1 \times N/(N \cap \Gamma(p \mathcal{O}_d)) \). If \( \psi \) is not onto, then by Proposition 21 there exist a group \( T \neq 1 \) and \( N/H_1 \xrightarrow{k_1} T \xrightarrow{k_2} N/(N \cap \Gamma(p \mathcal{O}_d)) \) such that \( k_1 \pi_1 = k_2 \pi_2 \). Suppose \( T = N/W, \) where \( W := \ker(k_1 \pi_1) \). Since \( T \neq 1, H_1 \leq W \leq N, \) and \( N \) is the congruence closure of \( H_1, W \) is not congruence. But \( N \cap \Gamma(p \mathcal{O}_d) \leq W \), a contradiction. Hence \( \psi \) is onto. Now for \( s \in N/H_1, \) there exists
Remark 26. One application of Theorem 23 is in the theory of Bianchi modular forms for imaginary quadratic fields of class number one. The Eichler-Shimura-Harder correspondence (see [10]) allows us to see these forms as classes in the cohomology of finite index subgroups of Bianchi groups. Theorem 23 can be used to deduce that the Hecke action on Bianchi modular forms for a non-congruence subgroup of a Bianchi group is essentially the same as the Hecke action on Bianchi modular forms for its congruence closure.

Acknowledgement

The author is indebted to Dr. Haluk Sengun for several discussions and valuable suggestions and corrections.
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Department of Mathematics, Heinrich-Heine University Düsseldorf, Düsseldorf, Germany

E-mail address: zarghani@math.uni-duesseldorf.de, zarghani.s@gmail.com