A NOTE ON DISCRETENESS OF F-JUMPING NUMBERS

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Abstract. Suppose that $R$ is a ring essentially of finite type over a perfect field of characteristic $p > 0$ and that $a \subseteq R$ is an ideal. We prove that the set of $F$-jumping numbers of $\tau_b(R; a^t)$ has no limit points under the assumption that $R$ is normal and $\mathbb{Q}$-Gorenstein – we make no assumption as to whether the $\mathbb{Q}$-Gorenstein index is divisible by $p$. Furthermore, we also show that the $F$-jumping numbers of $\tau_b(R; \Delta, a^t)$ are discrete under the more general assumption that $KR + \Delta$ is $\mathbb{R}$-Cartier.

1. Introduction

The test ideal is an important and subtle object associated to ideals $a$ in positive characteristic rings $R$. It measures the singularities of both the ambient ring and the elements of the ideal; see [HY03]. While the test ideal was initially introduced in the celebrated theory of tight closure of Hochster and Huneke (see [HH90]), more recent interest in the test ideal has been in regards to its connection with the multiplier ideal – a fundamental invariant of higher dimensional algebraic geometry in characteristic zero; see for example [Tak06] or [MY09].

Given a normal ring $R$ essentially of finite type over a perfect field of characteristic $p > 0$, an ideal $a \subseteq R$ and a real number $t \geq 0$, one can form the (big) test ideal $\tau_b(R; a^t)$ – an object which measures both algebraic and arithmetic properties of $R$ and $a$. Inspired by the test ideal’s close relation with the multiplier ideal $\mathcal{J}(R, a^t)$, people have studied the numbers $t_i$ where $\tau_b(R; a^{t_i})$ changes. That is, people have studied the $F$-jumping numbers (see [MTW05]), real numbers which are by definition the $t_i > 0$ such that for every $\varepsilon > 0$,

$$\tau_b(R; a^{t_i - \varepsilon}) \neq \tau_b(R; a^{t_i}).$$

One easy to observe fact about multiplier ideals is that their jumping numbers are discrete and rational, at least when $R$ is $\mathbb{Q}$-Gorenstein and normal; see [ELSV04]. Here, by discrete we mean that the set of jumping numbers with respect to a fixed ideal have no limit points. Because of this, various groups have recently worked to show that the $F$-jumping numbers of the test ideal are also discrete and rational; see [Har06], [BMS08], [BMS09], [KLZ09], and [BSTZ10]. In the most recent-mentioned work, the author, along with M. Blickle, S. Takagi, and W. Zhang, showed that...
the $F$-jumping numbers of test ideals formed a discrete set of rational numbers when $R$ is normal and $\mathbb{Q}$-Gorenstein with index not divisible by $p > 0$. Recall that the index of a $\mathbb{Q}$-Gorenstein ring $R$ is the smallest natural number $n$ where $\omega_R^{(n)} = O_{\text{Spec } R}(nK_R)$ is locally free.

The most fundamental case left open is the case when $R$ is $\mathbb{Q}$-Gorenstein but of arbitrary index; see [BSTZ10] Question 6.1. We answer this question at least for discreteness.

**Theorem 3.5** Suppose that $R$ is a normal domain essentially of finite type over an $F$-finite field. Further suppose that $a \subseteq R$ is an ideal and $\Delta$ is an $R$-divisor on $X = \text{Spec } R$ such that $K_X + \Delta$ is $R$-Cartier (for example, this holds if $\Delta = 0$ and $R$ is $\mathbb{Q}$-Gorenstein). Then, as $t$ varies, the $F$-jumping numbers of $\tau_t(R; \Delta, a^t)$ have no limit points — they are discrete.

We also point out why the existing proofs of rationality do not seem to work in the case that $R$ is $\mathbb{Q}$-Gorenstein with index divisible by $p$.

Recently, in [DH09], de Fernex and Hacon gave a definition of the multiplier ideal without the $\mathbb{Q}$-Gorenstein assumption and asked the question of whether discreteness and rationality of the $F$-jumping numbers still holds in this context. Following this, Urbinati showed that rationality need not hold but gave some evidence that discreteness may hold in general; see [Urb10]. This suggests that one should not expect rationality to hold in positive characteristic either.

## 2. Definition of the test ideal

We only give a very brief description of the big test ideal in this paper. Please see [BSTZ10] for a more detailed description of the test ideal.

First we fix some notation. Given a ring $R$ of characteristic $p > 0$ and $M$ an $R$-module, we set $F^e_RM$ to be the $R$-module which agrees with $M$ as an additive group but where the $R$-module structure is defined by the rule $r.m = r^e.m$. Also recall that $R$ is said to be $F$-finite if $F^e_RM$ is a finitely generated $R$-module.

**Convention.** Throughout this paper, all rings will be assumed to be $F$-finite.

Recall that an $R$-divisor on a normal scheme $X$ is a formal linear combination of prime Weil divisors $D_i$ with real coefficients. An $R$-divisor $D$ is called $R$-Cartier if it is equal to an $R$-linear combination of Cartier divisors.

We now define the test ideal $\tau_t(R; \Delta, a^t)$ to a new triple $(R, \Delta', a'^{t_1} \cdots b'^{t_m})$ which has the same test ideal.

**Definition 2.1** ([HH90], [Hoc07], [Sch10]). Suppose that $R$ is an $F$-finite normal domain, $\Delta \geq 0$ is an $R$-divisor on $X = \text{Spec } R$, $a, b_1, \ldots, b_m \subseteq R$ are non-zero ideals and $t, s_1, \ldots, s_m \geq 0$ are real numbers. Then the **big test ideal** $\tau_t(R; \Delta, a^t b_1^{s_1} \cdots b_m^{s_m})$ is defined to be the unique smallest non-zero ideal $J \subseteq R$ such that

\[
\phi \left( F^e_t(a^{(t^e-1)} b_1^{s_1 (p^e-1)}, \ldots, b_m^{s_m (p^e-1)}) \right) \subseteq J
\]

for every $e \geq 0$ and every $\phi \in \text{Hom}_R(F^e_tR((p^e-1)\Delta), R)$. This ideal always exists in the context described.

**Remark 2.2.** In the case that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, the **big test ideal** is known to equal the (finitistic) **test ideal** (which we will not define here); see [Tak04] and [BSTZ10] for details.
If all $b_i = R$, then we denote the associated big test ideal by $\tau_b(R; \Delta, \alpha^t)$. Likewise if $\Delta = 0$, then we denote the associated big test ideal using the notation $\tau_b(R; \alpha^t b_1^{s_1} \cdots b_m^{s_m})$. Finally, if the $b_i = (f_i)$ are principal, we denote the associated big test ideal by $\tau_b(R; \Delta, \alpha^t f_1^{s_1} \cdots f_m^{s_m})$.

**Remark 2.3.** Given a non-zero element $c \in \tau_b(R; \Delta, \alpha^t b_1^{s_1} \cdots b_m^{s_m})$ (such an element is called a big sharp test element), we note that

$$\tau_b(R; \Delta, \alpha^t b_1^{s_1} \cdots b_m^{s_m}) = \sum_{e \geq 0} \sum_{\phi} \left( F_\phi^c (ca^{(p^e-1)} b_1^{s_1(p^e-1)} \cdots b_m^{s_m(p^e-1)}) \right),$$

where the inner sum is over $\phi \in \text{Hom}_R(F_\phi^c R([(p^e - 1)\Delta]), R)$. To see this, simply note that the right side satisfies equation (1), and it is by definition the smallest ideal containing $c$ satisfying equation (1); note $a^0 = b_i^0 = R$.

Suppose that $X = \text{Spec } R$ is normal. Then given $\phi \in \text{Hom}_R(F_\phi^c R, R) \cong F_\phi^c \mathcal{O}_X((1-p^e)K_X)$, we may view $\phi$ as determining an effective Weil divisor linearly equivalent to $(1-p^e)K_X$.

**Definition 2.4.** We use $D_\phi$ to denote the Weil divisor associated to $\phi$ in this way.

Given an $\mathbb{R}$-divisor $\Delta \geq 0$ on $X$, one has an inclusion

$$\text{Hom}_R(F_\phi^c R([(p^e - 1)\Delta]), R) \subseteq \text{Hom}_R(F_\phi^c R, R).$$

The following lemma gives a nice interpretation of this submodule.

**Lemma 2.5.** An element $\phi \in \text{Hom}_R(F_\phi^c R, R)$ is contained inside the submodule $\text{Hom}_R(F_\phi^c R([(p^e - 1)\Delta]), R)$ if and only if $D_\phi \geq (p^e - 1)\Delta$.

**Proof.** Because all the modules are reflexive, the statement can be reduced to the case when $R$ is a discrete valuation ring and $\Delta = s \text{div}(x)$, where $x$ is the parameter for the DVR $R$ and $s \geq 0$ is a real number. In this case, the inclusion from equation (3) can be identified with the multiplication map $R \rightarrow R$ which sends 1 to $x^{[s(p^e-1)]}$. Thus, $\phi \in \text{Hom}_R(F_\phi^c R, R) \cong R$ is contained inside $\text{Hom}_R(F_\phi^c R([(p^e - 1)\Delta]), R) \cong x^{[s(p^e-1)]} R$ if and only if $D_\phi \geq [s(p^e-1)] \text{div}(x) = [(p^e - 1)\Delta]$. However, since $D_\phi$ is integral, it is harmless to remove the round-up $\lceil \cdot \rceil$.

### 3. Discreteness of $F$-jumping numbers

In this section we prove our main result. We accomplish this by perturbing our triples $(R, \Delta, \alpha^t)$ in order to reduce the discreteness statement to the case where the (log) $\mathbb{Q}$-Gorenstein index is not divisible by $p > 0$. First we need a lemma.

**Lemma 3.1.** Suppose that $(X = \text{Spec } R, \Delta, \alpha^t b_1^{s_1} \cdots b_m^{s_m})$ is a triple and that $\Delta = \Gamma + b \text{div}(f)$ for some $f \in R \setminus \{0\}$ and non-negative number $b \in \mathbb{R}$. Then

$$\tau_b(R; \Delta, \alpha^t b_1^{s_1} \cdots b_m^{s_m}) = \tau_b(R; \Gamma, f^b \alpha^t b_1^{s_1} \cdots b_m^{s_m}).$$

This type of statement is essentially obvious for multiplier ideals, but because of certain issues surrounding the construction of test ideals we have thus far presented, it is somewhat less obvious in this context. However, it is still quite straightforward, especially from the definition of the generalized test ideal by Hara-Yoshida-Takagi (the proof in that case uses the theory tight closure); see HY03 and Tak04. We provide a short proof here, certainly acknowledging that this statement is obvious to experts.
Thus, since \( \geq \)

Lemma 3.2 ([HT04, Theorem 4.1])

\[
\text{and so } \tau_x \in \Delta \text{, } a^t b_1^t \ldots b_m^t.
\]

We leave the proof to reader; see [HT04, Theorem 4.2] and [BSTZ10, Lemma 3.26]. \( \square \)

Now we can prove the following result.

Theorem 3.3. Suppose that \( R \) is an \( F \)-finite normal domain and further suppose that \( (X, \Delta, a^t) \) is a triple where \( K_X + \Delta \) is \( \mathbb{R} \)-Cartier. Then for each point \( x \in X \), there exists an open set \( U = \text{Spec } R' = \text{Spec } R[1/h] \) containing \( x \in X \).
with the following properties: There exists an effective \( \mathbb{Q} \)-divisor \( \Gamma \) on \( U \), elements \( f_1, \ldots, f_n \in R' \setminus \{0\} \) and non-negative real numbers \( b_1, \ldots, b_n \) such that

1. \( K_U + \Gamma \) is \( \mathbb{Q} \)-Cartier with index not divisible by \( p > 0 \), and, furthermore, \((p^e - 1)(K_U + \Gamma) \sim 0 \) for some integer \( e > 0 \).
2. The \( F \)-jumping numbers of \( \tau_b(U, \Delta|_U, (aR')^t) \) are the same as the \( F \)-jumping numbers of \( \tau_b(U, \Delta|_U, (aR')^t) \) (both sets of jumping numbers are with respect to \( t \)).

**Proof.** Choose a non-zero section \( \phi \) of \( \text{Hom}_R(F_R^eR, R) \) and set \( \Gamma = \frac{1}{p^e - 1} D_\phi \). It follows that \( K_X + \Gamma \) satisfies condition (1) on \( X \). Therefore, \((K_X + \Delta) - (K_X + \Gamma) = \Delta - \Gamma \) is \( \mathbb{R} \)-Cartier, and so we may write \( \Delta - \Gamma = d_1 D_1 + \cdots + d_m D_m \) for some integral effective Cartier divisors \( D_i \) and real numbers \( d_i \in \mathbb{R} \). We choose our open set \( U = \text{Spec } R[h^{-1}] = \text{Spec } R' \) to be any such set containing \( x \in X \) where all of the \( D_i|_U \) are principal divisors.

Now write \( D_i|_U = \text{div}(f_i) \) for some \( f_i \in R' \setminus \{0\} \) and also by abuse of notation denote \( \Gamma := \Gamma|_U \). Choose natural numbers \( l_i \) such that \( b_i := l_i + d_i > 0 \) for all \( i \) and set \( g := f_1^{l_1} \cdots f_m^{l_m} \in R' \). Notice that \((\Delta|_U + \text{div}(g)) - \Gamma = b_1 \text{div}(f_1) + \cdots + b_m \text{div}(f_m) \).

By Lemma 3.2 the \( F \)-jumping numbers of \( \tau_b(U; \Delta|_U, (aR')^t) \) and the \( F \)-jumping numbers of \( \tau_b(U; \Delta|_U + \text{div}(g), (aR')^t) \) coincide. Now using Lemma 3.1 we have

\[
\tau_b(U; \Delta|_U + \text{div}(g), (aR')^t) = \tau_b(U; \Gamma + b_1 \text{div}(f_1) + \cdots + b_m \text{div}(f_m), (aR')^t) = \tau_b(U; \Gamma, f_1^{l_1} \cdots f_m^{l_m} (aR')^t),
\]

which proves the theorem. \( \square \)

**Remark 3.4.** If, in Theorem 3.3 \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier, then one needs only a single \( f_1^{b_1} \) (and no other \( f_i^{b_i} \)). However, if the index of \( K_X + \Delta \) is divisible by \( p > 0 \), then it follows by construction that \( b_1 \) will be a rational number with its denominator divisible by \( p > 0 \).

We are now in a position to prove the discreteness of the \( F \)-jumping numbers in the case that \( X \) is essentially of finite type over a field. The proof idea follows the usual lines.

**Theorem 3.5.** Suppose that \( R \) is a normal domain essentially of finite type over an \( F \)-finite field. Further suppose that \( a \subseteq R \) is an ideal and \( \Delta \) is an \( \mathbb{R} \)-divisor on \( X = \text{Spec } R \) such that \( K_X + \Delta \) is \( \mathbb{R} \)-Cartier (for example, this holds if \( \Delta = 0 \) and \( R \) is \( \mathbb{Q} \)-Gorenstein). Then, as \( t \) varies, the \( F \)-jumping numbers of \( \tau_b(R; \Delta, a^t) \) have no limit points — they are discrete.

**Proof.** By [BSTZ10] Proposition 3.28, it is sufficient to answer this question on a finite affine cover of \( X \). Therefore, we reduce to the case that \( X \) is one of the charts from Theorem 3.3. In particular, it is sufficient to prove our result for triples of the form \( \tau_b(R; \Gamma, f_1^{b_1} \cdots f_m^{b_m} a^t) \), where \((p^e - 1)(K_X + \Gamma) \sim 0 \) for some \( e > 0 \). Using [BSTZ10] Lemma 4.2, Proposition 3.28, one can further assume that \( R \) is of finite type over an \( F \)-finite field of characteristic \( p > 0 \). One then has two options:

(a) Mimic the proof of the main result of [BSTZ10] Section 4. In other words, use the methods of \( F \)-adjunction (as worked out in [Sch09a] and [BSTZ10]) to reduce to the case where \( R \) is a polynomial ring and then use
degree bounding methods similar to those found in [BMS08]. Note that in [BSTZ10] one worked with triples \((R, \Delta, a^t)\) and not with the more complicated objects \((R, \Gamma, f_1^{b_1} \ldots f_m^{b_m} a^t)\), but the methods are easily generalized to our setting.

(b) Use the new language of [Bli09, Section 4]. We claim that the algebra of \(p^{-e}\)-linear maps associated to the triple \((R, \Gamma, f_1^{b_1} \ldots f_m^{b_m})\), as in [Sch09b, Remark 3.10], is “gauge bounded” (see [Bli09, Definition 4.7]). To see this claim, note that by [Sch09a, Lemma 3.9] or [Sch09b, Remark 4.4] the Cartier-algebra associated to \((R, \Gamma)\) is finitely generated and thus gauge bounded by [Bli09, Proposition 4.8]. It then follows from [Bli09, Proposition 4.13] that the Cartier-algebra associated to \((R, \Gamma, f_1^{b_1} \ldots f_m^{b_m})\) is also gauge bounded, as claimed. To finish the proof, apply [Bli09, Theorem 4.14]. In either case, the result follows easily from the theories previously developed. □

4. ON THE QUESTION OF RATIONALITY

Note that the usual way to prove the rationality of the \(F\)-jumping numbers employs the following theorem. First recall that a pair \((X, \Delta)\) is called \(\log \mathbb{Q}\)-Gorenstein with index \(n\) if \(n(K_X + \Delta)\) is Cartier and \(n > 0\) is the smallest integer with this property.

Theorem 4.1 ([BMS08, BSTZ10]). Suppose \((X, \Delta)\) is \(\log \mathbb{Q}\)-Gorenstein with index \(n\) such that \(n\) divides \((p^e - 1)\) for some fixed \(e > 0\). Further suppose that \(a\) is an ideal sheaf of \(X\). Then if \(t_0\) is an \(F\)-jumping number of \(\tau(X; \Delta, a^t)\), then \(p^e t_0\) is also an \(F\)-jumping number.

However, without the “index not divisible by \(p\)” assumption, this theorem is false. Consider the following example (which in some sense is typical by Remark 3.4).

Example 4.2. Set \(X = \mathbb{A}_k^1 = \text{Spec } k[x]\), \(\Delta = \frac{1}{p} \text{div}(x)\) and \(a = (x)\). Then the \(F\)-jumping numbers of \((X, \Delta, a^t) = (X, (x)^{1/p^e} a^t)\) with respect to \(t\) are

\[
p - 1, \quad \frac{2p - 1}{p}, \quad \frac{3p - 1}{p}, \quad \ldots
\]

In particular, \(p\) (or \(p^e\)) times any of them is not an \(F\)-jumping number.

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References


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