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A NOTE ON DISCRETENESS OF F-JUMPING NUMBERS

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ABSTRACT. Suppose that R is a ring essentially of finite type over a perfect field of characteristic $p > 0$ and that $\mathfrak{a} \subseteq R$ is an ideal. We prove that the set of F-jumping numbers of $\tau_b(R; \mathfrak{a}^t)$ has no limit points under the assumption that R is normal and $\mathbb{Q}$-Gorenstein – we make no assumption as to whether the $\mathbb{Q}$-Gorenstein index is divisible by p. Furthermore, we also show that the F-jumping numbers of $\tau_b(R; \Delta, \mathfrak{a}^t)$ are discrete under the more general assumption that $K_R + \Delta$ is $\mathbb{R}$-Cartier.

1. INTRODUCTION

The test ideal is an important and subtle object associated to ideals $\mathfrak{a}$ in positive characteristic rings R. It measures the singularities of both the ambient ring and the elements of the ideal; see [HY03]. While the test ideal was initially introduced in the celebrated theory of tight closure of Hochster and Huneke (see [HH90]), more recent interest in the test ideal has been in regards to its connection with the multiplier ideal – a fundamental invariant of higher dimensional algebraic geometry in characteristic zero; see for example [Tak06] or [MY09].

Given a normal ring R essentially of finite type over a perfect field of characteristic $p > 0$, an ideal $\mathfrak{a} \subseteq R$ and a real number $t \geq 0$, one can form the (big) test ideal $\tau_b(R; \mathfrak{a}^t)$ – an object which measures both algebraic and arithmetic properties of R and $\mathfrak{a}$. Inspired by the test ideal's close relation with the multiplier ideal $\mathcal{J}(R, \mathfrak{a}^t)$, people have studied the numbers t_i where $\tau_b(R; \mathfrak{a}^{t_i})$ changes. That is, people have studied the *F-jumping numbers* (see [MTW05]), real numbers which are by definition the $t_i > 0$ such that for every $\varepsilon > 0$,

$$\tau_b(R; \mathfrak{a}^{t_i - \varepsilon}) \neq \tau_b(R; \mathfrak{a}^{t_i}).$$

One easy to observe fact about multiplier ideals is that their jumping numbers are discrete and rational, at least when R is $\mathbb{Q}$-Gorenstein and normal; see [ELSV04]. Here, by discrete we mean that the set of jumping numbers with respect to a fixed ideal have no limit points. Because of this, various groups have recently worked to show that the F-jumping numbers of the test ideal are also discrete and rational; see [Har06], [BMS08], [BMS09], [KLZ09], and [BSTZ10]. In the most recent-mentioned work, the author, along with M. Blickle, S. Takagi, and W. Zhang, showed that

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the F-jumping numbers of test ideals formed a discrete set of rational numbers when R is normal and $\mathbb{Q}$-Gorenstein *with index not divisible by* $p > 0$. Recall that the *index* of a $\mathbb{Q}$-Gorenstein ring R is the smallest natural number n where $\omega_R^{(n)} = \mathcal{O}_{\operatorname{Spec} R}(nK_R)$ is locally free.

The most fundamental case left open is the case when R is $\mathbb{Q}$-Gorenstein but of arbitrary index; see [BSTZ10, Question 6.1]. We answer this question at least for discreteness.

Theorem 3.5. *Suppose that R is a normal domain essentially of finite type over an F-finite field. Further suppose that $\mathfrak{a} \subseteq R$ is an ideal and Δ is an $\mathbb{R}$-divisor on $X = \operatorname{Spec} R$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier (for example, this holds if $\Delta = 0$ and R is $\mathbb{Q}$-Gorenstein). Then, as t varies, the F-jumping numbers of $\tau_b(R; \Delta, \mathfrak{a}^t)$ have no limit points – they are discrete.*

We also point out why the existing proofs of *rationality* do not seem to work in the case that R is $\mathbb{Q}$-Gorenstein with index divisible by p.

Recently, in [DH09], de Fernex and Hacon gave a definition of the multiplier ideal without the $\mathbb{Q}$-Gorenstein assumption and asked the question of whether discreteness and rationality of the F-jumping numbers still holds in this context. Following this, Urbinati showed that rationality need not hold but gave some evidence that discreteness may hold in general; see [Urb10]. This suggests that one should not expect rationality to hold in positive characteristic either.

2. Definition of the test ideal

We only give a very brief description of the big test ideal in this paper. Please see [BSTZ10] for a more detailed description of the test ideal.

First we fix some notation. Given a ring R of characteristic $p > 0$ and M an R-module, we set $F^e_* M$ to be the R-module which agrees with M as an additive group but where the R-module structure is defined by the rule $r.m = r^{p^e} m$. Also recall that R is said to be *F-finite* if $F^e_* R$ is a finitely generated R-module.

Convention. Throughout this paper, all rings will be assumed to be F-finite.

Recall that an $\mathbb{R}$-divisor on a normal scheme X is a formal linear combination of prime Weil divisors D_i with real coefficients. An $\mathbb{R}$-divisor D is called *$\mathbb{R}$-Cartier* if it is equal to an $\mathbb{R}$-linear combination of Cartier divisors.

We now define the test ideal $\tau_b(R; \Delta, \mathfrak{a}^t \mathfrak{b}_1^{s_1} \dots \mathfrak{b}_m^{s_m})$. We work in this greater generality because when proving our main theorem, we perturb our initial triple $(R, \Delta, \mathfrak{a}^t)$ to a new triple $(R, \Delta', \mathfrak{a}^t \mathfrak{b}_1^{s_1} \dots \mathfrak{b}_m^{s_m})$ which has the same test ideal.

Definition 2.1 ([HH90], [Hoc07], [Sch10])**.** Suppose that R is an F-finite normal domain, $\Delta \geq 0$ is an $\mathbb{R}$-divisor on $X = \operatorname{Spec} R$, $\mathfrak{a}, \mathfrak{b}_1, \dots \mathfrak{b}_m \subseteq R$ are non-zero ideals and $t, s_1, \dots, s_m \geq 0$ are real numbers. Then the *big test ideal* $\tau_b(R; \Delta, \mathfrak{a}^t \mathfrak{b}_1^{s_1} \dots \mathfrak{b}_m^{s_m})$ is defined to be the unique smallest non-zero ideal $J \subseteq R$ such that

$$\phi\left(F^e_*(\mathfrak{a}^{\lceil t(p^e-1)\rceil} \mathfrak{b}_1^{\lceil s_1(p^e-1)\rceil} \dots \mathfrak{b}_m^{\lceil s_m(p^e-1)\rceil} J)\right) \subseteq J \tag{1}$$

for every $e \geq 0$ and every $\phi \in \operatorname{Hom}_R(F^e_* R(\lceil (p^e-1)\Delta \rceil), R)$. This ideal always exists in the context described.

Remark 2.2. In the case that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, the *big test ideal* is known to equal the *(finitistic) test ideal* (which we will not define here); see [Tak04] and [BSTZ10] for details.

If all $\mathfrak{b}_i = R$, then we denote the associated big test ideal by $\tau_b(R; \Delta, \mathfrak{a}^t)$. Likewise if $\Delta = 0$, then we denote the associated big test ideal using the notation $\tau_b(R; \mathfrak{a}^t \mathfrak{b}_1^{s_1} \dots \mathfrak{b}_m^{s_m})$. Finally, if the $\mathfrak{b}_i = (f_i)$ are principal, we denote the associated big test ideal by $\tau_b(R; \Delta, \mathfrak{a}^t f_1^{s_1} \dots f_m^{s_m})$

Remark 2.3. Given a non-zero element $c \in \tau_b(R; \Delta, \mathfrak{a}^t \mathfrak{b}_1^{s_1} \dots \mathfrak{b}_m^{s_m})$ (such an element is called a *big sharp test element*), we note that

$$(2)\quad \tau_b(R; \Delta, \mathfrak{a}^t \mathfrak{b}_1^{s_1} \dots \mathfrak{b}_m^{s_m}) = \sum_{e \geq 0} \sum_{\phi} \phi \left(F_*^e (c \mathfrak{a}^{\lceil t(p^e-1) \rceil} \mathfrak{b}_1^{\lceil s_1(p^e-1) \rceil} \dots \mathfrak{b}_m^{\lceil s_m(p^e-1) \rceil}) \right),$$

where the inner sum is over $\phi \in \operatorname{Hom}_R(F_*^e R(\lceil (p^e-1)\Delta \rceil), R)$. To see this, simply note that the right side satisfies equation (1), and it is by definition the smallest ideal containing c satisfying equation (1); note $\mathfrak{a}^0 = \mathfrak{b}_i^0 = R$.

Suppose that $X = \operatorname{Spec} R$ is normal. Then given $\phi \in \operatorname{Hom}_R(F_*^e R, R) \cong F_*^e \mathcal{O}_X((1-p^e)K_X)$, we may view ϕ as determining an effective Weil divisor linearly equivalent to $(1-p^e)K_X$.

Definition 2.4. We use D_ϕ to denote the Weil divisor associated to ϕ in this way.

Given an $\mathbb{R}$-divisor $\Delta \geq 0$ on X, one has an inclusion

$$(3)\qquad \operatorname{Hom}_R(F_*^e R(\lceil (p^e-1)\Delta \rceil), R) \subseteq \operatorname{Hom}_R(F_*^e R, R).$$

The following lemma gives a nice interpretation of this submodule.

Lemma 2.5. *An element $\phi \in \operatorname{Hom}_R(F_*^e R, R)$ is contained inside the submodule $\operatorname{Hom}_R(F_*^e R(\lceil (p^e-1)\Delta \rceil), R)$ if and only if $D_\phi \geq (p^e-1)\Delta$.*

Proof. Because all the modules are reflexive, the statement can be reduced to the case when R is a discrete valuation ring and $\Delta = s \operatorname{div}(x)$, where x is the parameter for the DVR R and $s \geq 0$ is a real number. In this case, the inclusion from equation (3) can be identified with the multiplication map $R \to R$ which sends 1 to $x^{\lceil s(p^e-1) \rceil}$. Thus, $\phi \in \operatorname{Hom}_R(F_*^e R, R) \cong R$ is contained inside $\operatorname{Hom}_R(F_*^e R(\lceil (p^e-1)\Delta \rceil), R) \cong x^{\lceil s(p^e-1) \rceil} R$ if and only if $D_\phi \geq \lceil s(p^e-1) \rceil \operatorname{div}(x) = \lceil (p^e-1)\Delta \rceil$. However, since D_ϕ is integral, it is harmless to remove the round-up $\lceil \cdot \rceil$. $\square$

3. Discreteness of F-jumping numbers

In this section we prove our main result. We accomplish this by perturbing our triples $(R, \Delta, \mathfrak{a}^t)$ in order to reduce the discreteness statement to the case where the (log) $\mathbb{Q}$-Gorenstein index is not divisible by $p > 0$. First we need a lemma.

Lemma 3.1. *Suppose that $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}^t \mathfrak{b}_1^{s_1} \dots \mathfrak{b}_m^{s_m})$ is a triple and that $\Delta = \Gamma + b \operatorname{div}(f)$ for some $f \in R \setminus \{0\}$ and non-negative number $b \in \mathbb{R}$. Then*

$$\tau_b(R; \Delta, \mathfrak{a}^t \mathfrak{b}_1^{s_1} \dots \mathfrak{b}_m^{s_m}) = \tau_b(R; \Gamma, f^b \mathfrak{a}^t \mathfrak{b}_1^{s_1} \dots \mathfrak{b}_m^{s_m}).$$

This type of statement is essentially obvious for multiplier ideals, but because of certain issues surrounding the construction of test ideals we have thus-far presented, it is somewhat less obvious in this context. However, it is still quite straightforward, especially from the definition of the generalized test ideal by Hara-Yoshida-Takagi (the proof in that case uses the theory tight closure); see [HY03] and [Tak04]. We provide a short proof here, certainly acknowledging that this statement is obvious to experts.

Proof. Choose c to be a big sharp test element for both $(X, \Delta, \mathfrak{a}^t\mathfrak{b}_1^{s_1}\dots\mathfrak{b}_m^{s_m})$ and $(X, \Gamma, f^b\mathfrak{a}^t\mathfrak{b}_1^{s_1}\dots\mathfrak{b}_m^{s_m})$. Then we know that the test ideal $\tau_b(R; \Gamma, f^b\mathfrak{a}^t\mathfrak{b}_1^{s_1}\dots\mathfrak{b}_m^{s_m})$ equals

$$\begin{aligned}
&\sum_{\phi,\, D_\phi \geq (p^e-1)\Gamma} \phi\left(F_*^e c f^{\lceil b(p^e-1)\rceil}\mathfrak{a}^{\lceil t(p^e-1)\rceil}\mathfrak{b}_1^{\lceil s_1(p^e-1)\rceil}\dots\mathfrak{b}_m^{\lceil s_m(p^e-1)\rceil}\right)\\
&= \sum_{\phi,\, D_\phi \geq (p^e-1)\Gamma + \operatorname{div} f^{\lceil b(p^e-1)\rceil}} \phi\left(F_*^e c \mathfrak{a}^{\lceil t(p^e-1)\rceil}\mathfrak{b}_1^{\lceil s_1(p^e-1)\rceil}\dots\mathfrak{b}_m^{\lceil s_m(p^e-1)\rceil}\right)\\
&\subseteq \sum_{\phi,\, D_\phi \geq (p^e-1)\Gamma + \lceil b(p^e-1)\operatorname{div}(f)\rceil} \phi\left(F_*^e c \mathfrak{a}^{\lceil t(p^e-1)\rceil}\mathfrak{b}_1^{\lceil s_1(p^e-1)\rceil}\dots\mathfrak{b}_m^{\lceil s_m(p^e-1)\rceil}\right)\\
&\subseteq \sum_{\phi,\, D_\phi \geq (p^e-1)\Gamma + b(p^e-1)\operatorname{div}(f)} \phi\left(F_*^e c \mathfrak{a}^{\lceil t(p^e-1)\rceil}\mathfrak{b}_1^{\lceil s_1(p^e-1)\rceil}\dots\mathfrak{b}_m^{\lceil s_m(p^e-1)\rceil}\right)\\
&= \tau_b(R; \Delta, \mathfrak{a}^t\mathfrak{b}_1^{s_1}\dots\mathfrak{b}_m^{s_m}),
\end{aligned}$$

and so $\tau_b(R; \Gamma, f^b\mathfrak{a}^t\mathfrak{b}_1^{s_1}\dots\mathfrak{b}_m^{s_m}) \subseteq \tau_b(R; \Delta, \mathfrak{a}^t\mathfrak{b}_1^{s_1}\dots\mathfrak{b}_m^{s_m})$. For the converse inclusion, observe first that

$$\operatorname{div}(f^{\lceil b(p^e-1)\rceil}) - b(p^e-1)\operatorname{div}(f) \leq \operatorname{div}(f).$$

Thus, since cf is also a test element,

$$\begin{aligned}
&\tau_b(R, \Delta, \mathfrak{a}^t\mathfrak{b}_1^{s_1}\dots\mathfrak{b}_m^{s_m}) = \sum_{\phi,\, D_\phi \geq (p^e-1)\Delta} \phi\left(F_*^e c f \mathfrak{a}^{\lceil t(p^e-1)\rceil}\mathfrak{b}_1^{\lceil s_1(p^e-1)\rceil}\dots\mathfrak{b}_m^{\lceil s_m(p^e-1)\rceil}\right)\\
&= \sum_{\phi,\, D_\phi \geq (p^e-1)\Gamma + b(p^e-1)\operatorname{div}(f)} \phi\left(F_*^e c f \mathfrak{a}^{\lceil t(p^e-1)\rceil}\mathfrak{b}_1^{\lceil s_1(p^e-1)\rceil}\dots\mathfrak{b}_m^{\lceil s_m(p^e-1)\rceil}\right)\\
&= \sum_{\phi,\, D_\phi \geq (p^e-1)\Gamma + b(p^e-1)\operatorname{div}(f) + \operatorname{div}(f)} \phi\left(F_*^e c \mathfrak{a}^{\lceil t(p^e-1)\rceil}\mathfrak{b}_1^{\lceil s_1(p^e-1)\rceil}\dots\mathfrak{b}_m^{\lceil s_m(p^e-1)\rceil}\right)\\
&\subseteq \sum_{\phi,\, D_\phi \geq (p^e-1)\Gamma + \operatorname{div}(f^{\lceil b(p^e-1)\rceil})} \phi\left(F_*^e c \mathfrak{a}^{\lceil t(p^e-1)\rceil}\mathfrak{b}_1^{\lceil s_1(p^e-1)\rceil}\dots\mathfrak{b}_m^{\lceil s_m(p^e-1)\rceil}\right)\\
&= \sum_{\phi,\, D_\phi \geq (p^e-1)\Gamma} \phi\left(F_*^e c f^{\lceil b(p^e-1)\rceil}\mathfrak{a}^{\lceil t(p^e-1)\rceil}\mathfrak{b}_1^{\lceil s_1(p^e-1)\rceil}\dots\mathfrak{b}_m^{\lceil s_m(p^e-1)\rceil}\right)\\
&= \tau_b(R, \Gamma, f^b\mathfrak{a}^t\mathfrak{b}_1^{s_1}\dots\mathfrak{b}_m^{s_m}),
\end{aligned}$$

and so $\tau_b(R; \Delta, \mathfrak{a}^t\mathfrak{b}_1^{s_1}\dots\mathfrak{b}_m^{s_m}) \subseteq \tau_b(R; \Gamma, f^b\mathfrak{a}^t\mathfrak{b}_1^{s_1}\dots\mathfrak{b}_m^{s_m})$, as desired. □

We also need a very special case of Skoda's theorem.

Lemma 3.2 ([HT04, Theorem 4.1])**.** *Suppose that $X = \operatorname{Spec} R$, $\Delta > 0$, $\mathfrak{a} \subseteq R$ and $t \geq 0$ is as above. Further suppose that $f \in R$ is a non-zero element. Then*

$$\tau_b(X; \Delta + \operatorname{div}(f), \mathfrak{a}^t\mathfrak{b}_1^{s_1}\dots\mathfrak{b}_m^{s_m}) = f\tau_b(X; \Delta, \mathfrak{a}^t\mathfrak{b}_1^{s_1}\dots\mathfrak{b}_m^{s_m}).$$

Proof. We leave the proof to reader; see [HT04, Theorem 4.2] and [BSTZ10, Lemma 3.26]. □

Now we can prove the following result.

Theorem 3.3. *Suppose that R is an F-finite normal domain and further suppose that $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}^t)$ is a triple where $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then for each point $x \in X$, there exists an open set $U = \operatorname{Spec} R' = \operatorname{Spec} R[h^{-1}]$ containing $x \in X$*

with the following properties: There exists an effective $\mathbb{Q}$-divisor Γ on U, elements $f_1, \dots f_m \in R' \setminus \{0\}$ and non-negative real numbers $b_1, \dots, b_m$ such that

(1) *$K_U + \Gamma$ is $\mathbb{Q}$-Cartier with index not divisible by $p > 0$, and, furthermore, $(p^e - 1)(K_U + \Gamma) \sim 0$ for some integer $e > 0$.*
(2) *The F-jumping numbers of $\tau_b(U, \Gamma, f_1^{b_1} \dots f_m^{b_m}(\mathfrak{a}R')^t)$ are the same as the F-jumping numbers of $\tau_b(U, \Delta|_U, (\mathfrak{a}R')^t)$ (both sets of jumping numbers are with respect to t).*

Proof. Choose a non-zero section ϕ of $\operatorname{Hom}_R(F^e_* R, R)$ and set $\Gamma = \frac{1}{p^e-1} D_\phi$. It follows that $K_X + \Gamma$ satisfies condition (1) on X. Therefore, $(K_X + \Delta) - (K_X + \Gamma) = \Delta - \Gamma$ is $\mathbb{R}$-Cartier, and so we may write $\Delta - \Gamma = d_1 D_1 + \dots + d_m D_m$ for some integral effective Cartier divisors D_i and real numbers $d_i \in \mathbb{R}$. We choose our open set $U = \operatorname{Spec} R[h^{-1}] = \operatorname{Spec} R'$ to be any such set containing $x \in X$ where all of the $D_i|_U$ are principal divisors.

Now write $D_i|_U = \operatorname{div}(f_i)$ for some $f_i \in R' \setminus 0$ and also by abuse of notation denote $\Gamma := \Gamma|_U$. Choose natural numbers l_i such that $b_i := l_i + d_i > 0$ for all i and set $g := f_1^{l_1} \dots f_m^{l_m} \in R'$. Notice that $(\Delta|_U + \operatorname{div}(g)) - \Gamma = b_1 \operatorname{div}(f_1) + \dots + b_m \operatorname{div}(f_m)$.

By Lemma 3.2, the F-jumping numbers of $\tau_b(U; \Delta|_U, (\mathfrak{a}R')^t)$ and the F-jumping numbers of $\tau_b(U; \Delta|_U + \operatorname{div}(g), (\mathfrak{a}R')^t)$ coincide. Now using Lemma 3.1 we have

$$\begin{aligned} &\tau_b(U; \Delta|_U + \operatorname{div}(g), (\mathfrak{a}R')^t) \\ =\, &\tau_b(U; \Gamma + b_1 \operatorname{div}(f_1) + \dots + b_m \operatorname{div}(f_m), (\mathfrak{a}R')^t) \\ =\, &\tau_b(U; \Gamma, f_1^{b_1} \dots f_m^{b_m} (\mathfrak{a}R')^t), \end{aligned}$$

which proves the theorem. □

Remark 3.4. If, in Theorem 3.3, $K_X + \Delta$ is $\mathbb{Q}$-Cartier, then one needs only a single $f_1^{b_1}$ (and no other $f_i^{b_i}$). However, if the index of $K_X + \Delta$ is divisible by $p > 0$, then it follows by construction that b_1 will be a rational number with its denominator divisible by $p > 0$.

We are now in a position to prove the discreteness of the F-jumping numbers in the case that X is essentially of finite type over a field. The proof idea follows the usual lines.

Theorem 3.5. *Suppose that R is a normal domain essentially of finite type over an F-finite field. Further suppose that $\mathfrak{a} \subseteq R$ is an ideal and Δ is an $\mathbb{R}$-divisor on $X = \operatorname{Spec} R$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier (for example, this holds if $\Delta = 0$ and R is $\mathbb{Q}$-Gorenstein). Then, as t varies, the F-jumping numbers of $\tau_b(R; \Delta, \mathfrak{a}^t)$ have no limit points – they are discrete.*

Proof. By [BSTZ10, Proposition 3.28], it is sufficient to answer this question on a finite affine cover of X. Therefore, we reduce to the case that X is one of the charts from Theorem 3.3. In particular, it is sufficient to prove our result for triples of the form $\tau_b(R; \Gamma, f_1^{b_1} \dots f_m^{b_m} \mathfrak{a}^t)$, where $(p^e - 1)(K_X + \Gamma) \sim 0$ for some $e > 0$. Using [BSTZ10, Lemma 4.2, Proposition 3.28], one can further assume that R is of finite type over an F-finite field of characteristic $p > 0$. One then has two options:

(a) Mimic the proof of the main result of [BSTZ10, Section 4]. In other words, use the methods of F-adjunction (as worked out in [Sch09a] and [BSTZ10]) to reduce to the case where R is a polynomial ring and then use degree bounding methods similar to those found in [BMS08]. Note that in [BSTZ10] one worked with triples $(R, \Delta, \mathfrak{a}^t)$ and not with the more complicated objects $(R, \Gamma, f_1^{b_1} \dots f_m^{b_m} \mathfrak{a}^t)$, but the methods are easily generalized to our setting.

(b) Use the new language of [Bli09, Section 4]. We claim that the algebra of p^{-e}-linear maps associated to the triple $(R, \Gamma, f_1^{b_1} \dots f_m^{b_m})$, as in [Sch09b, Remark 3.10], is "gauge bounded" (see [Bli09, Definition 4.7]). To see this claim, note that by [Sch09a, Lemma 3.9] or [Sch09b, Remark 4.4] the Cartier-algebra associated to (R, Γ) is finitely generated and thus gauge bounded by [Bli09, Proposition 4.8]. It then follows from [Bli09, Proposition 4.13] that the Cartier-algebra associated to $(R, \Gamma, f_1^{b_1} \dots f_m^{b_m})$ is also gauge bounded, as claimed. To finish the proof, apply [Bli09, Theorem 4.14].

In either case, the result follows easily from the theories previously developed. $\square$

4. On the question of rationality

Note that the usual way to prove the rationality of the F-jumping numbers employs the following theorem. First recall that a pair (X, Δ) is called *log $\mathbb{Q}$-Gorenstein with index n* if $n(K_X + \Delta)$ is Cartier and $n > 0$ is the smallest integer with this property.

Theorem 4.1 ([BMS08], [BSTZ10])**.** *Suppose (X, Δ) is log $\mathbb{Q}$-Gorenstein with index n such that n divides $(p^e - 1)$ for some fixed $e > 0$. Further suppose that $\mathfrak{a}$ is an ideal sheaf of X. Then if t_0 is an F-jumping number of $\tau(X; \Delta, \mathfrak{a}^t)$, then $p^e t_0$ is also an F-jumping number.*

However, without the "index not divisible by p" assumption, this theorem is false. Consider the following example (which in some sense is typical by Remark 3.4).

Example 4.2. Set $X = \mathbb{A}^1_k = \operatorname{Spec} k[x]$, $\Delta = \frac{1}{p}\operatorname{div}(x)$ and $\mathfrak{a} = (x)$. Then the F-jumping numbers of $(X, \Delta, \mathfrak{a}^t) = (X, (x)^{1/p}\mathfrak{a}^t)$ with respect to t are

$$\frac{p-1}{p}, \frac{2p-1}{p}, \frac{3p-1}{p}, \dots.$$

In particular, p (or p^e) times any of them is not an F-jumping number.

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