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SOBOLEV ESTIMATES FOR THE LOCAL EXTENSION OF $\bar{\partial}_b$-CLOSED $(0,1)$-FORMS ON REAL HYPERSURFACES IN $\mathbb{C}^n$ WITH TWO POSITIVE EIGENVALUES

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Abstract. Let $\mathcal{M}$ be a smooth real hypersurface in complex space of dimension $n \geq 3$, and assume that the Levi-form at z_0 on $\mathcal{M}$ has at least two positive eigenvalues. We estimate solutions of the local $\bar{\partial}$-closed extension problem near z_0 for $(0,1)$-forms in Sobolev spaces. Using this result, we estimate the local solution of tangential Cauchy-Riemann equations near z_0 for $(0,1)$-forms in Sobolev spaces.

1. Introduction

For a set $W \subset \mathbb{C}^n$, we denote the vector space of smooth (p,q)-forms on W by $\bigwedge^{p,q}(W)$. Let $\mathcal{M}$ be a smooth real hypersurface in $\mathbb{C}^n$ with a smooth defining function ρ, and let $\mathcal{B}^{p,q}(\mathcal{M})$ be the restriction of $\bigwedge^{p,q}(\mathbb{C}^n)$ to $\mathcal{M}$ which is pointwise orthogonal to the ideal generated by $\bar{\partial}\rho$. In the sequel, we let $z_0 \in \mathcal{M}$ be a fixed point and let V be a neighborhood of z_0 in $\mathbb{C}^n$ where ρ is defined. For each open set $U \subset V$, $z_0 \in U$, we set $U^- = \{z \in U; \rho(z) \leq 0\}$ and $U^+ = \{z \in U; \rho(z) \geq 0\}$.

If there exists a neighborhood $U \subset V$, $z_0 \in U$, such that for any $\alpha \in \mathcal{B}^{p,q}(\mathcal{M} \cap U)$ with $\bar{\partial}_b \alpha = 0$ on $\mathcal{M} \cap U$, there exist a smooth (p,q)-form $\tilde{\alpha} \in \bigwedge^{p,q}(U^-)$ with $\bar{\partial}\tilde{\alpha} = 0$ in U^- and $(\tilde{\alpha} - \alpha) \wedge \bar{\partial}\rho = 0$ on $\mathcal{M} \cap U$, then we say the one-sided weak $\bar{\partial}$-closed extension problem is locally solvable.

The $\bar{\partial}$-closed extension problem and the solvability of the tangential Cauchy-Riemann equation for functions have origins in the two papers by Hans Lewy [17, 18]. In [14], Kohn and Rossi introduced the $\bar{\partial}_b$-complex and extended these problems to differential forms and solved the global $\bar{\partial}$-closed extension problem for forms from $\mathcal{M} = b\Omega$ to the domain Ω. Later, these extension problems were studied by several authors [13, 16, 19, 20].

For the local extension problem, Andreotti and Hill [2] solved the local weak $\bar{\partial}$-closed extension problem when the Levi-form at $z_0 \in \mathcal{M}$ satisfies the condition $Y(q)$. Under the same assumption, Boggess and Shaw [4] proved the same result

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using the integral kernel method. Note that the estimates of the solutions of the local extension problem or local solvability of the $\bar{\partial}_b$-equation in various spaces such as in C^k, L^p, Lipschitz and in Sobolev spaces, have many applications in the study of complex analysis, for example, the embeddability of abstract CR structures [1, 5, 6, 15, 24].

In [21], Shaw showed the local solvability of $\bar{\partial}_b$ in L^p spaces when $\mathcal{M}$ is the boundary of strongly pseudoconvex domains in $\mathbb{C}^n$ and in the Sobolev spaces when $\mathcal{M}$ is the boundary of pseudoconvex domains of finite type [22]. In [23], Shaw also constructed homotopy formulas for $\bar{\partial}_b$ using integral kernel methods and showed the local solvability for $\bar{\partial}_b$ in Hölder space when the Levi-form at $z_0 \in \mathcal{M}$ has at least $(q+2)$ positive and $(q+2)$ negative eigenvalues. In [10], the author obtained estimates for the local one-sided extension problem and proved the local solvability of $\bar{\partial}_b$ equations for (p,q)-forms with estimates in the Sobolev spaces when the Levi-form has at least $(q+2)$ positive eigenvalues (and hence $n \geq 4$).

It would be much better if we could prove the above result under the assumption that the Levi-form at $z_0 \in \mathcal{M}$ has only $(q+1)$ positive eigenvalues. We are especially interested in the case where $q = 1$ and $n \geq 3$. If $n \geq 4$, and if we have only two positive eigenvalues (not a mixed one), we note that the condition $Y(q)$, for $q = 1$, is not satisfied.

In this paper, we prove the above extension problem for $(0,1)$-forms with estimates in Sobolev spaces when the Levi-form at $z_0 \in \mathcal{M}$ has at least 2 positive eigenvalues. For a set $W \subset \mathbb{C}^n$, we denote the Sobolev norm of order s on W by $\|\cdot\|_{s,W}$.

Theorem 1.1. *Let $\mathcal{M}$ be a smooth hypersurface in $\mathbb{C}^n$, $n \geq 3$, with smooth defining function ρ, and suppose that the Levi-form at $z_0 \in \mathcal{M}$ has at least 2 positive eigenvalues. Then there is a neighborhood U of z_0 such that for any neighborhood $U_0 \supset\supset U$, and for any $\alpha \in \mathcal{B}^{0,1}(\mathcal{M} \cap U_0)$ satisfying $\bar{\partial}_b\alpha = 0$ on $\mathcal{M} \cap U$, there exists $\tilde{\alpha} \in \Lambda^{0,1}(U^-)$ such that $\bar{\partial}\tilde{\alpha} = 0$ on U^- and $(\tilde{\alpha} - \alpha) \wedge \bar{\partial}\rho = 0$ on $\mathcal{M} \cap U$. Also, if we let $\chi \in C_0^\infty(U_0)$ with $\chi = 1$ on U, $\tilde{\alpha}$ satisfies the estimate*

$$\|\tilde{\alpha}\|_{s,U^-} \leq C_s \|\chi\alpha\|_{s+1,\mathcal{M}}, \tag{1.1}$$

for each real $0 \leq s < 1/2$, and

$$\|\tilde{\alpha}\|_{s,U^-} \leq C_s \|\chi\alpha\|_{s+1/2,\mathcal{M}}, \tag{1.2}$$

for each real $s \geq 1/2$, where C_s does not depend on α.

Note that the local $\bar{\partial}$-closed extension problem and the local solvability of the $\bar{\partial}_b$ equation are closely related [19, 20]. We also note that there are well-known nonsolvability results of the tangential Cauchy Riemann equation for $n = 2$ [18] and for $q = n - 1$ [11]. Using the results of Theorem 1.1, we solve the local $\bar{\partial}_b$ equation in Sobolev spaces.

Theorem 1.2. *Let $\mathcal{M}$ be a smooth hypersurface in $\mathbb{C}^n$, $n \geq 3$, and suppose that the Levi-form at $z_0 \in \mathcal{M}$ has at least 2 positive eigenvalues, and $\mathcal{M}$ is pseudoconvex near z_0. Then there is a neighborhood U of z_0 such that for any $\alpha \in \mathcal{B}^{0,1}(\mathcal{M} \cap U)$ such that $\bar{\partial}_b\alpha = 0$ on $\mathcal{M} \cap U$, and for any $W \subset\subset U$, and for each real $s \geq 0$, there exists $u_s \in \mathcal{B}^{(0,0)}(\mathcal{M} \cap W)$ such that $\bar{\partial}_b u_s = \alpha$ on $\mathcal{M} \cap W$ and satisfies the estimate*

$$\|u_s\|_{s,\mathcal{M}\cap W} \leq C_s \|\alpha\|_{s+1,\mathcal{M}\cap U}.$$

Remark 1.3. In Theorem 1.1, the differentiability assumption $\alpha \in C^\infty$ can be weakened to $\alpha \in H^{s+1}(\mathcal{M}\cap U)$ (or $\alpha \in H^{s+1/2}(\mathcal{M}\cap U)$ if $s \geq 1/2$) to get $\tilde\alpha \in H^s(M\cap U^-)$, and similarly for Theorem 1.2.

2. Preliminaries

Let Ω be a domain in $\mathbb{C}^n$ with smooth boundary, let $\bar\partial$ be the Cauchy-Riemann operator on Ω, and let ϑ be the formal adjoint of $\bar\partial$. Also, we let $N_{(p,q)}$ denote the Neumann operator for (p,q)-forms. The Hodge star operator $* : \bigwedge^{p,q}(\Omega) \to \bigwedge^{n-p,n-q}(\Omega)$ is defined by the equation $\psi\wedge *\phi =< \psi,\phi > dV$ where dV is the volume form on Ω. We set

$$\mathcal{C}^{p,q}(\Omega) = \{\phi \in \bigwedge^{p,q}(\Omega); \phi\wedge\bar\partial\rho = 0 \text{ on } b\Omega\},$$

where ρ is a defining function for Ω.

Let I be an open ball in $\mathbb{R}^d$, let $|I|$ denote the diameter of I, and let $H_{s,l}(\Omega\times I)$ be the Sobolev space of order s on Ω and of order l in I with the norm denoted by $\|\cdot\|_{s,l}$. We state a theorem for smooth dependence of solutions of the $\bar\partial$-Neumann problem with respect to a parameter $\tau \in I$ [9].

Theorem 2.1. *Let $\{\Omega_\tau\}_{\tau\in I}$ be a smooth family of diffeomorphic strongly pseudoconvex domains in $\mathbb{C}^n$ and suppose that $\{\alpha_\tau\}_{\tau\in I}$ is a family of (p,q)-forms on $\{\Omega_\tau\}_{\tau\in I}$ such that $\alpha_\tau \in R(\bar\partial_\tau)$, the range of $\bar\partial_\tau$, for each $\tau \in I$, where $|I|$ is sufficiently small. Then for each real number $s \geq -1/2$ and for each nonnegative integer l, there is $C_{s,l} > 0$ such that the Neumann solution U of $\Box U = \alpha$ and the canonical solution $u = \bar\partial^* U$ of $\bar\partial u = \alpha$ on $\{\Omega_\tau\}_{\tau\in I}$ satisfy*

$$\|U\|_{s+1,l}, \|u\|_{s+1/2,l} \leq C_{s,l}\sum_{r=0}^{l}\|\alpha\|_{s+l-r,r}, \tag{2.1}$$

where $\alpha \in H_{s+l-r,r}(\Omega\times I)$, $0 \leq r \leq l$.

Remark 2.2. Note that the weak extension problem is a Cauchy problem to preserve the boundary values in a tangential direction. This means that we have to solve the $\bar\partial^*$-equation instead of the $\bar\partial$-equation. Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^k$ and let $\alpha \in \mathcal{B}^{p,q}(b\Omega)$, where $0 \leq p \leq k$ and $1 \leq q \leq k-1$. Then a necessary and sufficient condition for the extension problem to be solved is

$$\int_{b\Omega}\alpha\wedge h = 0, \tag{2.2}$$

for every $h \in \bigwedge^{k-p,k-q-1}(\Omega)\cap Ker(\bar\partial)$, and (2.2) is equivalent to the condition $\bar\partial_b\alpha = 0$ for $q \leq k-2$. Then the explicit formula for the extension is given by $\tilde\alpha = \alpha' - u$, where α' is a smooth extension of α onto Ω, and $u = -*\bar\partial N_{(n-p,n-q)}*\bar\partial\alpha'$.

In the sequel, we let $B_\epsilon(z)$ be a ball of radius $\epsilon > 0$ centered at z. Also, for $z \in \mathbf{C}^n$, we write $z = (z', z'')$ where $z' \in \mathbb{C}^2$ and $z'' \in \mathbb{C}^{n-2}$. In [8], we filled a neighborhood of $z_0 = (z_0', z_0'') \in \mathcal{M}$ by strongly convex domains $\{\Omega_{z''}\}_{B_\sigma(z_0'')}$ whose diameters converge to zero as $z'' \to z_0''$. Therefore the estimates of the Neumann operators are not uniform in Sobolev spaces. For a remedy of this non-uniformity, we shall fill a neighborhood of z_0 in a different manner. That is, we slice the domain by subdomains of complex dimension two starting at $z_0 \in \mathcal{M}$ as follows:

Following the holomorphic changes of coordinates as in Section 2 of [10], we have special coordinates $z=(z',z'')$, $z'=(z_1,z_2)$, $z''=(z_3,\dots,z_n)$, $z(z_0)=(0,\dots,0,1)$, and a new local defining function $\tilde{\rho}(z)$, which can be written as

$$\tilde{\rho}(z)=|z_n|^2-1+\sum_{i=1}^{2}|z_i|^2+\sum_{i=3}^{n-1}\lambda_i|z_i|^2+\mathcal{O}(|z-z_0|^3), \tag{2.3}$$

near $z_0=(0,\dots,0,1)$. Let $\sigma>0$ be a small constant to be determined. We regard $z'':=t''=(t_3,\dots,t_n)\in\mathbb{C}^{n-2}$ as a parameter variable near $t''_0:=z''_0=(0,\dots,0,1)\in\mathbb{C}^{n-2}$ and we construct a family of strongly convex domains:

For each $|z'|<\sigma^{1/4}$ and $|t''-t''_0|<\sigma^{1/4}$, set $\rho(z',t'')=\tilde{\rho}(z',t_3,\dots,t_{n-1},t_n-\sigma^{1/3}z_1)$. In view of (2.3), we can write

$$\begin{aligned}\rho(z',t'')=&(|t_n|^2-1)-2\sigma^{1/3}Re(z_1\bar{t}_n)+\sigma^{2/3}|z_1|^2+\sum_{i=1}^{2}|z_i|^2\\&+\sum_{i=3}^{n-1}\lambda_i|t_i|^2+\mathcal{O}(|z-z_0|^3).\end{aligned}$$

For $t''\in B_\sigma(t''_0)$, we set

$$r^\sigma_{t''}(z'):=(1-|t_n|^2)+2\sigma^{1/3}Re(z_1\bar{t}_n)-\sum_{i=3}^{n-1}\lambda_i|t_i|^2+\mathcal{O}(|z-z_0|^3). \tag{2.4}$$

At $t''=t''_0=(0,\dots,0,1)$, i.e., at the center of the ball $B_\sigma(t''_0)$, we have

$$r^\sigma_{t''_0}(z')=2\sigma^{1/3}Re(z_1)+\mathcal{O}(|z'|^3),$$

because $t_n=1$. Therefore it follows that

$$\Omega_{t''_0}:=\{z'\in B_{\sigma^{1/4}}(z'_0);\sigma^{2/3}|z_1|^2+\sum_{i=1}^{2}|z_i|^2<r^\sigma_{t''_0}(z')\}$$

is a nonempty strongly convex domain contained in the side of $\rho\le 0$ because, for example, $(\sigma^{3/8},0)\in\Omega_{t''_0}$ provided $\sigma>0$ is sufficiently small. Note that $\Omega_{t''_0}$ is a small deformation of a ball whose radius is larger than or equal to $\sigma^{17/48}$. Also we see that $z_0\in b\Omega_{t''_0}\subset\mathcal{M}$ and $\Omega_{t''_0}$ is the central slice of the side $\rho\le 0$.

Similarly, for each $t''\in B_\sigma(t''_0)$, we set

$$\Omega_{t''}:=\{z'\in B_{\sigma^{1/4}}(z'_0);\sigma^{2/3}|z_1|^2+\sum_{i=1}^{2}|z_i|^2<r^\sigma_{t''}(z)\}.$$

Since $|t_n-1|\le\sigma$, we see that the second term in the right-hand side of (2.4), $2\sigma^{1/3}Re(z_1\bar{t}_n)$, is the major term. As above, $\Omega_{t''}\subset\{z'\in B_{\sigma^{1/4}}(z'_0);\rho(z',t'')\le 0\}$ and $b\Omega_{t''}\subset\mathcal{M}$, and $\Omega_{t''}$ is a small deformation of a ball whose radius is larger than or equal to $\sigma^{17/46}$.

Remark 2.3. Note that $Rez_1\lesssim\sigma^{1/3}$ if $z'\in\Omega_{t''}$ and this necessitates that $|z'|\lesssim\sigma^{1/3}$; that is, $\Omega_{t''}\subset B_{\sigma^{7/24}}(z_0)\subset B_{\sigma^{1/4}}(z_0)$ provided σ is sufficiently small.

Remark 2.4. With the special coordinates $z=(z',z'')$ defined in (2.3), set $D_\sigma=\{z\in\mathbb{C}^n;|z_j|<\sigma^{7/24},\ 1\le j\le n-1,|z_n-1|<\sigma^{1/4}\}$. Then $U^-\subset D^-_\sigma$. Also note that $r^\sigma_{t''}<0$, and hence $\Omega_{z''}$ is an empty set for appropriate $t''=z''=(z_3,\dots,z_n)$, say, when $\sigma^{7/24} < |z_n - 1| < \sigma^{1/4}$ and $|z_j| < \sigma$, $j = 3, \ldots, n-1$, provided $\sigma > 0$ is sufficiently small.

Remark 2.5. Let us fix $\sigma = \sigma_0$ satisfying the above conditions and set $I := B_{\sigma_0}(t_0'')$. Then $\{\Omega_{z''}\}_{z'' \in I}$ is a family of bounded strongly convex domains in $\mathbb{C}^2$ and they are a small deformation of a ball of radius bigger than $\sigma_0^{17/48}$ (and hence its diameters are bounded uniformly from below). Also, $\Omega_{z''}$ is diffeomorphic to $\Omega_{z_0''}$ for each $z'' \in I$ and foliate in the part $\rho \le 0$. Define a tubular neighborhood U^- of z_0 by

$$U^- := \bigcup_{z'' \in I} \overline{\Omega}_{z''} \times \{z''\}. \tag{2.5}$$

3. $\bar{\partial}$-closed extension for $(p,1)$-forms

To prove the local extension theorem, we use the local decomposition of the set U^- considered in Remark 2.5. We will solve the ϑ-equation and use the estimates (2.1) on parameter variables $z'' \in I$, where I is defined as in Remark 2.5. Set $\mathcal{K} = \{1,2\}$ and $\mathcal{K}^c = \{3, \cdots, n\}$. For a smooth function f defined in $\mathbb{C}^n$, we define

$$\bar{\partial}_{\mathcal{K}} f = \sum_{j=1}^{2} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \quad \text{and} \quad \bar{\partial}_{\mathcal{K}^c} f = \sum_{j=3}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.$$

We can extend this definition for arbitrary smooth forms.

Since p does not play an important role in the estimates, we set $p = 0$; i.e., we consider only the cases $\bigwedge^{0,1}(W)$ where W is an appropriate set. We recall that $\|\cdot\|_{s,k,W}$ is the Sobolev norm of order s in the $z' \in \mathbb{C}^2$ variable and of order k in the $z'' \in I = B_{\sigma_0}(z'') \subset \mathbb{C}^{n-2}$ variable. In the sequel, every summation will be over strictly increasing indices, and the constants such as C_s, $C_{s,k}$ will depend only on s or k and can vary line to line while we estimate.

Proposition 3.1. *Let $\mathcal{M}$ be a smooth real hypersurface in $\mathbb{C}^n$, $n \ge 3$, with smooth defining function ρ defined in a neighborhood of $z_0 \in \mathcal{M}$, and suppose that the Levi-form at z_0 has at least 2 positive eigenvalues. Then there is a neighborhood U of z_0 such that for any neighborhood $U_0 \supset\supset U$ and for any $\alpha \in \mathcal{B}^{0,1}(\mathcal{M} \cap U_0)$ satisfying $\bar{\partial}_b \alpha = 0$ on $\mathcal{M} \cap U$, there exist $\tilde{\alpha}_l \in \Lambda^{0,1}(U_0^-)$, $l = 0,1,2$, such that $(\tilde{\alpha}_l - \alpha) \wedge \bar{\partial}\rho = 0$ on $\mathcal{M} \cap U$ and $\bar{\partial}\tilde{\alpha}_l$ can be written as*

$$\bar{\partial}\tilde{\alpha}_l = \sum_{\substack{I \subset \mathcal{K}, J \subset \mathcal{K}^c \\ |I|+|J|=2, |J| \ge l}} \alpha_{IJ}^l d\bar{z}^I \wedge d\bar{z}^J \tag{3.1}$$

on U^- for some smooth functions α_{IJ}^l. Also, if we let $\chi \in C_0^\infty(U_0)$ with $\chi = 1$ on U, then for each real $s \ge 0$, $\tilde{\alpha}_l$ satisfies the estimates

$$\begin{aligned} &\|\tilde{\alpha}_l\|_{s,U^-} \le C_s \|\chi\alpha\|_{s+\frac{l-1}{2},\mathcal{M}}, \quad \text{for } l = 0,1, \text{ and} \\ &\|\tilde{\alpha}_2\|_{s,U^-} \le C_s \|\chi\alpha\|_{s+1,\mathcal{M}}. \end{aligned} \tag{3.2}$$

Also if $s \ge \frac{1}{2}$, we have

$$\|\tilde{\alpha}_2\|_{s,U^-} \le C_s \|\chi\alpha\|_{s+1/2,\mathcal{M}}. \tag{3.3}$$

Proof. Let V be a neighborhood of $z_0 \in \mathcal{M}$ where special frames are defined, and let $I = B_{\sigma_0}(z_0'')$ be the neighborhood of $z_0'' \in \mathbb{C}^{n-2}$ defined in Remark 2.5. By shrinking $\sigma_0 > 0$ if necessary, we may assume that there is a neighborhood U of $z_0 \in \mathcal{M}$ such that Theorem 2.1 holds on U where $U \subset D_{\sigma_0} \subset B_{\sigma_0^{1/4}}(z_0) \subset V$, and

$U^- = \bigcup_{z''\in I} \overline{\Omega}_{z''} \times \{z''\}$, where D_{σ_0} is defined in Remark 2.4. For any U_0 with $U \subset\subset U_0 \subset D_{\sigma_0}$, we choose a smooth cutoff function $\chi \in C_0^\infty(U_0)$ with $\chi = 1$ on U.

Assume $\alpha \in \mathcal{B}^{0,1}(\mathcal{M} \cap U_0)$ and $\bar\partial_b \alpha = 0$ on $\mathcal{M} \cap U$. Using Theorem 2.1, we shall construct $\tilde\alpha_l$ inductively so that $\tilde\alpha_l$ satisfies (3.1) and (3.2). If we replace α by $\chi\alpha$, then we may assume that $\alpha \in C_0^\infty(\mathcal{M} \cap U_0)$. From Lemma 9.3.4 in [7], there is $\tilde\alpha_0 := E\alpha \in \bigwedge^{0,1}(U_0)$, $E\alpha = \alpha$ and $\bar\partial E\alpha = \mathcal{O}(\rho^\infty)$ on $\mathcal{M} \cap U$ such that

$$\|\tilde\alpha_0\|_{s,U^-} \le C_s \|\chi\alpha\|_{s-1/2,\mathcal{M}}, \tag{3.4}$$

for each real $s \ge 0$. This proves (3.1) and (3.2) for $l = 0$.

Replacing $\tilde\alpha_0$ by $\chi\tilde\alpha_0$, we assume that $\tilde\alpha_0 \in C_0^\infty(U_0)$. Let us write

$$\bar\partial\tilde\alpha_0 = \sum_{1\le j<k\le n} \alpha^0_{jk} d\bar z_j \wedge d\bar z_k,$$

and set

$$g^0 = \alpha^0_{12} d\bar z_1 \wedge d\bar z_2 \ \text{ on } \ U^-, \ \text{ and } \ g^0 = 0 \ \text{ on } \ U^+.$$

Then it follows that $g^0 \in C^\infty(U)$ because $\bar\partial\tilde\alpha_0 = \mathcal{O}(\rho^\infty)$ on $\mathcal{M} \cap U$. Also note that g^0 comes from the components of $\bar\partial_{\mathcal{K}}\tilde\alpha_0$. Hence for each real $s \ge -1$, and nonnegative integer k, there are $\tilde C_{s,k}$ and $C_{s,k}$ such that

$$\|g^0\|_{s,k,U^-} \le \tilde C_{s,k} \|\tilde\alpha_0\|_{s+1,k,U^-} \le C_{s,k} \|\chi\alpha\|_{s+k+1/2,\mathcal{M}}. \tag{3.5}$$

To remove the g^0 term in $\bar\partial\tilde\alpha_0$, we try to solve $\bar\partial_{\mathcal{K}} u = g^0(\cdot, z'')$, for each $z'' \in I$, and set $\alpha_1 = \tilde\alpha_0 - u$. However, to preserve the boundary condition, it is required that $u \in \mathcal{C}^{0,1}(\Omega_{z''})$ for each $z'' \in I$. This means that we have to solve the $\bar\partial^*$-equation rather than the $\bar\partial$-equation.

Since $g^0(\cdot, z'')$ is a $(0,2)$-form, it becomes a top degree problem in $\mathbb{C}^2$, and hence it is required to satisfy (2.2); that is,

$$F_h(z'') := \int_{\Omega_{z''}} g^0(\cdot, z'') \wedge h = 0, \quad \text{for every} \ \ h \in C^\infty_{(2,0)}(\mathbb{C}^2) \cap ker(\bar\partial_{\mathcal{K}}). \tag{3.6}$$

To prove (3.6), we consider the coefficients of the term $d\bar z_1 \wedge d\bar z_2 \wedge d\bar z_k$ in $\bar\partial^2\tilde\alpha_0$. Then we see that

$$\frac{\partial\alpha^0_{12}}{\partial\bar z_k} - \frac{\partial\alpha^0_{1k}}{\partial\bar z_2} + \frac{\partial\alpha^0_{2k}}{\partial\bar z_1} = 0, \tag{3.7}$$

for $3 \le k \le n$. Since $g^0(\cdot, z'') = 0$ for z'' with $(z', z'') \in U^+ \subset D_\sigma$, it follows that F_h has compact support in $\mathbb{C}^{n-2}$. Therefore it follows from (3.7) that

$$\begin{aligned}\frac{\partial F_h(z'')}{\partial\bar z_k} &= \int_{\mathbb{C}^2} \frac{\partial g^0(\cdot, z'')}{\partial\bar z_k} \wedge h = \int_{\mathbb{C}^2} \left(\frac{\partial\alpha^0_{1k}}{\partial\bar z_2} - \frac{\partial\alpha^0_{2k}}{\partial\bar z_1}\right) \wedge h \\ &= -\int_{\mathbb{C}^2} \left(\alpha^0_{1k}\frac{\partial h}{\partial\bar z_2} - \alpha^0_{2k}\frac{\partial h}{\partial\bar z_1}\right) = 0\end{aligned}$$

for $3 \le k \le n$. In view of Remark 2.4, it follows that $F_h(z'') = 0$ for $z'' = (z_3, \dots, z_n)$ with $\sigma^{7/24} < |z_n - 1| < \sigma^{1/4}$, provided σ is sufficiently small. Therefore we see that (3.6) holds.

Set $u_0 = - *_{\mathcal{K}} \bar\partial_{\mathcal{K}} N^{\mathcal{K}}_{(2,0)} *_{\mathcal{K}} g^0(\cdot, z'')$ where $N^{\mathcal{K}}_{(p,q)}$ is the Neumann operator for (p,q)-forms, and $*_{\mathcal{K}}$ is the Hodge star operator on $\Omega_{z''}$. Then we have $\overline{\partial}_{\mathcal{K}} u_0 = g^0$

and $u_0 \in \mathcal{C}^{0,1}(\Omega_{z''})$ for each z''. Also for each real $s \geq 0$ and nonnegative integer k, it follows from (2.1), (3.4) and (3.5) that

$$\|u_0\|_{s,k,U^-} \lesssim \sum_{r=0}^{k} \|g^0\|_{s-1/2+k-r,r,U^-} \lesssim \|\tilde\alpha_0\|_{s+1/2+k,U^-} \lesssim \|\chi\alpha\|_{s+k,\mathcal{M}}. \tag{3.8}$$

Note that $u_0 \wedge \bar\partial_{\mathcal{K}}\rho = 0$ on $\mathcal{M}\cap U$ because $u_0 \in \mathcal{C}^{0,1}(\Omega_{z''})$ for each z''. We have to correct u_0 so that the corrected one, $\tilde u_0$, belongs to $\mathcal{C}^{0,1}(\Omega)$; that is, $\tilde u_0 \wedge \bar\partial\rho = 0$ on $\mathcal{M}\cap U$. Since $u_0 \wedge \bar\partial_{\mathcal{K}}\rho = 0$ on $\mathcal{M}\cap U$, we can write

$$u_0 = \delta^0 \wedge \bar\partial_{\mathcal{K}}\rho + \rho\gamma^0$$

for some $\delta^0 \in \mathcal{C}^{0,0}(\Omega_{z''})$ and $\gamma^0 \in \mathcal{C}^{0,1}(\Omega_{z''})$. We may assume that $\delta^0 \wedge \bar\partial_{\mathcal{K}}\rho$ and $\rho\gamma^0$ are disjoint, and hence it follows from the estimate in (3.8) that

$$\|\delta^0\|_{s,k,U^-} \lesssim \|u_0\|_{s,k,U^-} \lesssim \|\chi\alpha\|_{s+k,\mathcal{M}} \tag{3.9}$$

for each real $s \geq 0$ and nonnegative integer k.

Set $\tilde u_0 = u_0 + \delta^0 \wedge \bar\partial_{\mathcal{K}^c}\rho$. Then one obtains that

$$\tilde u_0 \wedge \bar\partial\rho = \left(\delta^0 \wedge \bar\partial_{\mathcal{K}}\rho + \rho\gamma^0 + \delta^0 \wedge \bar\partial_{\mathcal{K}^c}\rho\right) \wedge \bar\partial\rho = 0$$

on $\mathcal{M}\cap U$ and hence $\tilde u_0 \in \mathcal{C}^{0,1}(U^-)$. Now we set $\tilde\alpha_1 = \tilde\alpha_0 - \tilde u_0$. Then it follows that $(\tilde\alpha_1 - \alpha)\wedge\partial\rho = -\tilde u_0 \wedge \partial\rho = 0$ on $U\cap\mathcal{M}$, and we can write

$$\bar\partial\tilde\alpha_1 = \sum_{3\leq k\leq n}\left(\alpha^1_{1k}d\bar z_1 + \alpha^1_{2k}d\bar z_2\right)\wedge d\bar z_k + \sum_{3\leq i<j\leq n}\alpha^1_{ij}d\bar z_i \wedge d\bar z_j,$$

for some smooth functions α^1_{ij}.

In view of (3.4), (3.8) and (3.9) one obtains that

$$\|\tilde\alpha_1\|_{s,k,U^-} \lesssim \|\chi\alpha\|_{s+k,\mathcal{M}} \tag{3.10}$$

for each real $s \geq 0$ and nonnegative integer k. Thus (3.2) holds for $l = 1$ by the Sobolev interpolation theorem.

For each $3 \leq k \leq n$, set $g_k = \alpha^1_{1k}d\bar z_1 + \alpha^1_{2k}d\bar z_2$. If we consider the coefficients of $\bar\partial^2\tilde\alpha_1 = 0$, we see that $\bar\partial_{\mathcal{K}}g_k = 0$; that is, g_k is a $\bar\partial_{\mathcal{K}}$ closed $(0,1)$-form in $\mathbb{C}^2$. Since $\vartheta_{\mathcal{K}} = -*_{\mathcal{K}}\bar\partial_{\mathcal{K}}*_{\mathcal{K}}$, it follows that $*_{\mathcal{K}}g_k$ is a $\vartheta_{\mathcal{K}}$-closed $(2,1)$-form in $\mathbb{C}^2$. Set $u^k(\cdot,z'') = -*_{\mathcal{K}}\bar\partial_{\mathcal{K}}N^{\mathcal{K}}_{(2,1)}*_{\mathcal{K}}g_k(\cdot,z'')$. Then we have $\bar\partial_{\mathcal{K}}u^k = g_k$ and $u^k \in \mathcal{C}^{0,0}(\Omega_{z''})$ for each $z'' \in I$, and hence $u^k = 0$ on $U\cap\mathcal{M}$.

Set $\tilde u_1 = \sum_{k=3}^n u^k d\bar z_k$. Note that each g_k comes from the components of $\bar\partial\tilde\alpha_1$. Thus, for each real $s \geq 0$ and nonnegative integer k, it follows from (2.1) and (3.10) that

$$\|\tilde u_1\|_{s,k,U^-} \lesssim \sum_{r=0}^{k}\left(\|\tilde\alpha_1\|_{s+1/2+k-r,r,U^-} + \|\tilde\alpha_1\|_{s-1/2+k-r,r+1,U^-}\right) \lesssim \|\chi\alpha\|_{s+1+k,\mathcal{M}}. \tag{3.11}$$

Here we lose $1/2$ derivative because we need $s - 1/2 + k - r \geq 0$ to use (3.10). On the other hand, if $s \geq 1/2$, we have

$$\|\tilde u_1\|_{s,k,U^-} \lesssim \|\chi\alpha\|_{s+1/2+k,\mathcal{M}}. \tag{3.12}$$

Set $\tilde\alpha_2 = \tilde\alpha_1 - \tilde u_1$. Then we can write

$$\bar\partial\tilde\alpha_2 = \sum_{3\leq i<j\leq n}\alpha^2_{ij}d\bar z_i \wedge d\bar z_j,$$

for some smooth functions α^2_{ij}. It also follows that $(\tilde{\alpha}_2 - \tilde{\alpha}_1) \wedge \bar{\partial}\rho = -\tilde{u}_1 \wedge \bar{\partial}\rho = 0$ on $\mathcal{M} \cap U$ because $\tilde{u}_1 = 0$ on $\mathcal{M} \cap U$. In view of (3.11), we obtain that

$$\|\tilde{\alpha}_2\|_{s,k,U^-} \lesssim \|\chi\alpha\|_{s+1+k,\mathcal{M}}, \tag{3.13}$$

for each real $s \geq 0$ and nonnegative integer k.

Using (3.13) and the interpolation theorem in Sobolev spaces, we see that (3.2) holds for each real $s \geq 0$. Also, if $s \geq 1/2$, we see from (3.12) that (3.3) holds. This proves Proposition 3.1. □

Now we are ready to prove the weak $\bar{\partial}$-closed extension problem (Theorem 1.1). We recall that $U^- = \bigcup_{z'' \in I} \overline{\Omega}_{z''} \times \{z''\}$ as defined in (2.5).

Proof of Theorem 1.1. In view of Proposition 3.1, there exists $\tilde{\alpha}_2 \in \Lambda^{0,1}(U^-)$ such that

$$\bar{\partial}\tilde{\alpha}_2 = \sum_{3 \leq i < j \leq n} \tilde{\alpha}_{ij} d\bar{z}_i \wedge d\bar{z}_j, \tag{3.14}$$

for some smooth functions $\tilde{\alpha}_{ij}$. If we consider the coefficients of $d\bar{z}_l \wedge d\bar{z}_i \wedge d\bar{z}_j$ of $\bar{\partial}^2\tilde{\alpha}_2 = 0$, we see that

$$\frac{\partial \tilde{\alpha}_{ij}}{\partial \bar{z}_l} = 0,$$

for $l = 1, 2$. Hence $\tilde{\alpha}_{ij}$ is a holomorphic function on $\Omega_{z''}$ for each z''.

Also, note that $(\tilde{\alpha}_2 - \alpha) \wedge \bar{\partial}\rho = 0$ on $\mathcal{M} \cap U$. Therefore $\bar{\partial}(\tilde{\alpha}_2 - \alpha) \wedge \bar{\partial}\rho = 0$ and hence $\bar{\partial}\tilde{\alpha}_2 \wedge \bar{\partial}\rho = 0$ on $\mathcal{M} \cap U$ because $\bar{\partial}_b\alpha = 0$ on $\mathcal{M} \cap U$. Considering the coefficients of $d\bar{z}_l \wedge d\bar{z}_i \wedge d\bar{z}_j$ of $\bar{\partial}\tilde{\alpha}_2 \wedge \bar{\partial}\rho$, $3 \leq i < j \leq n$, one obtains, from (3.14), that

$$\frac{\partial \rho}{\partial \bar{z}_l}\tilde{\alpha}_{ij} = 0,$$

for $l = 1, 2$ on $\mathcal{M} \cap U$. Since $\bar{\partial}_{\mathcal{K}}\rho \neq 0$ on $b\Omega_{z''}$, at least one of $\partial\rho/\partial\bar{z}_l$ for $l = 1, 2$ is not equal to zero and hence $\tilde{\alpha}_{ij} = 0$ on $b\Omega_{z''}$. Since $\tilde{\alpha}_{ij}$ is a holomorphic function on $\Omega_{z''}$ for each $z'' \in I$, we see that $\tilde{\alpha}_{ij} \equiv 0$ and hence $\bar{\partial}\tilde{\alpha}_2 = 0$ on U^- by (3.14). If we set $\tilde{\alpha} = \tilde{\alpha}_2$, then $\tilde{\alpha}$ satisfies the estimates (1.1) and (1.2) by (3.2). This proves Theorem 1.1. □

Proof of Theorem 1.2. Let U^- be the neighborhood constructed in Theorem 1.1 and $\alpha \in \mathcal{B}^{0,1}(\mathcal{M} \cap U)$ satisfies $\bar{\partial}_b\alpha = 0$ on $\mathcal{M} \cap U$. By shrinking U if necessary, we may assume that $\mathcal{M} \cap U$ is pseudoconvex. Assume $W \subset\subset U$, $z_0 \in W$, and choose U_1 such that $W \subset\subset U_1 \subset\subset U$. In view of Theorem 1.1, there is a weak $\bar{\partial}$-closed extension $\tilde{\alpha}$ of α onto U_1^- satisfying the estimate (1.1) and (1.2) on U_1^-. By a minor modification of the proof in the lemma in section 4 of [3], we can construct a small pseudoconvex domain $B \subset U_1^-$ such that $W \cap \mathcal{M} \subset B \cap \mathcal{M}$.

For each real $s \geq 0$, set $\tilde{u}_s = \bar{\partial}^* N_s^B \tilde{\alpha}$ where N_s^B denotes the weighted $\bar{\partial}$-Neumann operator in B with weight $e^{-t_s|z|^2}$ for sufficiently large $t_s > 0$ depending on s. Then we have $\bar{\partial}\tilde{u}_s = \tilde{\alpha}$ in B, and it follows that

$$\|\tilde{u}_s\|_{s,B} \leq C_s\|\tilde{\alpha}\|_{s,B}. \tag{3.15}$$

Set $u_s = \tau\tilde{u}_s$ where τ is the projection in $\bigwedge^{0,1}(\mathcal{M})$ onto $\mathcal{B}^{0,1}(\mathcal{M})$ defined by first restricting a $(0,1)$-form ϕ in $\mathbf{C}^n$ to $\mathcal{M}$, then projecting the restriction to $\mathcal{B}^{p,1}(\mathcal{M})$.

Then $\bar\partial_b u_s = \alpha$ on W, and if we use the estimates (1.1), (1.2), (3.15) and the trace theorem in Sobolev spaces, we obtain that

$$\|u_s\|_{s,W} \le C_s\|D\tilde u_s\|_{s-1/2,B} \le C_s\|\tilde\alpha\|_{s+1/2,B} \le C_s\|\chi\alpha\|_{s+1,\mathcal{M}},$$

for each real $s \ge 0$, where $\chi \in C_0^\infty(U)$ satisfies $\chi = 1$ on $U_1 \subset\subset U$. This completes the proof of Theorem 1.2. □

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