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INVERTIBLE LINEAR MAPS ON SIMPLE LIE ALGEBRAS PRESERVING COMMUTATIVITY

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ABSTRACT. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank l over an algebraically closed field of characteristic zero. An invertible linear map φ on $\mathfrak{g}$ is called preserving commutativity in both directions if, for any $x, y \in \mathfrak{g}$, $[x, y] = 0 \Leftrightarrow [\varphi(x), \varphi(y)] = 0$. The group of all such maps on $\mathfrak{g}$ is denoted by $Pzp(\mathfrak{g})$. It is shown in this paper that, if $l = 1$, then $Pzp(\mathfrak{g}) = GL(\mathfrak{g})$; otherwise, $Pzp(\mathfrak{g}) = Aut(\mathfrak{g}) \times F^*I_{\mathfrak{g}}$, where $F^*I_{\mathfrak{g}}$ denotes the group of all nonzero scalar multiplication maps on $\mathfrak{g}$.

1. INTRODUCTION

A lot of attention has been paid to the commutativity preserver problem on associative F-algebras, particularly on matrix algebras. The earliest paper on such problems dates back to 1976, when Watkins [9] studied commutativity preserving maps on the full matrix algebra $M_n(F)$ over a field F. If $n \geq 3$, then every invertible linear commutativity preserving map φ on $M_n(F)$ was shown to be one of the two standard forms: $\varphi(A) = cTAT^{-1} + f(A)I,\ A \in M_n$, or $\varphi(A) = cTA^tT^{-1} + f(A)I,\ A \in M_n$. Here c is a nonzero element in F, T an invertible matrix, and f a linear function on $M_n(F)$. In 1999, Marcoux et al. [6] described commutativity preserving maps on $T_n(F)$ of all upper triangular matrices, and in 2002, Cao et al. [2] determined commutativity preserving maps on $N_n(F)$ of all strictly upper triangular matrices. It was Omladič who first considered commutativity preserving maps on infinite-dimensional algebras. In [7], he considered such maps on $\mathfrak{B}(X)$ of all linear operators on an infinite-dimensional Banach space X. In 1993, Brešar [1] improved Omladič's result by using a ring theoretic approach called commuting mappings. Recently, Šemrl [8] studied non-linear maps on $M_n(F)$ preserving commutativity, and he found that, without the linear condition, these types of maps can have wild behaviors. For a Lie algebra L with the bracket operation $[\cdot,\cdot]$, we say that x commutes with y if $[x, y] = 0$. An invertible linear map φ on L is called *preserving commutativity* in both directions if for any $x, y \in L$, $[x, y] = 0 \Leftrightarrow [\varphi(x), \varphi(y)] = 0$. One easily sees that such maps behave like automorphisms only on pairs of commuting elements. All such maps form a group under composition of maps, which we denote by $Pzp(L)$. Searching in the literature, we find that it was in 1981, when

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Wong [10] studied invertible linear maps on Lie algebras preserving commutativity. It is regrettable that he only studied such maps on simple Lie algebras of linear types. Now a problem is posed naturally:

- *What form does an invertible linear map on an arbitrary simple Lie algebra preserving commutativity have?*

Inspired by Wong's paper, we will attempt to answer the above question. What we obtain can be viewed as a generalization of Wong's result.

In this paper, the notations concerning Lie algebras follow mainly from [3]–[4]. Let F be an algebraically closed field of characteristic zero, let $\mathfrak{g}$ be an arbitrary finite-dimensional simple Lie algebra over F of rank l, let $\mathfrak{h}$ be a fixed Cartan subalgebra of $\mathfrak{g}$, let $\Phi \subseteq \mathfrak{h}^*$ be the corresponding root system of $\mathfrak{g}$, let Δ be a fixed base of Φ, and let Φ^+ (resp.,Φ^-) be the set of positive (resp., negative) roots relative to Δ. The roots in Δ are called *simple*. A root β can be written as $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$ with $k_\alpha \in \mathbb{Z}$. The integer $\sum_{\alpha\in\Delta} k_\alpha$ is called the *height* of β and is denoted by $ht\ \beta$. The root system Φ has a unique maximal root which we denote by θ. For $\alpha \in \Phi$, let $\mathfrak{g}_\alpha$ be the root space of $\mathfrak{g}$ relative to α, $\mathfrak{n} = \sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$, $\mathfrak{n}^- = \sum_{\alpha\in\Phi^-} \mathfrak{g}_\alpha$, and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. We denote by $ker\ \alpha$, for $\alpha \in \Phi$, the kernel of α in $\mathfrak{h}$. For each $\alpha \in \Phi^+$, let e_α be a non-zero element of $\mathfrak{g}_\alpha$. Then there is a unique element $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $e_\alpha, e_{-\alpha}, h_\alpha = [e_\alpha, e_{-\alpha}]$ span a three-dimensional simple subalgebra of $\mathfrak{g}$ isomorphic to $sl(2, F)$ via $e_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_{-\alpha} \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The set $\{h_\alpha, e_\beta, e_{-\beta} \mid \alpha \in \Delta, \beta \in \Phi^+\}$ forms a basis of $\mathfrak{g}$. If $\alpha, \beta, \alpha+\beta \in \Phi$, then $[e_\alpha, e_\beta]$ is a scalar multiple of $e_{\alpha+\beta}$ since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. We define $N_{\alpha,\beta}$ by $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$, which we call the *structure constants* of $\mathfrak{g}$. We can choose a basis $\{h_\alpha, e_\beta, e_{-\beta} \mid \alpha \in \Delta, \beta \in \Phi^+\}$ of $\mathfrak{g}$ such that all structure constants of $\mathfrak{g}$ are integers. We call this a *Chevalley basis* of $\mathfrak{g}$. In the remainder of this paper, the set $\{h_\alpha, e_\beta, e_{-\beta} \mid \alpha \in \Delta, \beta \in \Phi^+\}$ will always denote a Chevalley basis of $\mathfrak{g}$. For the base Δ of Φ, let $\mathfrak{d}_\Delta = \{d_\alpha \mid \alpha \in \Delta\}$ be the dual basis of $\mathfrak{h}$ relative to Δ. Namely, $\beta(d_\alpha)$ takes the value 0 when $\beta \neq \alpha \in \Delta$ and takes the value 1 when $\beta = \alpha \in \Delta$. A symmetric bilinear form (,) is defined on the l-dimensional real vector space spanned by Φ, which is dual to the Killing form on $\mathfrak{g}$. For $\alpha, \beta \in \Phi$, let $\langle \beta, \alpha \rangle = 2(\beta, \alpha)/(\alpha, \alpha)$. If $\alpha \neq \pm\beta$, let p, q be the greatest non-negative integers for which $\beta - p\alpha, \beta + q\alpha \in \Phi$. Then

$$\langle \beta, \alpha \rangle = p - q; \quad N_{\alpha,\beta} = \pm(p+1). \tag{1.0.1}$$

The main result of this paper is as follows:

Theorem 1.1. *Let $\mathfrak{g}$ be an arbitrary finite-dimensional simple Lie algebra of rank l over an algebraically closed field F of characteristic zero. Then*

(i) $Pzp(\mathfrak{g}) = GL(\mathfrak{g})$, *if* $l = 1$;

(ii) $Pzp(\mathfrak{g}) = Aut(\mathfrak{g}) \times F^* I_{\mathfrak{g}}$, *if* $l \geq 2$. □

Before giving the proof of the main theorem, we introduce several types of standard maps on $\mathfrak{g}$ preserving commutativity in both directions.

(i) For a nilpotent element x in $\mathfrak{g}$, the map $exp(ad\ x)$ is an automorphism of $\mathfrak{g}$. The group generated by all such automorphisms is called the inner automorphism group of $\mathfrak{g}$, which we denote by $Int(\mathfrak{g})$, and each element in it is called an *inner automorphism* of $\mathfrak{g}$. In particular, for any $\alpha \in \Phi$ and any $t \in F$, te_α is nilpotent in $\mathfrak{g}$, so the map $exp(ad\ te_\alpha)$, denoted by $\sigma_\alpha(t)$, belongs to $Int(\mathfrak{g})$. We denote by G the subgroup of $Int(\mathfrak{g})$ generated by the elements $\sigma_\alpha(t)$ for all $\alpha \in \Phi$, $t \in F$.

It is well known that G coincides with $Int(\mathfrak{g})$ (see [5], page 288). For $\alpha \in \Phi$, let X_α be the subgroup of G of the elements $\sigma_\alpha(t)$ for all $t \in F$, which we call the *root subgroup* of G relative to α. Let $\langle X_\alpha, X_{-\alpha}\rangle$ be the subgroup of G generated by X_α and $X_{-\alpha}$. For $\alpha \in \Phi^+$, there exists a homomorphism ϕ_α from the special linear group $SL(2,F)$ into $\langle X_\alpha, X_{-\alpha}\rangle$, sending $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ to $\sigma_\alpha(a)$ and $\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ to $\sigma_{-\alpha}(b)$. Set $\chi_\alpha(c) = \phi_\alpha \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}$ for $c \in F^*$, $\omega_\alpha = \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then $\omega_\alpha = \sigma_\alpha(1)\sigma_{-\alpha}(-1)\sigma_\alpha(1)$. We define H to be the subgroup of G generated by the elements $\chi_\alpha(c)$ for all $\alpha \in \Phi$, $c \in F^*$, N the subgroup of G generated by H together with the elements ω_α for all $\alpha \in \Phi$. We denote by $\mathscr{W}$ the Weyl group of $\mathfrak{g}$ generated by the reflections w_α, $\alpha \in \Phi$. There exists a homomorphism from N onto $\mathscr{W}$ with kernal H. Thus N/H is isomorphic to $\mathscr{W}$.

(ii) If ρ is a symmetry (nontrivial or trivial) of the Dynkin diagram of Φ, or equivalently, $\langle \rho(\alpha), \rho(\beta)\rangle = \langle \alpha, \beta\rangle$ for any $\alpha, \beta \in \Delta$, then ρ can be extended to an automorphism $\bar\rho$ of Φ in the following way:

$$\sum_{\alpha\in\Delta} k_\alpha \alpha \in \Phi \mapsto \sum_{\alpha\in\Delta} k_\alpha \rho(\alpha).$$

Using $\bar\rho$ we can define an automorphism φ_ρ of $\mathfrak{g}$ in the way:

$$\sum_{\alpha\in\Delta} a_\alpha h_\alpha + \sum_{\alpha\in\Phi} b_\alpha e_\alpha \mapsto \sum_{\alpha\in\Delta} a_\alpha h_{\rho(\alpha)} + \sum_{\alpha\in\Phi} b_\alpha r_\alpha e_{\bar\rho(\alpha)}, \qquad a_\alpha, b_\alpha \in F,$$

where $r_\alpha = \pm 1$ and it satisfies the following two conditions: (1) $r_\alpha r_{-\alpha} = 1$; (2) $N_{\bar\rho(\alpha),\bar\rho(\beta)} r_\alpha r_\beta = N_{\alpha,\beta} r_{\alpha+\beta}$ if $\alpha+\beta$ is also a root. We call φ_ρ a *graph automorphism* of $\mathfrak{g}$.

(iii) Let $P = \mathbb{Z}\Phi$ be the set of all $\mathbb{Z}$-linear combinations of the elements of Φ. It is a free abelian group of rank l and has Δ as its basis. A homomorphism λ from the additive group P into the multiplicative group F^* of non-zero elements of F is called a *character* of P. Each character λ of P gives rise to an automorphism φ_λ of $\mathfrak{g}$ by

$$h + \sum_{\alpha\in\Phi} x_\alpha e_\alpha \mapsto h + \sum_{\alpha\in\Phi} x_\alpha \lambda(\alpha) e_\alpha, \quad h \in \mathfrak{h},\ x_\alpha \in F.$$

φ_λ is called a *diagonal automorphism* of $\mathfrak{g}$. Actually, each φ_λ is just an inner automorphism of $\mathfrak{g}$ (see [3] or [5] for details).

(iv) For $c \in F^*$, define

$$\varphi_c : \mathfrak{g} \to \mathfrak{g}, \qquad x \mapsto cx, \quad \forall\ x \in \mathfrak{g}.$$

We call φ_c a *scalar multiplication map* on $\mathfrak{g}$.

It is clear that all the standard maps defined above are respectively invertible linear maps on $\mathfrak{g}$ preserving commutativity in both directions. Later on we will prove that each $\varphi \in Pzp(\mathfrak{g})$ is conversely a composition of the standard maps when $l \geq 2$.

2. Some elementary results

For a subalgebra $\mathfrak{s}$ and a subset A of $\mathfrak{g}$, we denote by $C_{\mathfrak{s}}(A)$ the centralizer of A in $\mathfrak{s}$. The terms $C_{\mathfrak{g}}(A)$ and $C_{\mathfrak{g}}(C_{\mathfrak{g}}(A))$ are abbreviated to A' and A'', respectively. If $\varphi \in Pzp(\mathfrak{g})$, then it is easy to verify that $\varphi(A') = [\varphi(A)]'$ and $\varphi(A'') = [\varphi(A)]''$.

Each element $x \in \mathfrak{g}$ can be uniquely written as the linear combination of the Chevalley basis in the form

$$x = \sum_{\alpha\in\Delta} a_\alpha h_\alpha + \sum_{\beta\in\Phi} b_\beta e_\beta.$$

Since each coefficient is uniquely determined by x and the element in the basis, we will denote the coefficient b_β of e_β in the expression of x by $\{x\}_\beta$ for brevity. For $\alpha \in \Phi^+$, set

$$X_\alpha = \{\beta \in \Phi^+ \mid \beta + \alpha \in \Phi\}; \quad Y_\alpha = \{\beta \in \Phi^+ \mid \beta + \alpha \notin \Phi\}.$$

Lemma 2.1. *Let $h \in \mathfrak{h}$, $\alpha \in \Phi^+$. If $\beta(h) = 0$ for all $\beta \in Y_\alpha$, then $h = 0$.*

Proof. Let $\Delta_1 = \Delta \cap Y_\alpha$ and $\Delta_2 = \Delta \cap X_\alpha$. By assumption, if $\beta \in \Delta_1$, then $\beta(h) = 0$. If $\beta \in \Delta_2$, let k be the maximal positive integer such that $\beta + k\alpha \in \Phi^+$, i.e., $\beta + k\alpha \in \Phi^+$ and $\beta + (k+1)\alpha \notin \Phi^+$. That is to say, $\beta + k\alpha \in Y_\alpha$, forcing $(\beta + k\alpha)(h) = 0$. Since $\alpha \in Y_\alpha$ we know $\alpha(h) = 0$, and it follows immediately that $\beta(h) = 0$. Therefore, $\beta(h) = 0$ for all $\beta \in \Delta$, leading to $h = 0$. □

Lemma 2.2. *$\{e_\alpha\}'' = \mathfrak{g}_\alpha$ for any $\alpha \in \Phi$.*

Proof. We may assume, without loss of generality, that $\alpha \in \Phi^+$. Obviously, $\mathfrak{g}_\alpha \subseteq \{e_\alpha\}''$. Conversely, it is easy to see that $ker\ \alpha + (\sum_{\beta\in Y_\alpha} \mathfrak{g}_\beta) \subseteq \{e_\alpha\}'$. Suppose $x = h_0 + n_0 \in \{e_\alpha\}''$, where $h_0 \in \mathfrak{h}, n_0 \in \mathfrak{n} + \mathfrak{n}^-$, then $[h_0 + n_0, e_\beta] = 0$ for all $\beta \in Y_\alpha$. This implies that $\beta(h_0) = 0$ for all $\beta \in Y_\alpha$, leading to $h_0 = 0$ (thanks to Lemma 2.1). Write n_0 as $n_0 = \sum_{\beta\in\Phi} a_\beta e_\beta$. For any $h \in ker\ \alpha$, by $[h, n_0] = 0$, we have that $\beta(h)a_\beta = 0$ for each $\beta \in \Phi$. If $\beta \neq \pm\alpha$, then there exists $h_0 \in \mathfrak{h}$ such that $\alpha(h_0) = 0$ and $\beta(h_0) \neq 0$. Then by $\beta(h_0)a_\beta = 0$ we have that $a_\beta = 0$. Now the expression of x has a reduced form as $x = a_\alpha e_\alpha + a_{-\alpha}e_{-\alpha}$. By $[x, e_\alpha] = 0$, we know $a_{-\alpha} = 0$. Thus $x = a_\alpha e_\alpha \in \mathfrak{g}_\alpha$. Finally, $\{e_\alpha\}'' = \mathfrak{g}_\alpha$. □

Lemma 2.3. *$\{d_\alpha\}'' = Fd_\alpha$ for each $\alpha \in \Delta$.*

Proof. Obviously, $\{d_\alpha\}' = \mathfrak{h} + \sum_{\beta\in\Gamma} \mathfrak{g}_\beta$, where $\Gamma = \{\gamma \in \Phi \mid \gamma(d_\alpha) = 0\}$. If $x = h_0 + n_0 \in \{d_\alpha\}''$, where $h_0 \in \mathfrak{h}, n_0 \in \mathfrak{n} + \mathfrak{n}^-$, then $n_0 = 0$, since $[\mathfrak{h}, n_0] = 0$. Thus $x = h_0 \in \mathfrak{h}$. Obviously, the simple root which is different from α lies in Γ. Thus $[h_0, \mathfrak{g}_\beta] = 0$, i.e., $\beta(h_0) = 0$, $\forall\ \beta \in \Delta \setminus \{\alpha\}$, and it follows that $h_0 \in Fd_\alpha$. So $x \in Fd_\alpha$; i.e., $\{d_\alpha\}'' \subseteq Fd_\alpha$. Obviously, $Fd_\alpha \subseteq \{d_\alpha\}''$. Therefore $\{d_\alpha\}'' = Fd_\alpha$. □

Lemma 2.4. *Let $x_0 = h_0 + n_0 \in \mathfrak{b}$, where $h_0 \in \mathfrak{h}, n_0 \in \mathfrak{n}$. If h_0 and n_0 are both nonzero and $[h_0, n_0] = 0$, then the dimension of $\{x_0\}''$ is at least two.*

Proof. To prove the result it suffices to prove that if $[x_0, y] = 0$ for some $y \in \mathfrak{g}$, then $[h_0, y] = [n_0, y] = 0$. Suppose that $y = h + n$ satisfies $[x_0, y] = 0$, where $h \in \mathfrak{h}, n \in \mathfrak{n} + \mathfrak{n}^-$, and assume that

$$n = \sum_{\beta\in\Phi} y_\beta e_\beta, \quad n_0 = \sum_{\alpha\in\Phi^+} x_\alpha e_\alpha.$$

If we can show that $[h_0, n] = 0$, then we have $[h_0, y] = [n_0, y] = 0$. Thus the aim is achieved. Otherwise, if

$$[h_0, n] = \sum_{\beta\in\Phi} y_\beta \beta(h_0) e_\beta \neq 0,$$

assume that α_0 is a root with the lowest height such that $y_{\alpha_0}\alpha_0(h_0) \neq 0$. To get a contradiction, we now show that the coefficient of e_{α_0} in the expression of $[h, n_0] = \sum_{\alpha\in\Phi^+} \alpha(h)x_\alpha e_\alpha$ and the coefficient of e_{α_0} in the expression of $[n_0, n]$ (written as the linear combination of the Chevally basis) are respectively zero.

If $\alpha_0 \in \Phi^+$, since $x_{\alpha_0} = 0$ (recall that $[h_0, n_0] = 0$ and $\alpha_0(h_0) \neq 0$), the term $\alpha_0(h)x_{\alpha_0}e_{\alpha_0}$, in the expression of $[h, n_0]$, is just zero. If $\alpha_0 \in \Phi^-$, then the term e_{α_0} does not emerge in the expression of $[h, n_0] = \sum_{\alpha\in\Phi^+} \alpha(h)x_\alpha e_\alpha$.

Writing $[n_0, n]$ as the linear combination of the Chevally basis of $\mathfrak{g}$ and considering the coefficient of e_{α_0} in the expression, we find that

$$\{[n_0, n]\}_{\alpha_0} = \sum_{\alpha\in\Phi^+,\beta\in\Phi,\alpha+\beta=\alpha_0} x_\alpha y_\beta N_{\alpha,\beta}. \tag{2.4.1}$$

We show that it also takes the value 0. For $\alpha \in \Phi^+, \beta \in \Phi$, satisfying $\alpha + \beta = \alpha_0$, if $\alpha(h_0) \neq 0$, then $x_\alpha = 0$ (recall that $[h_0, n_0] = 0$); thus $x_\alpha y_\beta N_{\alpha,\beta} = 0$. Suppose $\alpha(h_0) = 0$. Then $\beta(h_0) \neq 0$ (recall that $\alpha_0(h_0) \neq 0$). If $y_\beta \neq 0$, then β is a root with lower height than that of α_0 such that $y_\beta\beta(h_0) \neq 0$. This contradicts the way that we choose α_0. So $y_\beta = 0$. Then we also have $x_\alpha y_\beta N_{\alpha,\beta} = 0$. Thus $\{[n_0, n]\}_{\alpha_0} = 0$.

Now by

$$\{[h_0, n]\}_{\alpha_0} \neq 0, \quad \{[h, n_0]\}_{\alpha_0} = 0, \quad \{[n_0, n]\}_{\alpha_0} = 0,$$

we have that $[h_0, n]+[n_0, h]+[n_0, n] \neq 0$, contradicting the assumption that $[x_0, y] = 0$. □

For an arbitrary finite-dimensional Lie algebra L over F, $x \in L$ is called strongly ad-nilpotent if there exists $y \in L$ and a non-zero eigenvalue a such that $x \in L_a(ad\ y)$. Let $\mathcal{N}(L)$ denote the set of all strongly ad-nilpotent elements of L, and let $\mathcal{E}(L)$ be the subgroup of $Int(L)$ generated by all $exp(ad\ x), x \in \mathcal{N}(L)$. The well-known Winter's theorem (see [4, page 84, Theorem 16.4]) says that all Borel subalgebras of L are conjugate under $\mathcal{E}(L)$.

Let $\mathfrak{d}$ be a given subspace of $\mathfrak{h}$, and define Ψ (relative to $\mathfrak{d}$) to be the subset of Φ consisting of $\alpha \in \Phi$ satisfying $\alpha(d) = 0$ for all $d \in \mathfrak{d}$. Set

$$\Psi^+ = \Psi \cap \Phi^+, \quad \Psi^i = \{\alpha \in \Psi \mid ht\ \alpha = i\} \ \text{ where } i > 0.$$

Then $\mathfrak{d}' = \mathfrak{h} + \sum_{\alpha\in\Psi} \mathfrak{g}_\alpha$. It is not difficult to see that $\mathfrak{d}'$ is a subalgebra of $\mathfrak{g}$ with a Borel subalgebra $\mathfrak{h} + \sum_{\alpha\in\Psi^+} \mathfrak{g}_\alpha$. Here we denote $\sum_{\alpha\in\Psi^+} \mathfrak{g}_\alpha$ by $\mathfrak{n}_\Psi$, and $\mathfrak{h} + \mathfrak{n}_\Psi$ by $\mathfrak{b}_\Psi$. One can easily see that any $\sigma \in \mathcal{E}(\mathfrak{d}')$ fixes each element in $\mathfrak{d}$. Define G_Ψ^+ to be the subgroup of $\mathcal{E}(\mathfrak{d}')$ generated by all $\sigma_\alpha(t)$ for $\alpha \in \Psi^+$, $t \in F$.

For a given $x \in \mathfrak{b}_\Psi$, it has a unique decomposition as $x = h_x + n_x$ with $h_x \in \mathfrak{h}$ and $n_x \in \mathfrak{n}_\Psi$. Define

$$\Omega_x = \{\alpha \in \Psi^+ \mid \{n_x\}_\alpha \cdot \alpha(h_x) \neq 0\};$$
$$\Gamma_x = \{\alpha \in \Psi^+ \mid \{n_x\}_\alpha \cdot \alpha(h_x) = 0\}.$$

Then $\Omega_x \cup \Gamma_x = \Psi^+$ and $\Omega_x \cap \Gamma_x = \emptyset$. If $\Omega_x \neq \emptyset$, i.e., $[h_x, n_x] \neq 0$, we call the minimal height of the roots in Ω_x the *Jordan degree* of x, and we denote it by D_x. Otherwise, if $\Omega_x = \emptyset$, i.e., $[h_x, n_x] = 0$, we define the *Jordan degree* of x to be $ht\ \theta + 1$.

Lemma 2.5. (i) *For an element x in $\mathfrak{b}_\Psi$, we can find some $\sigma \in G_\Psi^+$ such that the Jordan degree of $\sigma(x)$ is $ht\ \theta + 1$; i.e., $\Omega_{\sigma(x)} = \emptyset$.*

(ii) *For an element x in $\mathfrak{d}'$, we can find some $\sigma \in \mathcal{E}(\mathfrak{d}')$ such that $\sigma(x) = h + n$, where $h \in \mathfrak{h}$, $n \in \mathfrak{n}_\Psi$ satisfy $[h, n] = 0$.*

Proof. Let $G^+_\Psi(x)$ denote the set of $\sigma(x)$ for all $\sigma \in G^+_\Psi$. We now use decreasing induction on the Jordan degree of the elements in $G^+_\Psi(x)$ to prove the lemma. If D_x takes the maximal value $ht\ \theta + 1$, then we choose σ to be the identity element in G^+_Ψ and $\sigma(x)$ is as desired. Suppose that the Jordan degree of x, assumed to be k, is strictly smaller than $ht\ \theta + 1$, and assume that $x = h + n$ with $h \in \mathfrak{h}$, $n \in \mathfrak{n}_\Psi$. Since $D_x = k$, we can express n in the form

$$n = \sum_{\beta \in \Gamma_x} a_\beta e_\beta + \sum_{ht\ \alpha = k}^{\alpha \in \Omega_x} a_\alpha e_\alpha + \sum_{ht\ \alpha \geq k+1}^{\alpha \in \Omega_x} a_\alpha e_\alpha .$$

Choose $\sigma_1 = \prod_{ht\ \alpha=k}^{\alpha \in \Omega_x} \sigma_\alpha(a_\alpha \cdot \alpha(h)^{-1}) \in G^+_\Psi$, where the product is taken according to any fixed order. Considering the action of σ_1 on x, we have

$$\sigma_1(x) = \sigma_1(h) + \sigma_1(\sum_{\beta \in \Gamma_x} a_\beta e_\beta) + \sigma_1(\sum_{ht\ \alpha = k}^{\alpha \in \Omega_x} a_\alpha e_\alpha) + \sigma_1(\sum_{ht\ \alpha \geq k+1}^{\alpha \in \Omega_x} a_\alpha e_\alpha). \tag{2.5.1}$$

For a positive integer i, we denote $\sum_{ht\ \alpha \geq i}^{\alpha \in \Psi} \mathfrak{g}_\alpha$ by $\mathfrak{n}^i_\Psi$. It is easy to see that

$$\sigma_1(h) \equiv h - \sum_{ht\ \alpha = k}^{\alpha \in \Omega_x} a_\alpha e_\alpha\ (mod\ \mathfrak{n}^{k+1}_\Psi), \tag{2.5.2}$$

$$\sigma_1(\sum_{\alpha \in \Gamma_x} a_\beta e_\beta) \equiv \sum_{\alpha \in \Gamma_x} a_\beta e_\beta\ (mod\ \mathfrak{n}^{k+1}_\Psi), \tag{2.5.3}$$

$$\sigma_1(\sum_{ht\ \alpha = k}^{\alpha \in \Omega_x} a_\alpha e_\alpha) \equiv \sum_{ht\ \alpha = k}^{\alpha \in \Omega_x} a_\alpha e_\alpha\ (mod\ \mathfrak{n}^{k+1}_\Psi), \tag{2.5.4}$$

$$\sigma_1(\sum_{ht\ \alpha \geq k+1}^{\alpha \in \Omega_x} a_\alpha e_\alpha) \in \mathfrak{n}^{k+1}_\Psi. \tag{2.5.5}$$

Substituting Equations 2.5.2–2.5.5 into 2.5.1 we have that

$$\sigma_1(x) \equiv h + \sum_{\alpha \in \Gamma_x} a_\beta e_\beta\ (mod\ \mathfrak{n}^{k+1}_\Psi), \tag{2.5.6}$$

from which we find that $D_{\sigma_1(x)} > D_x$. Then we conclude, by induction, that there exists $\sigma_2 \in G^+_\Psi$ such that the Jordan degree of $\sigma_2\sigma_1(x)$ is $ht\ \theta + 1$. In other words, $\Omega_{\sigma_2\sigma_1(x)} = \emptyset$. Taking σ to be $\sigma_2\sigma_1$, we complete the proof of (i).

For $x \in \mathfrak{d}'$, by Winter's theorem, there exists σ_0 in $\mathcal{E}(\mathfrak{d}')$ such that $\sigma_0(x) \in \mathfrak{b}_\Psi$. By (i) of this lemma we can find $\sigma_1 \in G^+_\Psi \leq \mathcal{E}(\mathfrak{d}')$ such that $\Omega_{\sigma_1\sigma_0(x)} = \emptyset$. That is to say, if we assume that $\sigma_1\sigma_0(x) = h + n$ with $h \in \mathfrak{h}$, $n \in \mathfrak{n}_\Psi$, then $[h, n] = 0$. □

Let $\delta = \sum_{\beta \in \Delta} \beta$. For any given $\alpha \in \Delta$, we can write δ as the sum of the simple roots in the form $\delta = \alpha_1 + \alpha_2 + \ldots + \alpha_l$ such that the first simple root is just α and each partial sum $\alpha_1 + \alpha_2 + \ldots + \alpha_i$ is a root (which we denote by δ_i). For δ_i $(1 \leq i \leq l)$, assume k_i is the maximal non-negative integer such that $\delta_i + k_i\alpha_i$ is a root, and denote $\delta_i + k_i\alpha_i$ by $\bar{\delta_i}$. If $i \geq 2$, then $\bar{\delta_i} + \alpha_i$ is not a root and $\bar{\delta_i} - \alpha_i$ is a positive root. Set $M^+_\alpha = \{\bar{\delta_1}, \bar{\delta_2}, \ldots, \bar{\delta_l}\}$, $M^-_\alpha = \{-\beta \mid \beta \in M^+_\alpha\}$ and $M_\alpha = M^+_\alpha \cup M^-_\alpha$.

Lemma 2.6. *Let $\alpha \in \Delta$ be a simple root and let M_α be as defined above. If $\beta \in M_\alpha$, then we can find some $\eta \in \Phi$ such that $\beta + \eta \notin \Phi \cup \{0\}$, $\beta - \eta \in \Phi$, and $\eta(d_\alpha) = 0$.*

Proof. If $\beta = \bar{\delta_1}$ (exactly α itself), then we choose $\eta = -\alpha_2$. If $\beta = \bar{\delta_i}$ with $i \geq 2$, we choose $\eta = \alpha_i$. If $\beta = -\bar{\delta_1}$, we choose $\eta = \alpha_2$. If $\beta = -\bar{\delta_i}$ with $i \geq 2$, we choose $\eta = -\alpha_i$. Then one will see that $\beta + \eta \notin \Phi$, $\beta - \eta \in \Phi$, and $\eta(d_\alpha) = 0$. □

Lemma 2.7. *Let $l \geq 2$, $\alpha \in \Delta$ and let $\varphi \in Pzp(\mathfrak{g})$. Then*

(i) *for each $\beta \in M_\alpha$, there exist $t_\beta \in \mathfrak{h}$, $a_\beta \in F$, such that $[\varphi(d_\alpha), \varphi(e_\beta)] = \varphi(t_\beta) + a_\beta \varphi(e_\beta)$;*

(ii) *$[\varphi(d_\alpha), \varphi(e_\alpha)] = b_\alpha \varphi(d_\alpha) + a_\alpha \varphi(e_\alpha)$ for some $b_\alpha \in F$;*

(iii) *the inner derivation induced by $\varphi(d_\alpha)$ has non-zero eigenvalue.*

Proof. Because $\varphi(\mathfrak{h})$ and $\varphi(e_\gamma), \gamma \in \Phi$, span $\mathfrak{g}$, we may assume that

$$[\varphi(d_\alpha), \varphi(e_\beta)] = \varphi(t_\beta) + \sum_{\gamma \in \Phi} a_\gamma \varphi(e_\gamma), \quad t_\beta \in \mathfrak{h},\ a_\gamma \in F. \tag{2.7.1}$$

If there exists some $\beta_0 \in \Phi$, distinct with $\pm\beta$, such that $a_{\beta_0} \neq 0$, we choose $h \in \mathfrak{h}$ such that $\beta(h) = 0$ and $\beta_0(h) \neq 0$. Then by Jacobi's identity we have that

$$[\varphi(h), [\varphi(d_\alpha), \varphi(e_\beta)]] = 0. \tag{2.7.2}$$

That is,

$$[\varphi(h), \varphi(t_\beta) + \sum_{\gamma \in \Phi} a_\gamma \varphi(e_\gamma)] = 0. \tag{2.7.3}$$

It follows that

$$[h, t_\beta + \sum_{\gamma \in \Phi} a_\gamma e_\gamma] = \sum_{\gamma \in \Phi} a_\gamma \gamma(h) e_\gamma = 0. \tag{2.7.4}$$

So $a_\gamma \gamma(h) = 0$ for each $\gamma \in \Phi$, contradicting the fact that $a_{\beta_0}\beta_0(h) \neq 0$. So equation (2.7.1) has a reduced form as

$$[\varphi(d_\alpha), \varphi(e_\beta)] = \varphi(t_\beta) + a_\beta \varphi(e_\beta) + a_{-\beta}\varphi(e_{-\beta}). \tag{2.7.5}$$

To complete the proof of (i) we need to show that $a_{-\beta} = 0$. By Lemma 2.6, we can choose $\eta \in \Phi$ such that $\beta + \eta \notin \Phi \cup \{0\}$, $\beta - \eta \in \Phi$, and $\eta(d_\alpha) = 0$. Applying the inner derivation *ad* $\varphi(e_\eta)$ on two sides of equation (2.7.5) we get

$$[\varphi(e_\eta), [\varphi(d_\alpha), \varphi(e_\beta)]] = [\varphi(e_\eta), \varphi(t_\beta) + a_\beta \varphi(e_\beta) + a_{-\beta}\varphi(e_{-\beta})]. \tag{2.7.6}$$

It is not difficult to see that the left side of (2.7.6) is just zero (thanks to Jacobi's identity), so

$$[\varphi(e_\eta), \varphi(t_\beta) + a_\beta \varphi(e_\beta) + a_{-\beta}\varphi(e_{-\beta})] = 0, \tag{2.7.7}$$

and it follows that $[e_\eta, t_\beta + a_\beta e_\beta + a_{-\beta} e_{-\beta}] = 0$, leading to $a_{-\beta} = 0$. Thus (2.7.5) is reduced to

$$[\varphi(d_\alpha), \varphi(e_\beta)] = \varphi(t_\beta) + a_\beta \varphi(e_\beta). \tag{2.7.8}$$

This completes the proof of (i).

By (i), we know that $[\varphi(d_\alpha), \varphi(e_\alpha)] = \varphi(t_\alpha) + a_\alpha \varphi(e_\alpha)$ for certain $t_\alpha \in \mathfrak{h}$. For each simple root η, distinct in α, Jacobi's identity shows that $[\varphi(e_{-\eta}), [\varphi(d_\alpha), \varphi(e_\alpha)]] = 0$. That is, $[\varphi(e_{-\eta}), \varphi(t_\alpha) + a_\alpha \varphi(e_\alpha)] = 0$, and it follows that $[e_{-\eta}, t_\alpha + a_\alpha e_\alpha] = \eta(t_\alpha) e_{-\eta} = 0$. So $\eta(t_\alpha) = 0$ for each simple root $\eta (\neq \alpha)$, forcing $t_\alpha \in F d_\alpha$. This completes the proof of (ii).

For (iii), we claim that there exists at least one $\beta \in M_\alpha$ such that a_β (in equation (2.7.8)) is not zero. Otherwise, if $a_\beta = 0$ for all $\beta \in M_\alpha$, then $[\varphi(d_\alpha), \varphi(e_\beta)] = \varphi(t_\beta),\ \forall\ \beta \in M_\alpha$. Assume that $\sum_{\beta\in M_\alpha} k_\beta t_\beta = 0$ for $k_\beta \in F$. Then

$$[\varphi(d_\alpha), \varphi(\sum_{\beta\in M_\alpha} k_\beta e_\beta)] = \varphi(\sum_{\beta\in M_\alpha} k_\beta t_\beta) = 0,$$

and it follows that $[d_\alpha, \sum_{\beta\in M_\alpha} k_\beta e_\beta] = 0$. That is, $\sum_{\beta\in M_\alpha} k_\beta \beta(d_\alpha) e_\beta = 0$. Since the set $\{e_\beta \mid \beta \in M_\alpha\}$ is linearly independent we have that $k_\beta \beta(d_\alpha) = 0$ for all $\beta \in M_\alpha$. Thus $k_\beta = 0$ for all $\beta \in M_\alpha$ since $\beta(d_\alpha) \neq 0,\ \forall\ \beta \in M_\alpha$. This shows that $\{t_\beta \mid \beta \in M_\alpha\}$ is linearly independent, contradicting the fact that M_α has $2l$ elements and $dim(\mathfrak{h}) = l$. Assume that $a_{\beta_0} \neq 0$ for some $\beta_0 \in M_\alpha$. Then by

$$[\varphi(d_\alpha), [\varphi(d_\alpha), \varphi(e_{\beta_0})]] = a_{\beta_0}[\varphi(d_\alpha), \varphi(e_{\beta_0})],$$

we know that $[\varphi(d_\alpha), \varphi(e_{\beta_0})]$ is just an eigenvector of $\varphi(d_\alpha)$ relative to the non-zero eigenvalue a_{β_0}. □

The following lemma is fundamental for this paper.

Lemma 2.8. *Let $l \geq 2$ and $\varphi \in Pzp(\mathfrak{g})$. There exists $\sigma \in Int(\mathfrak{g})$ such that $(\sigma \cdot \varphi)(\mathfrak{h}) = \mathfrak{h}$.*

Proof. Arrange the simple roots in Δ in any given order: $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$, and let $\mathfrak{d}_\Delta = \{d_1, d_2, \ldots, d_l\}$ be the dual basis of $\mathfrak{h}$ relative to Δ; namely, $\alpha_i(d_j) = 1$ when $i = j$; $\alpha_i(d_j) = 0$ when $i \neq j$. We now consider the action of φ on $d_1, d_2, \ldots, d_l$ in turn. For $\varphi(d_1)$, we can find (by Lemma 2.5) $\sigma_1 \in \mathcal{E}(\mathfrak{g}) \leq Int(\mathfrak{g})$ such that $(\sigma_1 \cdot \varphi)(d_1) = h_1 + n_1$, where $h_1 \in \mathfrak{h}$, $n_1 \in \mathfrak{n}$, and $[h_1, n_1] = 0$. Denote $\sigma_1 \cdot \varphi$ by φ_1. Since $\{d_1\}'' = Fd_1$ is one dimensional, we have that $\{h_1 + n_1\}''$ is also one dimensional. Thus either $h_1 = 0$ or $n_1 = 0$ (applying Lemma 2.4). By Lemma 2.7, we know that the inner derivation induced by $\varphi_1(d_1)$ has non-zero eigenvalue, so $n_1 = 0$ and $h_1 \neq 0$ (note that n_1 is ad-nilpotent). Denote by $\mathfrak{d}_1$ the subspace of $\mathfrak{h}$ spanned by h_1. For $\varphi_1(d_2) \in \mathfrak{d}_1'$, by Lemma 2.5, we can find $\sigma_2 \in \mathcal{E}(\mathfrak{d}_1') \leq \mathcal{E}(\mathfrak{g}) \leq Int(\mathfrak{g})$ such that $(\sigma_2 \cdot \varphi_1)(d_2) = h_2 + n_2$, where $h_2 \in \mathfrak{h}$, $n_2 \in \mathfrak{n}$, and $[h_2, n_2] = 0$. Similarly, we know that $n_2 = 0$ and $h_2 \neq 0$. Denote $\sigma_2 \cdot \varphi_1$ by φ_2 and denote by $\mathfrak{d}_2$ the subspace of $\mathfrak{h}$ spanned by h_1, h_2. Obviously, $\varphi_2(d_j) = h_j,\ j = 1, 2$. Generally, suppose that we have found $\sigma_1 \in \mathcal{E}(\mathfrak{g})$, $\sigma_2 \in \mathcal{E}(\mathfrak{d}_1'), \ldots, \sigma_{k-1} \in \mathcal{E}(\mathfrak{d}_{k-2}')$ (where $2 \leq k \leq l$) such that

$$(\sigma_{k-1} \cdot \sigma_{k-2} \cdots \sigma_1 \cdot \varphi)(d_i) = h_i \in \mathfrak{h}, \quad i = 1, 2, \ldots, k-1.$$

Denote by $\mathfrak{d}_{k-1}$ the subspace of $\mathfrak{h}$ spanned by $h_1, h_2, \ldots, h_{k-1}$ and denote $\sigma_{k-1} \cdot \sigma_{k-2} \cdots \sigma_1 \cdot \varphi$ by φ_{k-1}. We now intend to find $\sigma_k \in \mathcal{E}(\mathfrak{d}_{k-1}')$ such that $(\sigma_k \cdot \varphi_{k-1})(d_j) \in \mathfrak{h}$ for $j = 1, 2, \ldots, k$. By $[d_k, d_i] = 0$ for $i = 1, 2, \ldots, k-1$, we know that $[\varphi_{k-1}(d_k), \mathfrak{d}_{k-1}] = 0$. Thus $\varphi_{k-1}(d_k) \in \mathfrak{d}_{k-1}'$. For $\varphi_{k-1}(d_k)$, we can find $\sigma_k \in \mathcal{E}(\mathfrak{d}_{k-1}') \leq \mathcal{E}(\mathfrak{g}) \leq Int(\mathfrak{g})$ such that $(\sigma_k \cdot \varphi_{k-1})(d_k) = h_k + n_k$, where $h_k \in \mathfrak{h}$, $n_k \in \mathfrak{n}$, and $[h_k, n_k] = 0$. Similarly to the first step, we also have $n_k = 0$. So $(\sigma_k \cdot \varphi_{k-1})(d_k) = h_k \in \mathfrak{h}$. Note that σ_k fixes each h_i for $i = 1, 2, \ldots, k-1$. So $(\sigma_k \cdot \varphi_{k-1})(d_i) = h_i \in \mathfrak{h}$, for $i = 1, 2, \ldots k$. Finally, we conclude by induction that there exist $\sigma_1, \sigma_2, \ldots, \sigma_l \in Int(\mathfrak{g})$, such that

$$(\sigma_l \cdot \sigma_{l-1} \cdots \sigma_1 \cdot \varphi)(d_i) \in \mathfrak{h}, \quad i = 1, 2, \ldots, l.$$

Setting $\sigma = \sigma_l \cdot \sigma_{l-1} \cdots \sigma_1$ we have that $(\sigma \cdot \varphi)(\mathfrak{h}) = \mathfrak{h}$. □

Lemma 2.9. *Let $l \geq 2$. If $\varphi \in Pzp(\mathfrak{g})$ stabilizes $\mathfrak{h}$, then for each $\alpha \in \Phi$, there exists $\gamma \in \Phi$ such that $\varphi(\mathfrak{g}_\alpha) = \mathfrak{g}_\gamma$.*

Proof. First we prove that, for each $\alpha \in \Delta$, there exists $\gamma \in \Phi$ such that $\varphi(\mathfrak{g}_\alpha) = \mathfrak{g}_\gamma$. For a given $\alpha \in \Delta$, suppose $\varphi(e_\alpha) = h + \sum_{\beta \in \Phi} a_\beta e_\beta,\ h \in \mathfrak{h},\ a_\beta \in F$. If $a_\gamma \neq 0$ and $a_\xi \neq 0$ for some linearly independent $\gamma,\ \xi \in \Phi$, then $C_\mathfrak{h}(\varphi(e_\alpha))$ has dimension $\leq l-2$, since it is contained in $(ker\ \gamma) \cap (ker\ \xi)$. On the other hand, since $C_\mathfrak{h}(\varphi(e_\alpha)) = \varphi(ker\ \alpha)$, it has dimension $l-1$, which is absurd. So $\varphi(e_\alpha) \in \mathfrak{h} + \mathfrak{g}_{\beta_0} + \mathfrak{g}_{-\beta_0}$ for certain $\beta_0 \in \Phi^+$. Now the expression of $\varphi(e_\alpha)$ is reduced to $\varphi(e_\alpha) = h + a_{\beta_0} e_{\beta_0} + a_{-\beta_0} e_{-\beta_0}$. We consider $[\varphi(d_\alpha), \varphi(e_\alpha)]$. On one hand, we may assume (by Lemma 2.7) that

$$[\varphi(d_\alpha), \varphi(e_\alpha)] = b_\alpha \varphi(d_\alpha) + c_\alpha \varphi(e_\alpha). \tag{2.9.1}$$

That is,

$$[\varphi(d_\alpha), \varphi(e_\alpha)] = b_\alpha \varphi(d_\alpha) + c_\alpha h + c_\alpha a_{\beta_0} e_{\beta_0} + c_\alpha a_{-\beta_0} e_{-\beta_0}. \tag{2.9.2}$$

On the other hand,

$$[\varphi(d_\alpha), \varphi(e_\alpha)] = a_{\beta_0} \beta_0(\varphi(d_\alpha)) e_{\beta_0} - a_{-\beta_0} \beta_0(\varphi(d_\alpha)) e_{-\beta_0}. \tag{2.9.3}$$

Comparing (2.9.2) with (2.9.3), we have that $h = -c_\alpha^{-1} b_\alpha \varphi(d_\alpha)$ (noting that $c_\alpha \neq 0$) and $a_{\beta_0} a_{-\beta_0} = 0$. Thus $\varphi(e_\alpha) = h + a_\gamma e_\gamma$, where $\gamma = \beta_0$ or $-\beta_0$. To complete the proof of this lemma, we only need to show that $h = 0$. Suppose $h \neq 0$. We will give a contradiction. Obviously, $a_\gamma \neq 0$. If $[h, e_\gamma] = 0$, then the dimension of $\{h + a_\gamma e_\gamma\}''$ is at least 2, contradicting the fact that $dim(\{e_\alpha\}'') = 1$ (recall Lemma 2.2). If $[h, e_\gamma] \neq 0$, choosing $\sigma_\gamma(\gamma(h)^{-1} a_\gamma) \in Int(\mathfrak{g})$, we then have $(\sigma_\gamma(\gamma(h)^{-1} a_\gamma) \cdot \varphi)(e_\alpha) = h$. Thus $(\varphi^{-1} \cdot \sigma_\gamma(-\gamma(h)^{-1} a_\gamma))(h) = e_\alpha$. By Lemma 2.7 we know that $(\varphi^{-1} \cdot \sigma_\gamma(-\gamma(h)^{-1} a_\gamma))(h)$ has non-zero eigenvalue (recall that $h = -c_\alpha^{-1} b_\alpha \varphi(d_\alpha)$), which is absurd. So $h = 0$.

Now for any given $\beta \in \Phi$, there exist $w \in \mathscr{W}$ and $\alpha \in \Delta$ such that $\beta = w(\alpha)$. Let $\phi : N \mapsto \mathscr{W}$ be the natural homomorphism, and let $\omega \in N$ be an original image of w in N. Then ω permutes the root spaces $\mathfrak{g}_\xi,\ \xi \in \Phi$, in the same way as w permutes Φ, and $\omega(\mathfrak{h}) = \mathfrak{h}$. Noting that $(\varphi \cdot \omega)(\mathfrak{h}) = \mathfrak{h}$, and applying the result of the above paragraph, we get $\varphi(\mathfrak{g}_\beta) = (\varphi \cdot \omega)(\mathfrak{g}_\alpha) = \mathfrak{g}_\gamma$ for certain $\gamma \in \Phi$. □

Lemma 2.9 shows that if $\varphi \in Pzp(\mathfrak{g})$ stabilizes $\mathfrak{h}$, then it induces a permutation ρ_φ on Φ in such a way: $\varphi(\mathfrak{g}_\alpha) = \mathfrak{g}_{\rho_\varphi(\alpha)},\ \alpha \in \Phi$.

Lemma 2.10. *Suppose that $\varphi \in Pzp(\mathfrak{g})$ stabilizes $\mathfrak{h}$.*

(i) *For any linearly independent roots α, β, we have $[\varphi(\mathfrak{g}_\alpha), \varphi(\mathfrak{g}_\beta)] = \varphi([\mathfrak{g}_\alpha, \mathfrak{g}_\beta])$;*

(ii) *For $\alpha \in \Phi$, $\rho_\varphi(-\alpha) = -\rho_\varphi(\alpha)$;*

(iii) *If ρ_φ sends each simple root to a positive root, then it permutes the sets Φ^+ and Φ^-, respectively.*

Proof. For (i), if $\alpha + \beta$ is not a root, the assertion holds obviously. Assume $\alpha + \beta$ is a root and denote it by γ. Then either $\gamma + \alpha$ or $\gamma + \beta$ fails to be a root. Suppose $\gamma + \alpha$ is not a root (without loss of generality). Choose $h \in \mathfrak{h}$ such that $\beta(h) = 0$ and $\gamma(h) = -N_{\alpha,\beta}$. Then

$$[h + e_\alpha, e_\beta + e_\gamma] = 0. \tag{2.10.1}$$

Applying φ we have

$$[\varphi(h) + \varphi(e_\alpha), \varphi(e_\beta) + \varphi(e_\gamma)] = 0. \tag{2.10.2}$$

It follows that

$$[\varphi(e_\alpha),\varphi(e_\beta)] = -[\varphi(h),\varphi(e_\gamma)]. \tag{2.10.3}$$

So

$$\begin{aligned}[\varphi(\mathfrak{g}_\alpha),\varphi(\mathfrak{g}_\beta)] &= [\varphi(h),\varphi(\mathfrak{g}_\gamma)] \\ &= \varphi(\mathfrak{g}_\gamma) = \varphi([\mathfrak{g}_\alpha,\mathfrak{g}_\beta]).\end{aligned}$$

For (ii), if $\rho_\varphi(-\alpha) \neq -\rho_\varphi(\alpha)$, then $[\varphi(\mathfrak{g}_{-\alpha}),\varphi(\mathfrak{g}_\alpha)] = \mathfrak{g}_\beta$ for certain $\beta \in \Phi$. Assume that $\rho_\varphi(\gamma) = \beta$ for some $\gamma \in \Phi$. Then $\varphi(\mathfrak{g}_\gamma) = [\varphi(\mathfrak{g}_{-\alpha}),\varphi(\mathfrak{g}_\alpha)]$, where $\gamma \neq \pm\alpha$. Find $h \in \mathfrak{h}$ such that $\gamma(h) \neq 0$ and $\alpha(h) = 0$. Then $[\varphi(h),[\varphi(\mathfrak{g}_{-\alpha}),\varphi(\mathfrak{g}_\alpha)]] = 0$ (thanks to Jacobi's identity). On the other hand, $[\varphi(h),\varphi(\mathfrak{g}_\gamma)] \neq 0$, which is absurd.

(iii) immediately follows from (i). □

Lemma 2.11. *Suppose $\varphi \in Pzp(\mathfrak{g})$ stabilizes $\mathfrak{h}$. There exists $\omega \in N$ such that the permutation on Φ induced by $\varphi \cdot \omega$ stabilizes Φ^+ and Φ^-, respectively.*

Proof. Obviously, if Φ^+ is stabilized by ρ_φ, then so is Φ^-. Now let $|\rho_\varphi|$ be the number of positive roots sent by ρ_φ into Φ^-. We will give the proof by induction on $|\rho_\varphi|$. If $|\rho_\varphi| = 0$, then the assertion already holds (choose ω to be the identity map). Now assume that the assertion holds for $|\rho_\varphi| = k-1$ $(1 \leq k)$. For the case where $|\rho_\varphi| = k$, there exists at least one $\beta \in \Delta$ such that $\rho_\varphi(\beta) \in \Phi^-$ (thanks to Lemma 2.10). Suppose that $\alpha \in \Delta$ is such a simple root. By Lemma 2.10, we know $\rho_\varphi(-\alpha) = -\rho_\varphi(\alpha) \in \Phi^+$. Take $\omega_\alpha = \sigma_\alpha(1)\sigma_{-\alpha}(-1)\sigma_\alpha(1) \in N$. One will see that ω_α stabilizes $\mathfrak{h}$, it sends $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\alpha}$, and it permutes the set $\{\mathfrak{g}_\beta \mid \beta \in \Phi^+ \setminus \{\alpha\}\}$. Thus $(\varphi \cdot \omega_\alpha)(\mathfrak{g}_\alpha) = \varphi(\mathfrak{g}_{-\alpha}) = \mathfrak{g}_{\rho_\varphi(-\alpha)}$, where $\rho_\varphi(-\alpha) \in \Phi^+$. Denote $\varphi \cdot \omega_\alpha$ by φ_1. Then we have that $|\rho_{\varphi_1}| = |\rho_\varphi| - 1$. By induction assumption, we can find $\omega_1 \in N$ such that $(\rho_{\varphi_1 \cdot \omega_1})(\beta) \in \Phi^+$ for all $\beta \in \Phi^+$. Finally, setting $\omega = \omega_\alpha \cdot \omega_1$, we complete the proof. □

Lemma 2.12. *Assume that $\varphi \in Pzp(\mathfrak{g})$ stabilizes $\mathfrak{h}$ and ρ_φ permutes Φ^+. Then ρ_φ permutes Δ.*

Proof. Let θ be the unique maximal root in Φ^+, and write it as $\theta = \alpha_1 + \alpha_2 + \cdots + \alpha_m$, where $\alpha_i \in \Delta$ and all partial sum $\beta_j = \alpha_1 + \alpha_2 + \ldots + \alpha_j,\ j = 1, 2, \ldots, m$, are roots. Then by applying φ on $\mathfrak{g}_\theta$ we have that

$$\begin{aligned}\mathfrak{g}_{\rho_\varphi(\theta)} &= \varphi(\mathfrak{g}_\theta) = [\varphi(\mathfrak{g}_{\beta_{m-1}}),\varphi(\mathfrak{g}_{\alpha_m})] \\ &= [[\varphi(\mathfrak{g}_{\beta_{m-2}}),\varphi(\mathfrak{g}_{\alpha_{m-1}})],\varphi(\mathfrak{g}_{\alpha_m})] \\ &= \ldots\ldots \\ &= [[[[\varphi(\mathfrak{g}_{\alpha_1}),\varphi(\mathfrak{g}_{\alpha_2})],\varphi(\mathfrak{g}_{\alpha_3})],\ldots,],\varphi(\mathfrak{g}_{\alpha_m})] \\ &= [[[[\mathfrak{g}_{\rho_\varphi(\alpha_1)},\mathfrak{g}_{\rho_\varphi(\alpha_2)}],\mathfrak{g}_{\rho_\varphi(\alpha_3)}],\ldots,],\mathfrak{g}_{\rho_\varphi(\alpha_m)}] \\ &= \mathfrak{g}_{\rho_\varphi(\alpha_1)+\rho_\varphi(\alpha_2)+\ldots+\rho_\varphi(\alpha_m)}.\end{aligned}$$

It follows that

$$\rho_\varphi(\theta) = \rho_\varphi(\alpha_1) + \rho_\varphi(\alpha_2) + \ldots + \rho_\varphi(\alpha_m) \in \Phi^+.$$

Comparing the height of $\rho_\varphi(\theta)$ and the height of each $\rho_\varphi(\alpha_i)$, one easily sees that $\rho_\varphi(\alpha_i) \in \Delta$ for $i = 1, 2, \ldots, m$. Finally we get $\rho_\varphi(\alpha) \in \Delta$ for each $\alpha \in \Delta$. □

For the purpose of proving Lemma 2.14 we need a lemma for determining the sign of each structure constant $N_{r,s}$ for $r, s \in \Phi$. Suppose we are given a total ordering on the space containing the roots. An ordered pair (r, s) of roots will be called a *special pair* if $r+s \in \Phi$ and $0 \prec r \prec s$. An ordered pair (r, s) is called *extraspecial* if (r, s) is a special pair and if for all special pairs (r_1, s_1) with $r+s = r_1+s_1$ we have $r \preceq r_1$. Then every root in Φ^+ that is the sum of two roots in Φ^+ can be expressed uniquely as the sum of an extraspecial pair. Since every positive root that is not simple has this property, the extraspecial pairs are in 1–1 correspondence with the roots in $\Phi^+ \setminus \Delta$.

Lemma 2.13 ([3, page 58, Proposition 4.2.2])**.** *The sign of the structure constants $N_{r,s}$ may be chosen arbitrarily for extraspecial pairs of roots (r, s), and then the structure constants for all pairs are uniquely determined.*

Lemma 2.14. *Let $\varphi \in Pzp(\mathfrak{g})$ and assume that φ fixes each element in $\mathfrak{h}$.*
(i) *If φ fixes each e_α for $\alpha \in \Delta$, then φ fixes each e_β for $\beta \in \Phi^+$;*
(ii) *If φ fixes each $e_{-\alpha}$ for $\alpha \in \Delta$, then φ fixes each $e_{-\beta}$ for $\beta \in \Phi^+$.*

Proof. We only prove (i). If Φ is not of the type G_2, we wish to show, by induction on $ht\,\beta$, that φ fixes all e_β for $\beta \in \Phi^+$. If $ht\,\beta = 1$, the result already holds. Assume that $\varphi(e_\beta) = e_\beta$ for $\beta \in \Phi^+$ with height $k-1$ (where $2 \le k \le ht\,\theta$). Let $ht\,\beta = k$. Since Φ is not of the type G_2, we can find $\alpha \in \Delta$ such that $\gamma = \beta - \alpha \in \Phi^+$, but $\beta + \alpha \notin \Phi^+$. Choose $h \in \mathfrak{h}$ such that $\gamma(h) = 0$ and $\beta(h) = -N_{\alpha,\gamma}$. Then by $[e_\alpha + h, e_\gamma + e_\beta] = 0$ and $\varphi(e_\gamma) = e_\gamma$, we have that $[e_\alpha + h, e_\gamma + \varphi(e_\beta)] = 0$. This implies that $\varphi(e_\beta) = e_\beta$. Thus the assertion holds (by induction).

If Φ is of the type G_2, assume that the base of Φ consists of α_1 and α_2, where α_1 is long and α_2 is short. We define the total ordering in the real linear space spanned by Δ in such a way that $a\alpha_1 + b\alpha_2 \preceq c\alpha_1 + d\alpha_2$ if and only if one of the following conditions holds: (1) $a < c$, (2) $a = c$ and $b \le d$. Under the definition of this ordering, one will find that the pairs (α_2, α_1), $(\alpha_2, \alpha_1 + \alpha_2)$, $(\alpha_2, \alpha_1 + 2\alpha_2)$ are all extraspecial. So the signs of the corresponding structure constants may be chosen arbitrarily (thanks to Lemma 2.13). Now we may assume (recalling equation (2.0.1)) that $N_{\alpha_2,\alpha_1} = 1$, $N_{\alpha_2,\alpha_1+\alpha_2} = 2$, $N_{\alpha_2,\alpha_1+2\alpha_2} = 3$. Let $h_0 = -d_{\alpha_2}$. Then $(\alpha_1 + k\alpha_2)(h_0) = -k$. It is not difficult to see that

$$[e_{\alpha_2} + h_0, e_{\alpha_1} + e_{\alpha_1+\alpha_2} + e_{\alpha_1+2\alpha_2} + e_{\alpha_1+3\alpha_2}] = 0,$$

which implies that

$$[e_{\alpha_2} + h_0, e_{\alpha_1} + \varphi(e_{\alpha_1+\alpha_2}) + \varphi(e_{\alpha_1+2\alpha_2}) + \varphi(e_{\alpha_1+3\alpha_2})] = 0,$$

from which we have that $\varphi(e_{\alpha_1+k\alpha_2}) = e_{\alpha_1+k\alpha_2}$ for $k = 1, 2, 3$. Now consider the action of φ on e_θ. Let θ_1 denote the root $\theta - \alpha_1$ and choose $h \in \mathfrak{h}$ such that $\theta_1(h) = 0$ and $\theta(h) = -N_{\alpha_1,\theta_1}$. Then by $[h + e_{\alpha_1}, e_{\theta_1} + e_\theta] = 0$, we obtain $[h + e_{\alpha_1}, e_{\theta_1} + \varphi(e_\theta)] = 0$. So $\varphi(e_\theta) = e_\theta$. Also we have $\varphi(e_\beta) = e_\beta$ for all $\beta \in \Phi^+$. □

3. Proof of the main theorem

If $l = 1$, then Φ is of the type A_1. In this case, $[x, y] = 0$ if and only if x and y are linearly dependent. If φ is an invertible linear map on $\mathfrak{g}$, then $[x, y] = 0 \Leftrightarrow [\varphi(x), \varphi(y)] = 0$, showing that $\varphi \in Pzp(\mathfrak{g})$. Hence $Pzp(\mathfrak{g}) = GL(\mathfrak{g})$.

Now suppose $rank(\mathfrak{g}) \ge 2$. Let $\varphi \in Pzp(\mathfrak{g})$. By Lemma 2.8, we can find some $\sigma \in Int(\mathfrak{g})$ such that $(\sigma \cdot \varphi)(\mathfrak{h}) = \mathfrak{h}$. By Lemma 2.11, we can choose $\omega \in N \le Int(\mathfrak{g})$ such that the permutation on Φ induced by $\sigma \cdot \varphi \cdot \omega$ stabilizes Φ^+ and Φ^-, respectively. Denote $\sigma \cdot \varphi \cdot \omega$ by φ_1 and denote the permutation on Φ

induced by φ_1 by ρ (for brevity). Then ρ permutes Δ (thanks to Lemma 2.12). Now we shall show that $\langle\rho(\alpha),\rho(\beta)\rangle = \langle\alpha,\beta\rangle$ for all $\alpha,\beta\in\Delta$. If $(\alpha,\beta)=0$, then $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]=0$. Applying Lemma 2.10, we have $[\mathfrak{g}_{\rho(\alpha)},\mathfrak{g}_{\rho(\beta)}]=0$, which leads to $(\rho(\alpha),\rho(\beta))=0$. Thus $\langle\rho(\alpha),\rho(\beta)\rangle=\langle\alpha,\beta\rangle=0$. So the assertion holds. Now suppose $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle=k>0$, and assume, without loss of generality, that $\langle\alpha,\beta\rangle=-k$ and $\langle\beta,\alpha\rangle=-1$. Then $\alpha+k\beta$, denoted by γ, is a positive root, but $\alpha+(k+1)\beta$ fails to be a root. Applying Lemma 2.10 repeatedly, we have

$$\begin{aligned}
\mathfrak{g}_{\rho(\gamma)} &= \varphi_1(\mathfrak{g}_\gamma)\\
&= [\varphi_1(e_\beta),\varphi_1(\mathfrak{g}_{\alpha+(k-1)\beta})]\\
&= [[\varphi_1(e_\beta),[\varphi_1(e_\beta),\varphi_1(\mathfrak{g}_{\alpha+(k-2)\beta})]]]\\
&= \cdots\\
&= (ad\ \varphi_1(e_\beta))^k(\varphi_1(\mathfrak{g}_\alpha))\\
&= (ad\ e_{\rho(\beta)})^k(\mathfrak{g}_{\rho(\alpha)})\\
&= \mathfrak{g}_{\rho(\alpha)+k\rho(\beta)}.
\end{aligned}$$

So $\rho(\gamma)=\rho(\alpha)+k\rho(\beta)$ is a positive root. By $[\mathfrak{g}_\beta,\mathfrak{g}_\gamma]=0$, we have that $[\mathfrak{g}_{\rho(\beta)},\mathfrak{g}_{\rho(\gamma)}]=0$. Thus $\rho(\beta)+\rho(\gamma)$ is not a root. This shows that $\langle\rho(\alpha),\rho(\beta)\rangle=-k=\langle\alpha,\beta\rangle$. So the aim is achieved. Now we see that ρ is just a symmetry of the Dynkin diagram of Φ. Using ρ we construct the graph automorphism φ_ρ of $\mathfrak{g}$. Then $(\varphi_\rho)^{-1}\cdot\varphi_1$ stabilizes each $\mathfrak{g}_\alpha$, $\alpha\in\Delta$. Furthermore, one will easily see that $(\varphi_\rho)^{-1}\cdot\varphi_1$ stabilizes each $\mathfrak{g}_\beta$, $\beta\in\Phi^+$ (thanks to Lemma 2.10). Denote $(\varphi_\rho)^{-1}\cdot\varphi_1$ by φ_2.

For any given $\alpha\in\Delta$, since

$$Fd_\alpha=\bigcap_{\beta\in\Delta\setminus\{\alpha\}} ker\ \beta=\bigcap_{\beta\in\Delta\setminus\{\alpha\}} C_{\mathfrak{h}}(\mathfrak{g}_\beta)$$

and

$$\varphi_2(C_{\mathfrak{h}}(\mathfrak{g}_\beta))=C_{\mathfrak{h}}(\varphi_2(\mathfrak{g}_\beta))=C_{\mathfrak{h}}(\mathfrak{g}_\beta),\quad \text{for } \forall\ \beta\in\Delta,$$

we have that $\varphi_2(Fd_\alpha)=Fd_\alpha$ for $\alpha\in\Delta$. Now suppose $\varphi_2(d_\alpha)=c_\alpha d_\alpha$ for $\alpha\in\Delta$. We wish to show that all c_α, $\alpha\in\Delta$, are equal. Write θ as the linear combination of the simple roots: $\theta=\sum_{\alpha\in\Delta}k_\alpha\alpha$, where all k_α are positive integers. We know that

$$C_{\mathfrak{h}}(\mathfrak{g}_\theta)=ker\ \theta=\{\sum_{\alpha\in\Delta}x_\alpha d_\alpha\in\mathfrak{h}\mid \sum_{\alpha\in\Delta}k_\alpha x_\alpha=0\},$$

which is an $(l-1)$-dimensional subspace of $\mathfrak{h}$. If $\sum_{\alpha\in\Delta}k_\alpha x_\alpha=0$, then $\sum_{\alpha\in\Delta}x_\alpha d_\alpha\in C_{\mathfrak{h}}(\mathfrak{g}_\theta)$. Thus

$$\sum_{\alpha\in\Delta}c_\alpha x_\alpha d_\alpha=\varphi_2(\sum_{\alpha\in\Delta}x_\alpha d_\alpha)\in\varphi_2(C_{\mathfrak{h}}(\mathfrak{g}_\theta))=C_{\mathfrak{h}}(\varphi_2(\mathfrak{g}_\theta))=C_{\mathfrak{h}}(\mathfrak{g}_\theta).$$

It follows that $\sum_{\alpha\in\Delta}c_\alpha k_\alpha x_\alpha=0$. So the equation (in variables x_α, $\alpha\in\Delta$)

$$\sum_{\alpha\in\Delta}k_\alpha x_\alpha=0$$

and the equations

$$\begin{cases}\sum_{\alpha\in\Delta}k_\alpha x_\alpha=0,\\ \sum_{\alpha\in\Delta}c_\alpha k_\alpha x_\alpha=0\end{cases}$$

have the same solutions. So all $\frac{c_\alpha k_\alpha}{k_\alpha}$ $(= c_\alpha)$ are equal for $\alpha \in \Delta$. Now we denote the common value by c. Using c we construct the scalar multiplication map φ_c. Then we see that $\varphi_c^{-1} \cdot \varphi_2$ fixes each element in $\mathfrak{h}$. Denote $\varphi_c^{-1} \cdot \varphi_2$ by φ_3.

Now suppose that $\varphi_3(e_\alpha) = b_\alpha e_\alpha$ for $\alpha \in \Delta$, and define

$$\lambda: \ P = \mathbb{Z}\Phi \to F^*, \quad \sum_{\alpha \in \Delta} k_\alpha \alpha \mapsto \prod_{\alpha \in \Delta} b_\alpha^{k_\alpha}.$$

Then λ is an F-character of P. Using it we construct the diagonal automorphism φ_λ of $\mathfrak{g}$. Then $\varphi_\lambda^{-1} \cdot \varphi_3$ will further fix each e_α for $\alpha \in \Delta$. Denote $\varphi_\lambda^{-1} \cdot \varphi_3$ by φ_4. We know from Lemma 2.14 that φ_4 fixes each e_β for $\beta \in \Phi^+$.

For $\alpha \in \Delta$, choose $\beta \in \Delta$ such that $\alpha + \beta \in \Phi^+$, and choose $h \in \mathfrak{h}$ such that $\beta(h) = -N_{-\alpha,\alpha+\beta}$ and $(\alpha+\beta)(h) = 0$. Then $[e_{-\alpha} + h, e_{\alpha+\beta} + e_\beta] = 0$. Applying φ_4, we have $[\varphi_4(e_{-\alpha}) + h, e_{\alpha+\beta} + e_\beta] = 0$, from which we get $\varphi_4(e_{-\alpha}) = e_{-\alpha}$. Furthermore, we have, by Lemma 2.14, that φ_4 fixes each $e_{-\beta}$ for $\beta \in \Phi^+$. So φ_4 is just the identity map on $\mathfrak{g}$. Finally we see that $\varphi_\lambda^{-1} \cdot \varphi_c^{-1} \cdot \varphi_\rho^{-1} \cdot \sigma \cdot \varphi \cdot \omega = I_{\mathfrak{g}}$. So

$$\varphi = \sigma^{-1} \cdot \varphi_\rho \cdot \varphi_\lambda \cdot \omega^{-1} \cdot \varphi_c \in Aut(\mathfrak{g}) \times F^* I_{\mathfrak{g}}.$$

This completes the proof of (ii).

References

[1] M. Brešar, Commuting traces of biadditive mappings, commutativity preserving mappings, and Lie mappings, *Trans. Amer. Math. Soc.,* 335(1993) 525-546. MR1069746 (93d:16044)

[2] Y. Cao, Z. Chen, C. Huang, Commutativity preserving linear maps and Lie automorphisms of strictly triangular matrix space, *Lin. Alg. Appl.,* 350(2002) 41-66. MR1906746 (2003c:15002)

[3] R. W. Carter, Simple Groups of Lie Type, Wiley Interscience, New York, 1972. MR0407163 (53:10946)

[4] J. E. Humphreys, Introduction to Lie algebras and representation theory, Springer, 1972. MR0323842 (48:2197)

[5] N. Jacobson, Lie Algebras, Interscience Publishers, New York–London, 1962. MR0143793 (26:1345)

[6] L. W. Marcoux, A. R. Sourour, Commutativity preserving linear maps and Lie automorphisms of triangular matrix algebras, *Lin. Alg. Appl.,* 288(1999) 89-104. MR1670535 (99k:16072)

[7] M. Omladič, On operators preserving commutativity, *J. Funct. Anal.,* 66(1986) 105-122. MR829380 (87k:47080)

[8] P. Šemrl, Non-linear commutativity preserving maps, *Acta Sci. Math. (Szeged)*, 71(2005) 781-819. MR2206609 (2006j:47060)

[9] W. Watkins, Linear maps that preserve commuting pairs of matrices, *Lin. Alg. Appl.,* 14(1976) 29-35. MR0480574 (58:732)

[10] W. J. Wong, Maps on simple algebras preserving zero products, II: Lie algebras of linear type, *Pacific J. Math.,* 92(1981) 469-488. MR618078 (82k:15002b)

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