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RELATING DIAMETER AND MEAN CURVATURE FOR RIEMANNIAN SUBMANIFOLDS

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ABSTRACT. Given an m-dimensional closed connected Riemannian manifold M smoothly isometrically immersed in an n-dimensional Riemannian manifold N, we estimate the diameter of M in terms of its mean curvature field integral under some geometric restrictions, and therefore generalize a recent work of P. M. Topping in the Euclidean case (Comment. Math. Helv., 83 (2008), 539–546).

1. Introduction

Let $M \to N$ be an isometric immersion of Riemannian manifolds of dimension m and n, respectively. In this paper, we estimate the intrinsic diameter of the closed submanifold M in terms of its mean curvature vector integral, under some geometric restrictions involving the volume of M, the sectional curvatures of N, and the injectivity radius of N. In particular, we can estimate the intrinsic diameter of the closed submanifold M in terms of its mean curvature vector integral without any geometric restriction, provided the sectional curvatures of the ambient manifold N is non-positive. Our work was inspired by the following result of P. M. Topping [11], who treated the case $N = \mathbb{R}^n$.

Theorem A (Topping [11]). For $m \geq 1$, suppose that M is an m-dimensional closed (compact, no boundary) connected manifold smoothly immersed in $\mathbb{R}^n$. Then there exists a constant $C(m)$ dependent only on m such that its intrinsic diameter d_{int} and mean curvature H are related by

$$d_{int} \leq C(m) \int_M |H|^{m-1} d\mu, \tag{1.1}$$

where $d_{int} := \max_{x,y \in M} dist_M(x,y)$ and μ is the measure on M induced by the ambient space. In particular, we can take $C(2) = \frac{32}{\pi}$.

Prior to Topping's work, L. M. Simon in [6] (see also [8]) derived an interesting estimate of the external diameter $d_{ext} := \max_{x,y \in M^2 \hookrightarrow \mathbb{R}^3} |x-y|_{\mathbb{R}^3}$ of a closed

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connected surface M^2 immersed in $\mathbb{R}^3$ in terms of its area and Willmore energy:

$$d_{ext} < \frac{2}{\pi}\left(Area(M^2)\cdot\int_{M^2}|H|^2 d\mu\right)^{\frac{1}{2}}. \tag{1.2}$$

At the core of the proof of (1.2) is the following assertion that one cannot simultaneously have small area and small mean curvature in a ball within the surface. In other words, for all $r>0$, we have

$$\pi \le \frac{A_{ext}(x,r)}{r^2}+\frac{1}{4}\int_{B_{ext}(x,r)}|H|^2 d\mu, \tag{1.3}$$

where $B_{ext}(x,r)$ and $A_{ext}(x,r)$ denote the subset of M^2 immersed inside the open extrinsic ball in $\mathbb{R}^3$ centred at x of radius $r>0$ and its area, respectively. This type of estimate is from [6], and with these sharp constants from [8]. Combining this fact with a simple covering argument, one can derive (1.2). Note that if M^2 is a surface of constant mean curvature H immersed in $\mathbb{R}^3$, Topping in [7] used a different method and established the following inequality:

$$d_{ext} \le \frac{A|H|}{2\pi}.$$

Equality is achieved when M^2 is a sphere.

Following the idea of Simon's proof, Topping in [11] proved Theorem A by considering a refined version of (1.3) for any dimensional manifold immersed in $\mathbb{R}^n$. Roughly speaking, Topping asserted that the maximal function and volume ratio (see their definitions in Section 2) cannot be simultaneously smaller than a fixed dimensional constant. This assertion can be confirmed by means of the Michael-Simon Sobolev inequality for submanifolds of Euclidean space [4]. Then using this assertion and a covering lemma, one can derive (1.1) immediately. As an application, H.-Z. Li in a recent paper [3] used Theorem A to discuss the convergence of the volume-preserving mean curvature flow in Euclidean space under some initial integral pinching conditions.

On the other hand, as we all know, D. Hoffman and J. Spruck in [2] extended the Michael-Simon result [4] to a general Sobolev inequality for submanifolds of a Riemannian manifold under some geometric restrictions. To formulate their result, we need some notation from [2]. Let $M \to N$ be an isometric immersion of Riemannian manifolds of dimension m and n, respectively. We denote the sectional curvatures of N by K_N. The mean curvature vector field of the immersion is given by H. We write $\bar{R}(M)$ for the injectivity radius of N restricted to M (or minimum distance to the cut locus in N for all points in M). Let us denote by ω_m the volume of the unit ball in $\mathbb{R}^m$, and let b be a positive real number or a pure imaginary one.

Theorem B (D. Hoffman and J. Spruck [2])**.** *Let $M \to N$ be an isometric immersion of Riemannian manifolds of dimension m and n, respectively. Some notations are adopted as above. Assume $K_N \le b^2$, and let h be a non-negative C^1 function on M vanishing on ∂M. Then*

$$\left(\int_M h^{m/(m-1)} d\mu\right)^{(m-1)/m} \le c(m)\int_M \left[|\nabla h| + h|H|\right] d\mu, \tag{1.4}$$

provided

$$b^2(1-\beta)^{-2/m}\left(\omega_m^{-1} Vol(\mathrm{supp} h)\right)^{2/m} \le 1 \tag{1.5}$$

and

$$2\rho_0 \le \bar{R}(M), \tag{1.6}$$

where

$$\rho_0 = \begin{cases} b^{-1}\sin^{-1}\left[b(1-\beta)^{-1/m}\left(\omega_m^{-1}Vol(\mathrm{supp}h)\right)^{1/m}\right] & \text{for } b \text{ real,} \\ (1-\beta)^{-1/m}\left(\omega_m^{-1}Vol(\mathrm{supp}h)\right)^{1/m} & \text{for } b \text{ imaginary.} \end{cases}$$

Here β is a free parameter, $0<\beta<1$, and

$$c(m) := c(m,\beta) = \pi\cdot 2^{m-1}\beta^{-1}(1-\beta)^{-1/m}\frac{m}{m-1}\omega_m^{-1/m}. \tag{1.7}$$

Remark 1.1. In Theorem B, we may replace the assumption $h\in C^1(M)$ by $h\in W^{1,1}(M)$. As mentioned in the remark in [2], the optimal choice of β to minimize c is $\beta = m/(m+1)$. When b is real we may replace condition (1.6) by the stronger condition $\bar{R}\ge \pi b^{-1}$. When b is a pure imaginary number and the Riemannian manifold N is simply connected and complete, $\bar{R}(M)=+\infty$. Hence conditions (1.5) and (1.6) are automatically satisfied.

Motivated by the work of Topping, it is natural to expect that there exists a general geometric inequality for submanifolds of a Riemannian manifold, which is similar to Theorem A. Fortunately, following closely the lines of Topping's proof of Theorem A in [11], we can employ the Hoffman-Spruck Sobolev inequality for submanifolds of a Riemannian manifold together with a covering lemma to derive the desired results.

Theorem 1.2. *For $m\ge 1$, suppose that M is an m-dimensional closed connected Riemannian manifold smoothly isometrically immersed in an n-dimensional complete Riemannian manifold N with $K_N\le b^2$. For any $0<\alpha<1$, if*

$$b^2(1-\alpha)^{-2/m}\left(\omega_m^{-1}Vol(M)\right)^{2/m}\le 1 \tag{1.8}$$

and

$$2\rho_0 \le \bar{R}(M), \tag{1.9}$$

where

$$\rho_0 = \begin{cases} b^{-1}\sin^{-1}\left[b(1-\alpha)^{-1/m}\left(\omega_m^{-1}Vol(M)\right)^{1/m}\right] & \text{for } b \text{ real,} \\ (1-\alpha)^{-1/m}\left(\omega_m^{-1}Vol(M)\right)^{1/m} & \text{for } b \text{ imaginary,} \end{cases}$$

then there exists a constant $C(m,\alpha)$ dependent only on m and α such that

$$d_{int}\le C(m,\alpha)\int_M |H|^{m-1}d\mu.$$

In particular, we can take $C(2,\alpha)=\frac{576\pi}{\alpha^2(1-\alpha)}$.

Remark 1.3. In Theorem 1.2, the coefficients $C(m,\alpha)$ are not identical to (but strongly dependent on) the coefficients $c(m)$ in Theorem B. From (2.7) and (3.3) we can find that $C(m,\alpha)$ can still arrive at the minimum when $\alpha=\frac{m}{m+1}$. The conditions of (1.8) and (1.9) are similar to the restrictions of (1.5) and (1.6) in Theorem B, and they guarantee that the Hoffman-Spruck Sobolev inequality for submanifolds of a Riemannian manifold can be applied in the proof of our theorem. When b is real we may replace condition (1.9) by the stronger condition $\bar{R}\ge\pi b^{-1}$.

When b is a pure imaginary number and the Riemannian manifold N is simply connected and complete, $\bar{R}(M) = +\infty$, and hence conditions (1.8) and (1.9) are automatically satisfied.

In particular, when $N = \mathbb{R}^n$, $K_N \equiv 0$ and $\bar{R}(M) = +\infty$, and hence there are also no volume restrictions on M. Combining this with Remark 1.3, if b is pure imaginary or zero, then we see that conditions (1.8) and (1.9) are automatically satisfied, and hence we conclude that

Corollary 1.4. *For $m \geq 1$, suppose that M is an m-dimensional closed connected Riemannian manifold smoothly isometrically immersed in an n-dimensional simply connected, complete, non-positively curved Riemannian manifold N ($K_N \leq 0$). For any $0 < \alpha < 1$, there exists a constant $C(m, \alpha)$ dependent only on m and α such that*

$$d_{int} \leq C(m, \alpha) \int_M |H|^{m-1} d\mu,$$

where $\min_{0<\alpha<1} C(m, \alpha) = C(m, \frac{m}{m+1})$. *In particular, we can take* $\min_{0<\alpha<1} C(2, \alpha) = C(2, \frac{2}{3}) = 3888\pi$.

We remark that the constants $C(2, \alpha)$ in Theorem 1.2 and Corollary 1.4 are not optimal in general. The proof of Theorem 1.2 follows the proof in the Euclidean case [11]. Theorem 1.2 and Corollary 1.4 may have many interesting applications which we have not discussed here. For example, we may borrow Li's idea from [3] and apply our Theorem 1.2 to study the convergence problem of the volume-preserving mean curvature flow in Riemannian manifolds. We will explore this aspect in the future.

Besides the above works, the closest precedent for our theorem is another work of Topping on diameter estimates for intrinsic manifolds evolving under the Ricci flow [9]. In the Ricci flow case, Topping explored a log-Sobolev inequality of the Ricci flow (see Theorem 3.4 in [9]), which can be derived by the monotonicity of Perelman's $\mathcal{W}$-functional (see [1], [5], [10]). However, a core tool of proving Theorem 1.2 is the Hoffman-Spruck Sobolev inequality.

The rest of this paper is organized as follows. In Section 2, we will prove Lemma 2.1. The proof needs the key Hoffman-Spruck Sobolev inequality. In Section 3, we will finish the proof of Theorem 1.2 using Lemma 2.1 of Section 2 and a covering lemma.

2. Estimates for maximal function and volume ratio

In this section we first introduce two useful geometric quantities: the maximal function and the volume ratio. Then we apply the Hoffman-Spruck Sobolev inequality to prove the following important Lemma 2.1, which is essential in the proof of Theorem 1.2.

Given $x \in M^m$, with respect to a given metric, we denote the open geodesic ball in M^m centred at x and of intrinsic radius $r > 0$ by $B(x, r)$ and its volume by

$$V(x, r) := Vol(B(x, r)).$$

Following Topping's definitions in [11], when $m \geq 2$, we introduce the *maximal function*

$$M(x, R) := \sup_{r \in (0, R]} r^{-\frac{1}{m-1}} [V(x, r)]^{-\frac{m-2}{m-1}} \int_{B(x,r)} |H| d\mu \tag{2.1}$$

and the *volume ratio*

$$\kappa(x, R) := \inf_{r\in(0,R]} \frac{V(x,r)}{r^m} \tag{2.2}$$

for any $R > 0$.

Similar to Lemma 1.2 in [11], we have the following general result.

Lemma 2.1. *For $m \geq 2$, suppose that M is an m-dimensional Riemannian manifold smoothly isometrically immersed in an n-dimensional Riemannian manifold N with $K_N \leq b^2$, which is complete with respect to the induced metric. For any $0 < \alpha < 1$, if conditions* (1.8) *and* (1.9) *are satisfied, then there exists a constant $\delta > 0$ dependent only on m and α such that for any $x \in M$ and $R > 0$ at least one of the following is true:*

(1) $M(x, R) \geq \delta$;
(2) $\kappa(x, R) > \delta$.

In the case of closed surfaces ($m = 2$) *in N, we can choose $\delta = \frac{\alpha^2(1-\alpha)}{144\pi}$.*

Remark 2.2. In Lemma 2.1, when b is a pure imaginary number and the Riemannian manifold N is simply connected and complete, $\bar{R}(M) = +\infty$. Hence conditions (1.8) and (1.9) are automatically satisfied.

Now we will finish the proof of Lemma 2.1.

Proof of Lemma 2.1. We follow the ideas of the proof of Lemma 1.2 in [11]. Suppose that $M(x, R) < \delta$ for some constant $\delta > 0$, which will be chosen later. According to the definition of the maximal function $M(x, R)$, we know that for all $r \in (0, R]$

$$\int_{B(x,r)} |H| d\mu < \delta r^{\frac{1}{m-1}} [V(x, r)]^{\frac{m-2}{m-1}}. \tag{2.3}$$

Note that for fixed x, $V(r) := V(x, r)$ is differentiable for almost all $r > 0$. For such $r \in (0, R]$, and any $s > 0$, we define a Lipschitz cut-off function h on M by

$$h(y) = \begin{cases} 1 & y \in B(x, r), \\ 1 - \frac{1}{s}(dist_M(x, y) - r) & y \in B(x, r + s) \setminus B(x, r), \\ 0 & y \notin B(x, r + s). \end{cases} \tag{2.4}$$

Since function $\sin^{-1} x$ is increasing on $[0, 1]$ and $Vol(\mathrm{supp} h) \leq Vol(M)$, we easily see that conditions (1.8) and (1.9) guarantee the function h of (2.4) to satisfy conditions (1.5) and (1.6), where $\beta = \alpha$. Substituting this function to the Hoffman-Spruck Sobolev inequality from Theorem B, we derive that

$$\begin{aligned} V(r)^{(m-1)/m} &\leq \left(\int_M h^{m/(m-1)} d\mu \right)^{(m-1)/m} \\ &\leq c(m) \left[\frac{V(r+s) - V(r)}{s} + \int_{B(x,r+s)} |H| d\mu \right], \end{aligned}$$

where $c(m) := c(m, \alpha) = \pi \cdot 2^{m-1} \alpha^{-1} (1-\alpha)^{-1/m} \frac{m}{m-1} \omega_m^{-1/m}$. Letting $s \downarrow 0$, we conclude that

$$V(r)^{(m-1)/m} \leq c(m) \left[\frac{dV}{dr} + \int_{B(x,r)} |H| d\mu \right].$$

Combining this with (2.3), we have

$$\frac{dV}{dr}+\delta r^{\frac{1}{m-1}}V(r)^{\frac{m-2}{m-1}}-c(m)^{-1}V(r)^{\frac{m-1}{m}}>0. \tag{2.5}$$

Now we assume that $\delta>0$ is sufficiently small so that $\delta<\omega_m$, and define another smooth function

$$v(r):=\delta r^m.$$

Then a straightforward computation yields

$$\frac{dv}{dr}+\delta r^{\frac{1}{m-1}}v(r)^{\frac{m-2}{m-1}}-c(m)^{-1}v(r)^{\frac{m-1}{m}}=\left(m\delta+\delta^{\frac{2m-3}{m-1}}-c(m)^{-1}\delta^{\frac{m-1}{m}}\right)r^{m-1}. \tag{2.6}$$

We can see that

$$\frac{dv}{dr}+\delta r^{\frac{1}{m-1}}v(r)^{\frac{m-2}{m-1}}-c(m)^{-1}v(r)^{\frac{m-1}{m}}\le 0 \tag{2.7}$$

as long as $\delta>0$ is sufficiently small, depending only on m and α.

Notice that $V(r)/r^m\to\omega_m$ as $r\downarrow 0$, while $v(r)/r^m=\delta<\omega_m$. Combining inequalities (2.5) and (2.7), we conclude that

$$V(r)>v(r)$$

for all $r\in(0,R]$. Otherwise, there exists a fixed r_0 such that $V(r_0)=v(r_0)$ and $V(r)>v(r)$ for all $r\in(0,r_0)$. Then from (2.5) and (2.7), we can derive

$$\frac{dV}{dr}\Big|_{r=r_0}>\frac{dv}{dr}\Big|_{r=r_0}.$$

Namely,

$$\frac{dV}{dr}>\frac{dv}{dr}$$

in any sufficiently small neighborhood of r_0, which is impossible since $V(r_0)=v(r_0)$ and $V(r)>v(r)$ for all $r\in(0,r_0)$.

Therefore

$$\kappa(x,R):=\inf_{r\in(0,R]}\frac{V(x,r)}{r^m}>\delta,$$

which completes the proof of Lemma 2.1.

In the case of closed surfaces ($m=2$) in N, we can choose $\delta=\frac{c(2,\alpha)^{-2}}{9}=\frac{\alpha^2(1-\alpha)}{144\pi}$ to satisfy (2.7) and the constraint condition $\delta<\omega_2=\pi$. □

3. Diameter control

In this section we can follow the lines of [11] or [9] and easily prove Theorem 1.2 by using Lemma 2.1 and a covering lemma. For the completeness of this paper, here we will give the detailed proof of Theorem 1.2.

Proof of Theorem 1.2. We may assume $m\ge 2$ since the case $m=1$ is trivial. Now we choose $R>0$ sufficiently large so that the total volume of the closed manifold M is less than δR^m, where δ is given by Lemma 2.1 (notice that δ does not depend on R). In particular, for all $z\in M$, we must have

$$\kappa(z,R)\le\frac{V(z,R)}{R^m}\le\delta.$$

Hence by Lemma 2.1, as long as conditions (1.8) and (1.9) are satisfied, we must have the maximal function $M(x,R) \geq \delta$. Namely, there exists $r = r(z)$ such that

$$\begin{aligned} \delta &\leq r^{-\frac{1}{m-1}} V(z,r)^{-\frac{m-2}{m-1}} \int_{B(z,r)} |H| d\mu \\ &\leq r^{-\frac{1}{m-1}} \left(\int_{B(z,r)} |H|^{m-1} d\mu \right)^{\frac{1}{m-1}}, \end{aligned} \tag{3.1}$$

where we used the Hölder inequality for the second inequality above. Hence

$$r(z) \leq \delta^{1-m} \int_{B(z,r(z))} |H|^{m-1} d\mu. \tag{3.2}$$

Now we have to pick appropriate points z at which to apply (3.2). Let $x_1, x_2 \in M$ be extremal points in M. This means that $d_{int} = dist_M(x_1, x_2)$. Let Σ be a shortest geodesic connecting x_1 and x_2. Obviously, Σ is covered by the balls $\{B(z, r(z)) : z \in \Sigma\}$. By a modification of the covering lemma (see Lemma 5.2 in [9]), there exists a countable (possibly finite) set of points $\{z_i \in \Sigma\}$ such that the balls $\{B(z_i, r(z_i))\}$ are disjoint and cover at least a fraction ρ, where $\rho \in (0, \frac{1}{2})$ of Σ:

$$\rho d_{int} \leq \sum_i 2r(z_i).$$

Combining this with (3.2), we have

$$\begin{aligned} d_{int} &\leq \frac{2}{\rho} \sum_i r(z_i) \\ &\leq \frac{2}{\rho} \delta^{1-m} \sum_i \int_{B(z_i, r(z_i))} |H|^{m-1} d\mu \\ &\leq \frac{2}{\rho} \delta^{1-m} \int_M |H|^{m-1} d\mu, \end{aligned}$$

where $\delta > 0$ is sufficiently small, depending only on m and α. Letting $\rho \to \frac{1}{2}$, we arrive at

$$d_{int} \leq 4\delta^{1-m} \int_M |H|^{m-1} d\mu. \tag{3.3}$$

Hence the desired theorem follows. If $m = 2$, we can choose $4\delta^{1-m} = \frac{576\pi}{\alpha^2(1-\alpha)}$, since $\delta = \frac{\alpha^2(1-\alpha)}{144\pi}$. □

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